Homological Algebra

Sheet 1 — MT26

Section A

1. A, B, C $\in \mathrm{Mod}_R$. Show that there exist canonical R-module isomorphisms

$$\mathrm{Hom}(A \oplus B, C) \simeq \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$$
$$\mathrm{Hom}(A, B \oplus C) \simeq \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C).$$

More generally, prove

$$\mathrm{Hom}\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I} \mathrm{Hom}(M_i, N)$$
$$\textit{and} \qquad \mathrm{Hom}\left(M, \prod_{i\in I} N_i\right) \simeq \prod_{i\in I} \mathrm{Hom}(M, N_i).$$

Solution: We reduce to the general case by noting the equivalence between finite products and finite coproducts in abelian categories (in particular, in Mod_R). Now the canonical isomorphisms are precisely the content of the universal properties of products and coproducts.

Explicitly, the natural maps $p_j : \prod_{i\in I} N_i \to N_j$ and $q_j : M_j \to \bigoplus_{i\in I} M_i$ yield

$$\mathrm{Hom}(M, \prod_{i\in I} N_i) \to \prod_{i\in I} \mathrm{Hom}(M, N_i)$$

via $\phi \mapsto (p_j \circ \phi)_{j\in I}$ and $\mathrm{Hom}(\bigoplus_{i\in I} M_i, N) \to \prod_{i\in I} \mathrm{Hom}(M_i, N)$ via $\phi \mapsto (\phi \circ q_j)_{j\in I}$. The inverse maps are provided by the universal properties.

In the setting of Mod_R, these canonical bijections clearly preserve the R-module structures, e.g. because the explicit natural maps above do. We've implicitly used that the product in Mod_R is the product in Set (which of course follows from the right-adjointness of the forgetful functor (a left adjoint is the free functor).

2. A monomorphism is a morphism f satisfying $[f \circ g_1 = f \circ g_2] \Rightarrow [g_1 = g_2]$. An epimorphism is a morphism satisfying $[g_1 \circ f = g_2 \circ f] \Rightarrow [g_1 = g_2]$.

Given $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show (using the language of category theory) that f is a monomorphism and g is an epimorphism.

Pierrick Bousseau: pierrick.bousseau@maths.ox.ac.uk

Solution: We know that $f = \ker g$ and so is an equalizer.

Claim: Any equalizer is a monomorphism. Let $X \xrightarrow{f} Y$ be the equalizer of $Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$, i.e. for any W, $\mathrm{Hom}(W, X) \cong \{\phi \in \mathrm{Hom}(W, Y) : g_1 \circ \phi = g_2 \circ \phi\}$, i.e. X represents the functor mapping W to $\mathrm{Hom}(W, Y) \times_{g_1 \circ -, g_2 \circ -} \mathrm{Hom}(W, Y)$. The bijection is induced by $\psi \mapsto f \circ \psi$. Thus, $f \circ - : \mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ is injective for all W, i.e. f is injective. By applying $(-)^{op}$, we get the dual result that any co-equalizer is an epimorphism, and note that $\mathrm{coker}(f) = g$.

Pierrick Bousseau: pierrick.bousseau@maths.ox.ac.uk