Lecture 7

Limits and colimits

I : small category $F: I \to \mathcal{A}$ functor

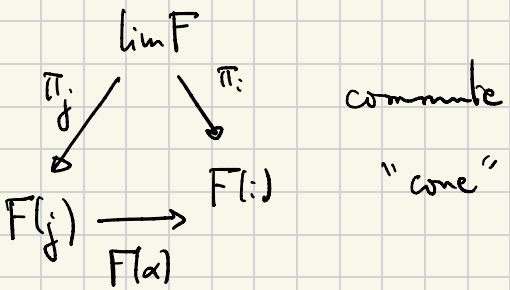
$\mathcal{A}$ any category

"diagram in $\mathcal{A}$"

$\varprojlim_I F$: object L of $\mathcal{A}$

with maps $\pi_i : L \to F(i) \quad i \in I$

s.t. $\forall (\alpha : j \to i) \in \mathrm{Hom}_I(j,i)$



commute

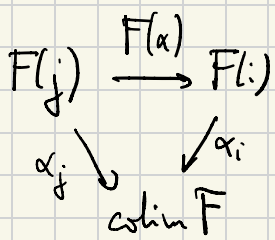
"cone"

+ universal with this property (final obj in category of cones)

Ex: I discrete (only id morphisms), $f: I \to \mathcal{A}$, $\{F(i)\}_{i \in I}$

$$\varprojlim F = \prod_{i \in I} f(i)$$

Dually: $\varinjlim F$



cocone.

Initial cocone.

Ex: I discrete

$$\varinjlim F = \coprod_{i \in I} f(i)$$

Ex: Equalizers & coequalizers.

The category Set has all limits & colimits.

Thm: $\mathcal{C}$ has all limits $\Leftrightarrow$ $\mathcal{C}$ has all products and equalizers.

——— colimits $\Leftrightarrow$ ——— coproducts and coequalizers.

Exercise:

$$\mathrm{Hom}_{\mathcal{A}}(X, \lim_{i \in I} F(i)) \xrightarrow{\sim} \lim_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X, F(i))$$

$$\mathrm{Hom}_{\mathcal{A}}(\operatorname*{colim}_{i \in I} F(i), X) \xrightarrow{\sim} \lim_{i \in I} \mathrm{Hom}_{\mathcal{A}}(F(i), X)$$

Let $L: \mathcal{A} \to \mathcal{B}$, $R: \mathcal{B} \to \mathcal{A}$ adjoint pair

Claim: $L(\operatorname*{colim}_{i \in I} A_i) \simeq \operatorname*{colim}_{i \in I} L(A_i)$.

$$R(\lim_{i \in I} B_i) \simeq \lim_{i \in I} R(B_i)$$

Proof: $\mathrm{Hom}_{\mathcal{A}}(X, R(\lim_{i \in I} B_i))$

$= \mathrm{Hom}_{\mathcal{B}}(L(X), \lim_{i \in I} B_i)$

$= \lim_{i \in I} \mathrm{Hom}_{\mathcal{B}}(L(X), B_i)$

$$= \lim_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X, R(B_i))$$

$$= \mathrm{Hom}_{\mathcal{A}}(X, \lim_{i \in I} R(B_i)) \quad + \text{Yoneda.}$$

□

Cor: L is right exact (preserves Cofer)

R is left exact (preserves Ker)

Ex: R commutative ring $- \otimes_R M$ left adjoint of Hom so commutes with colim eg $\bigoplus$

Cofer (right exact)

Want to calculate $N \otimes_R M$: $\exists$ presentation $a_{ij} \in R$

$$R^{\oplus J} \xrightarrow{(a_{ij})} R^{\oplus I} \to N \to 0$$

$- \otimes_R M$:

$$M^{\oplus J} \to M^{\oplus I} \to N \otimes_R M \to 0$$

$$R^{\oplus I} \otimes_R M = (R \otimes_R M)^{\oplus I} = M^{\oplus I}$$

$\uparrow$ $- \otimes_R M$ commute with $\oplus$

$$M^{\oplus J} \xrightarrow{(a_{ij})} M^{\oplus I} \to N \otimes_R M \to 0.$$

Ex: I: ideal of R generated by $\{x_j\}_{j \in J}$

$$R^{\oplus J} \to R \to R/I \to 0$$

$$M^{\oplus J} \to M \to R/I \otimes_R M \to 0 \Rightarrow \boxed{R/I \otimes_R M = M/IM}$$

—

R non-commutative M left-module

$- \otimes_R M : \underbrace{\text{mod-}R}_{\text{right-modules}} \to Ab$ left-adjoint to $\text{Hom}_{Ab}(M, -)$: $Ab \to \text{mod-}R$

Prop: If B is an R-S-bimodule and C a right S-module then $\text{Hom}_S(B, C)$ is a right R-module:

$$\text{Hom}_S(B, -) : \text{mod-}S \to \text{mod-}R$$

is right adjoint to $- \otimes_R B : \text{mod-}R \to \text{mod-}S$:

$$\text{Hom}_S(A \otimes_R B, C) \simeq \text{Hom}_R(A, \text{Hom}_S(B, C))$$

—

$\Rightarrow - \otimes_R B$ right exact and commutes with all colimits

Proof: Exercise.

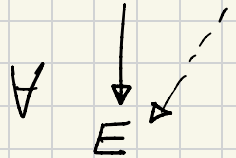
More on injectives: $\mathcal{A}$ abelian category

Recall: P projective: $\text{Hom}_{\mathcal{A}}(P, -)$ exact

I injective: $\text{Hom}_{\mathcal{A}}(-, I)$ exact.

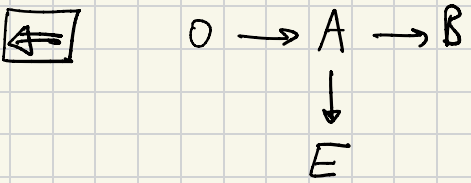
Baer's criterion E right R-module

E injective $\iff \forall J \subset R$ right ideal

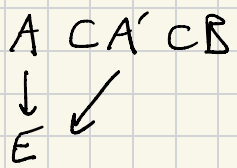


Proof: $\Rightarrow$ Clear

$\Leftarrow$



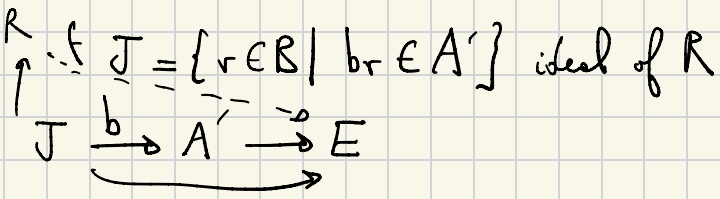
Look at partially ordered set



Zorn's lemma: $\exists$ max $A' \subset B$ (every chain as an upper-bound)



Assume $\exists b \in B \setminus A'$



can be extended to $f: R \to E$

$A'' := A' + bR$ $\quad \alpha'': A'' \to E$

$a + br \mapsto \alpha'(a) + f(r)$

well-defined (check)

Contradiction with max of A'. $\square$

Cor: If R PID, R-mod injective $\iff$ M divisible

M

$\forall m \in M \quad \exists r' \in M$

$r \in R \quad r = mr'$