Lecture 6

$$R^J \xrightarrow{(r_{ij})} R^I \longrightarrow M \longrightarrow 0 \qquad r_{ij} \in R = End_A(P)^{op}$$

$M \mapsto \mathrm{Coker}\left(P^J \xrightarrow{(r_{ij})} P\right)$ Check well-defined

Functor: $R\text{-mod} \longrightarrow \mathcal{A}$

can be written as "$P \otimes_R -$"

Check: both compositions are isom to Id (using assumptions on P). □

Freyd-Mitchell embedding thm: If $\mathcal{A}$ is a (small) abelian category. Then $\exists$ ring R and an exact fully faithful functor:

$$\mathcal{A} \longrightarrow R\text{-mod}.$$

Idea of proof: Yoneda embedding:

$$Y: \mathcal{A} \longrightarrow Fun(\mathcal{A}^{op}, Ab)$$

$$X \mapsto Hom_{\mathcal{A}}(-, X)$$

∀ category:
Yoneda lemma
Fully faithful

not exact ↗ Can be fixed.

Abelian category ↗ Not R-mod

Snake lemma:

In R-mod:

commutative diagram

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \xrightarrow{f} & C' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C \end{array}$$

If the rows are exact then $\exists$ exact sequence:

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h$$

If $c' \in \ker(h)$

$$\partial(c') = i^{-1} g p^{-1}(c')$$

$p^{-1}(c')$: some preimage.

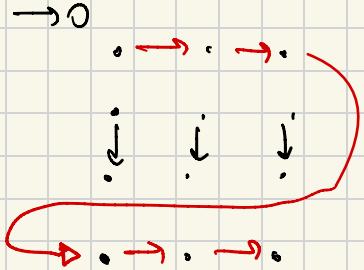
Well-defined.

"Diagram chasing":

If $0 \to \cdots \to 0$ then $0 \to \ker \to \cdots \to 0$.

$0 \to \cdots \to 0$

Snake:



True in any abelian category by embedding.

Projective / Injective resolution

P proj: $\mathrm{Hom}_{\mathcal{A}}(P, -)$ exact

I inj $\mathrm{Hom}_{\mathcal{A}}(-, I)$ exact

Df: $\mathcal{A}$ has enough proj if $\forall X \in \mathcal{A}$, $\exists\ P \to X \to 0$ epi, P proj.

$\mathcal{A}$ ——— inj if $\forall X \in \mathcal{A}$, $\exists\ 0 \to X \to I$ mono, I inj.

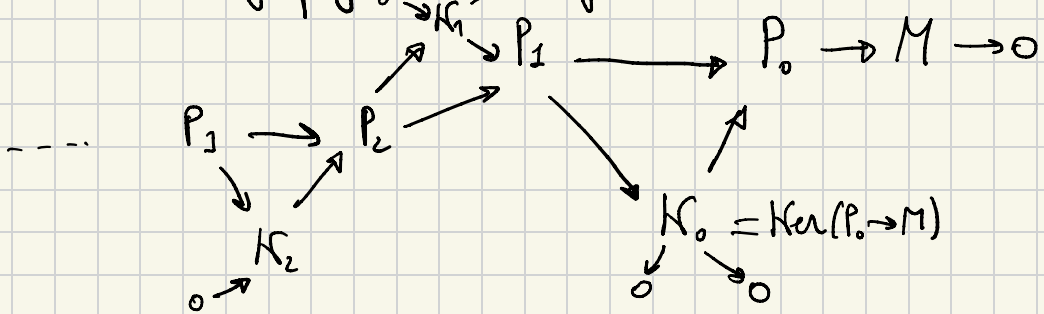
Ex:

R ring R-mod has enough proj: $\forall M, \exists I,\ R^{\oplus I} \to M \to 0$

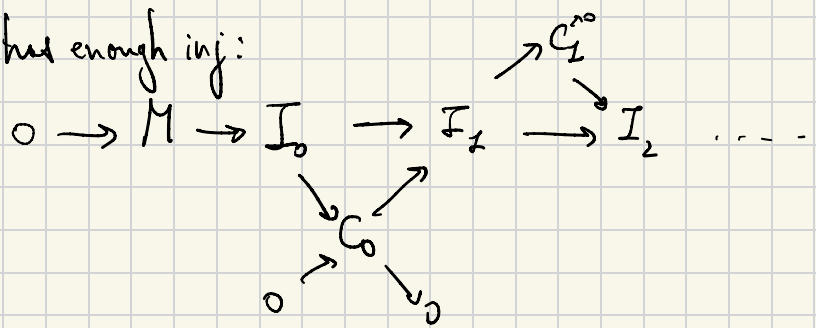
Can show: R-mod has enough injectives.

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Assume $\mathcal{A}$ has enough proj. Let $M \in \mathrm{obj}\,\mathcal{A}$



If $\mathcal{A}$ has enough inj:



Exact! $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ exact

$0 \to M \to I_0 \to I_1 \to \cdots$ exact