Lecture 8

Ex: $R = \mathbb{Z}$ $\quad \mathbb{Q}, \ \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \mathbb{Z}_{p^\infty}$ (p prime) $\quad \mathbb{Q}/\mathbb{Z}$ injectives

I, abelian $\quad I = I_{tors} \oplus I_{tf}$ $\quad I_{tf}$: $\mathbb{Q}$-vector space

I inj; I_{tf}: Torsion free

I_{tors} = direct sum of $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$

Claim: Ab has enough injectives.

Pf: A : ab. group $\quad I(A) := \prod_{\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ ← injective as product of injectives

[Coproduct of proj is proj]

$\exists$ canonical map

$$e_A : A \longrightarrow I(A), \quad a \longmapsto (f \longmapsto f(a))$$

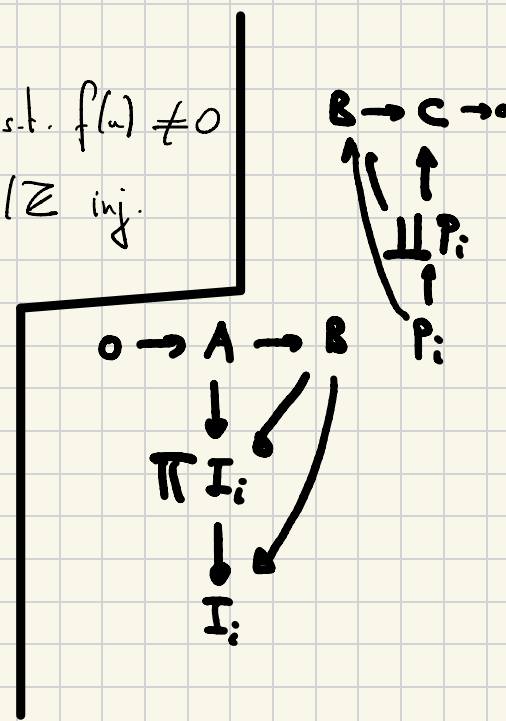
e_A is injective : $\forall a \neq 0$, $\exists f : A \to \mathbb{Q}/\mathbb{Z}$ s.t. $f(a) \neq 0$ because $\mathbb{Q}/\mathbb{Z}$ inj.

$\langle a \rangle \subset A$

$\exists \neq 0 \to \mathbb{Q}/\mathbb{Z}$

[If a not torsion: $a \mapsto$ anything $\neq 0$

If $na = 0$ smallest n $\quad a \mapsto \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$]



Prop: If an additive functor $R: \mathcal{B} \to \mathcal{A}$ is right adjoint to an exact functor and I inj in $\mathcal{B}$, then $R(I)$ injective in $\mathcal{A}$.

Dually: if $L: \mathcal{A} \to \mathcal{B}$ left-adj to an exact functor, sends proj to proj.

Pf: $\mathrm{Hom}_{\mathcal{A}}(-, R(I)) = \mathrm{Hom}_{\mathcal{B}}(L(-), I)$

composition of two exact functors: $L(-)$ and $\mathrm{Hom}_{\mathcal{B}}(-, I)$ so exact. □

Cor: If I injective ab. group then $\mathrm{Hom}_{Ab}(R, I)$ is an injective right R-module.

Pf: $\mathrm{Hom}_{Ab}(R, -) : Ab \to \text{mod-}R$

right adjoint to $\text{mod-}R \to Ab$, $M \mapsto M$ exact. □

M R-module $I(M) = \prod_{\mathrm{Hom}_R(M, \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})$

$e_M : M \to I(M)$ injective (∏ inj)

So: **Claim:** R-mod has enough injectives.

Chain complexes Let $\mathcal{A}$ be an abelian category

A chain complex is a family of objects $(C_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ with morphisms $d_n : C_n \to C_{n-1}$ s.t. $d^2 = 0$, that is $d_{n-1} \circ d_n = 0$.

$d = (d_n)_{n\in\mathbb{Z}}$ "differential"

$B_n = \mathrm{Im}(d_{n+1}) \hookrightarrow C_n$ n-boundaries.

$Z_n = \mathrm{Ker}(d_n)$ n-cycles $\quad B_n \hookrightarrow Z_n \hookrightarrow C_n$

n-Homology $H_n(C_\cdot) = \mathrm{Coker}(B_n \hookrightarrow Z_n)$ $(= Z_n/B_n)$

$\mathrm{Ch}(\mathcal{A})$: category, objects: chain complexes.

morphisms: $u : C_\cdot \to D_\cdot$

$(=$"chain map"$)$ $\quad \{u_n : C_n \to D_n\}$

s.t. $C_n \xrightarrow{d} C_{n-1}$, $C_n \xrightarrow{u_n} D_n$, $C_{n-1} \xrightarrow{u_{n-1}} D_{n-1}$, $D_n \xrightarrow{d} D_{n-1}$ commutes $\forall n$

Abelian Category

Exo: Show that $u : C_\cdot \to D_\cdot$ induces a map $u_* : H_n(C_\cdot) \to H_n(D_\cdot)$

morphism of chain complexes

s.t. $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ is a functor.

Def: $u: C_\bullet \to D_\bullet$ map of chain complexes is a **quasi-isomorphism** if $H_n(C_\bullet) \xrightarrow[u_n]{\sim} H_n(D_\bullet)$ $\forall n$.

Ex: The following are equivalent:

1. $C_\bullet$ is exact at every C_n
2. $C_\bullet$ is acyclic: $H_n(C_\bullet) = 0$ $\forall n$
3. $0 \to C_\bullet$ is a quasi-isomorphism.

Def: Cochain complex $\{C^n\}_{n \in \mathbb{Z}}$ $d^n: C^n \to C^{n+1}$ $d^2 = 0$

$Z^n = \ker d^n$ n-cocycles $\quad B^n = \operatorname{Im} d^{n-1}$ n-coboundaries $\quad H^n(C) = Z^n / B^n$ n-cohomology

Ex: X top space

R ring $\quad S_k(X) =$ free R-module on the basis given by continuous map $\Delta_k \to X$

Restriction to i-th face $S_k \xrightarrow{\partial_i} S_{k-1}$

$$\begin{array}{ccc} \Delta_k & \to & X \\ \uparrow j_i & \nearrow & \\ \Delta_{k-1} & & \end{array}$$

$d = \sum_i (-1)^i \partial_i$ chain complex of R-modules.

$$H_n(X, R) := H_n(S_\bullet)$$