Part A Graph Theory

Marc Lackenby

Trinity Term 2022

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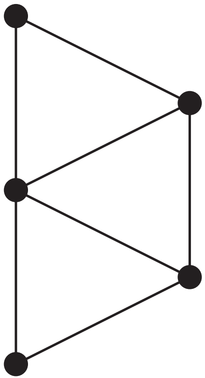

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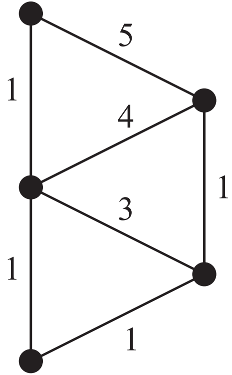

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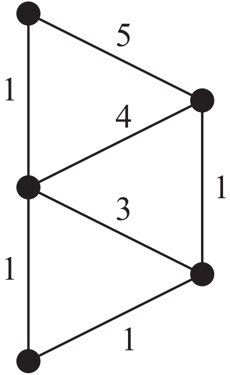

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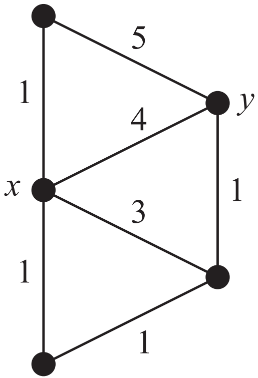

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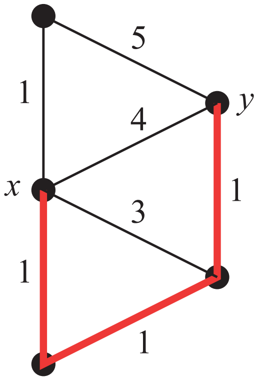

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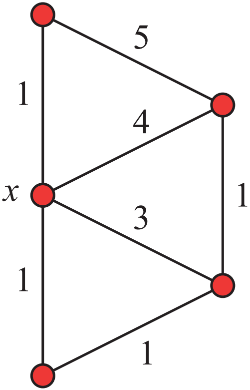

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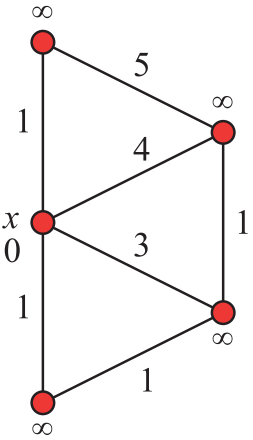

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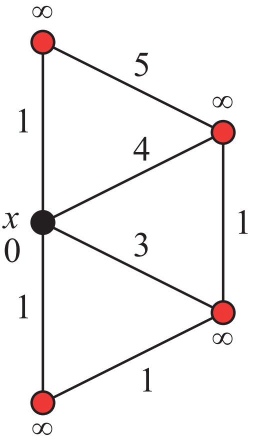

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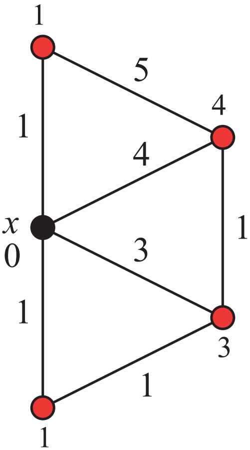

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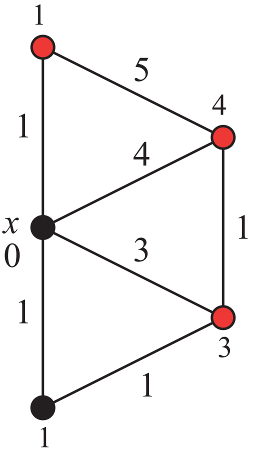

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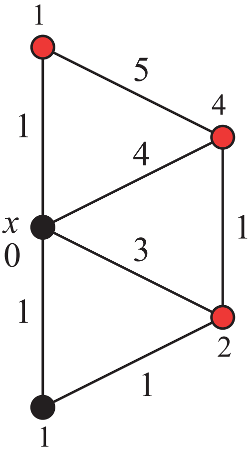

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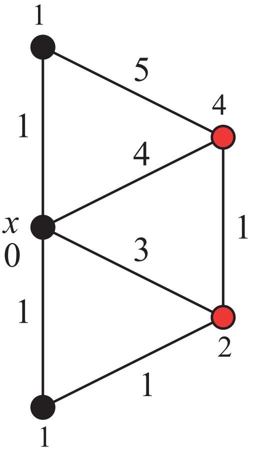

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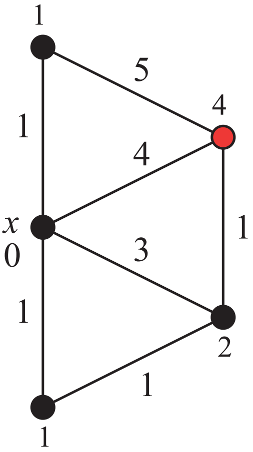

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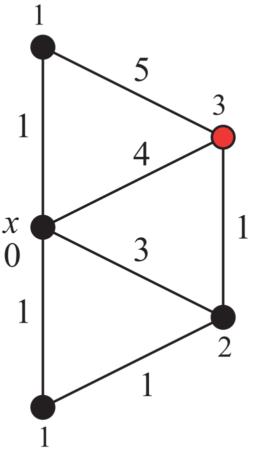

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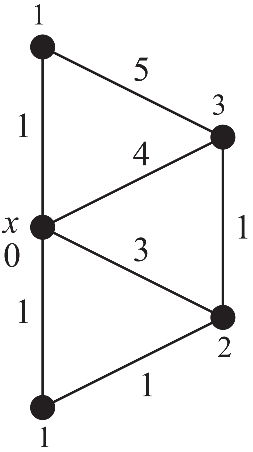

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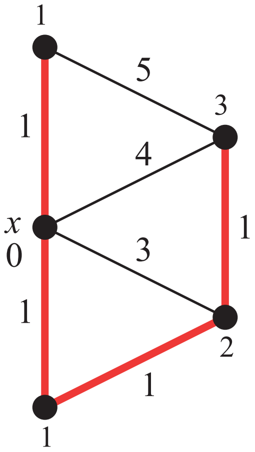

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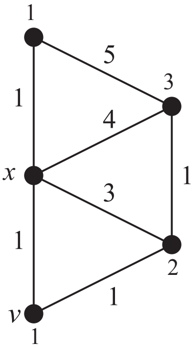

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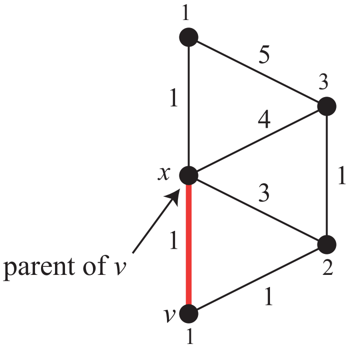

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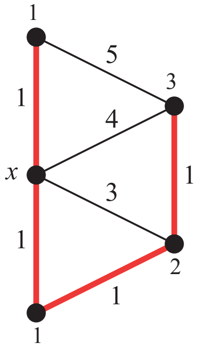

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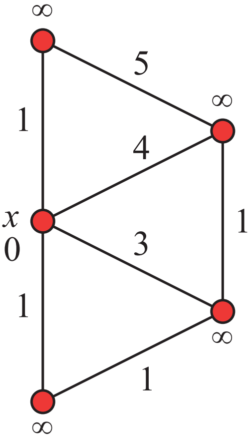

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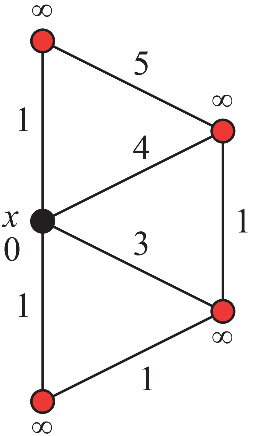

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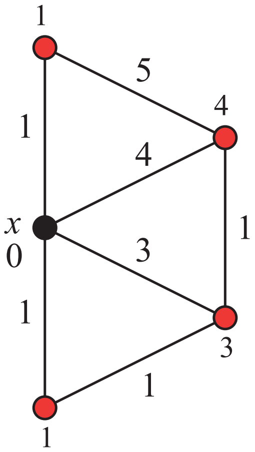

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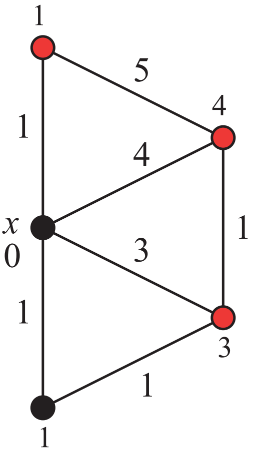

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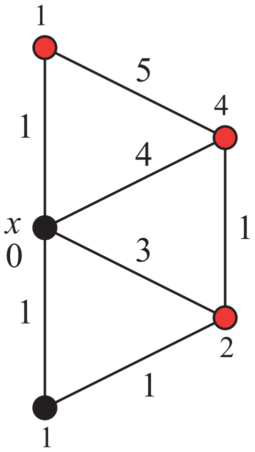

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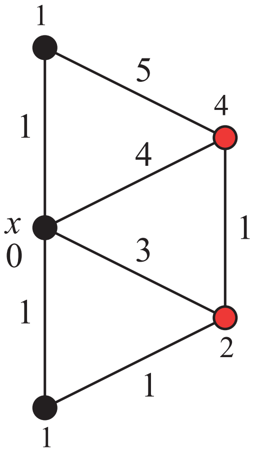

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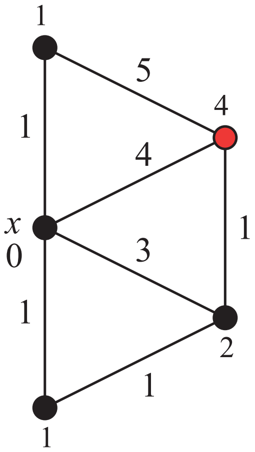

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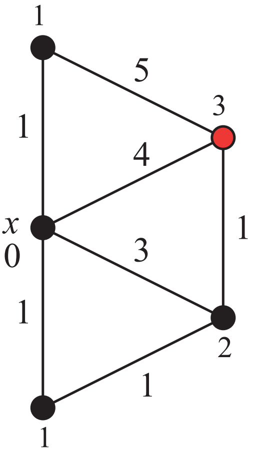

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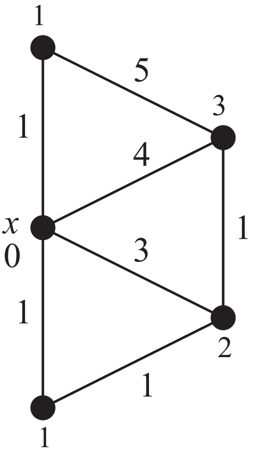

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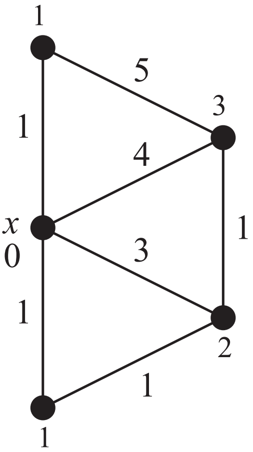

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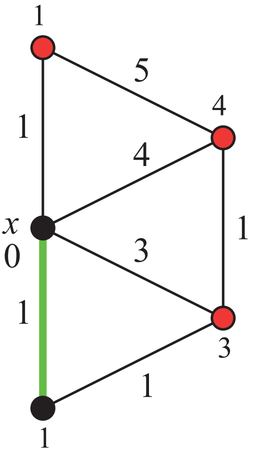

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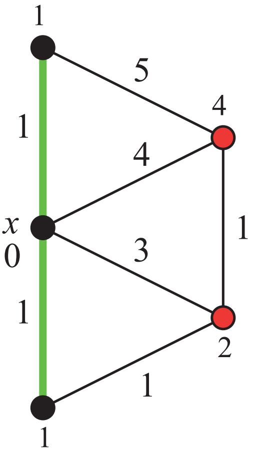

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Lemma 14. T is a tree, and for each $u \in V(G)$ we have $D(u) = \ell(P_u)$ where P_u is the unique xu-path in T.

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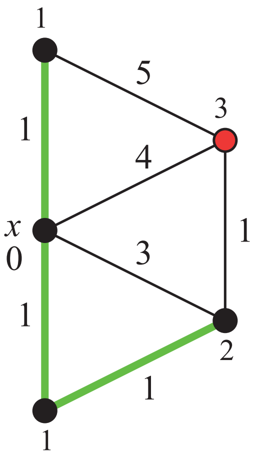

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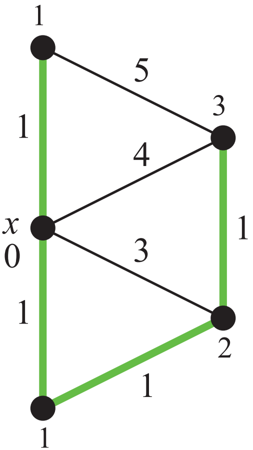

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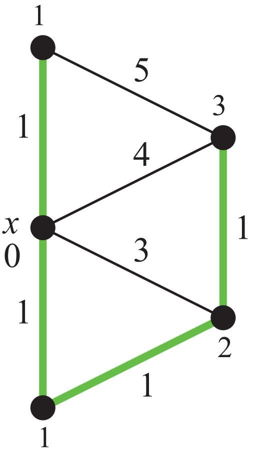

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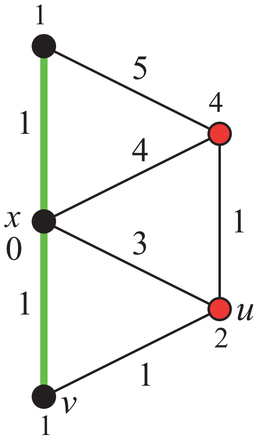

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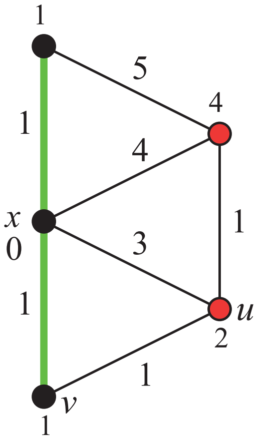

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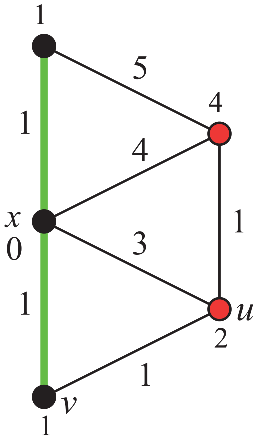

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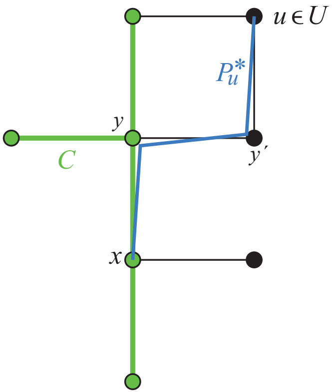

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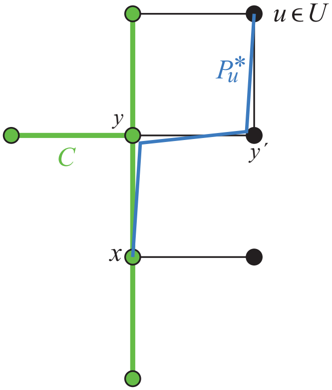

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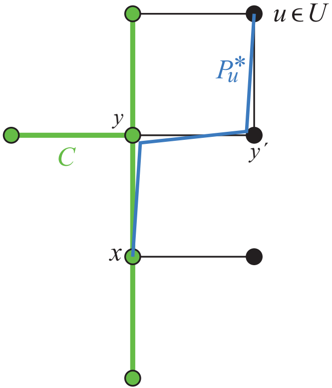

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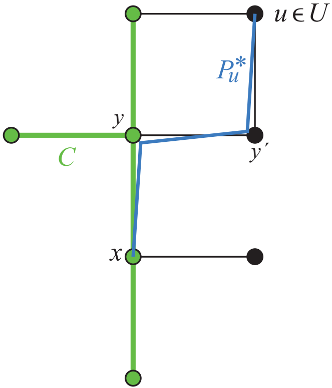

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However, $y' \in U$ with $D(y') < D(u)$ contradicts the choice of u in the algorithm. So $D(u) = D^*(u)$. $\square$

Running time

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A better implementation (which we omit) gives a running time of $O(|E(G)| + |V(G)| \log |V(G)|)$.

Matchings

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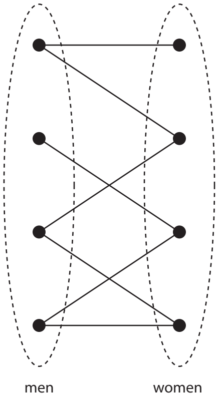

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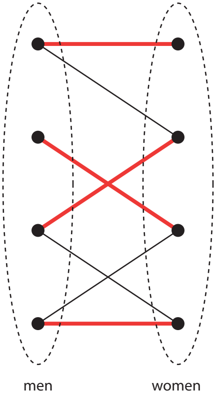

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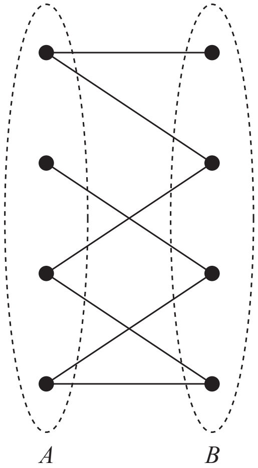
A graph G is *bipartite* if we can partition $V(G)$ into two sets A and B so that every edge of G crosses between A and B.



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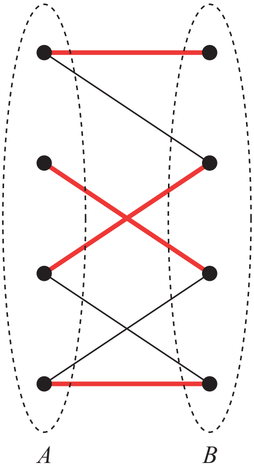

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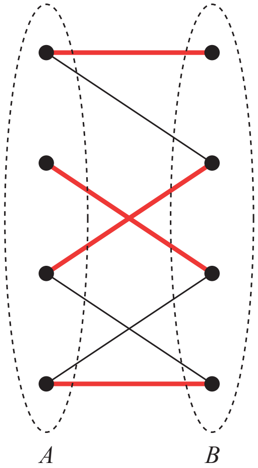

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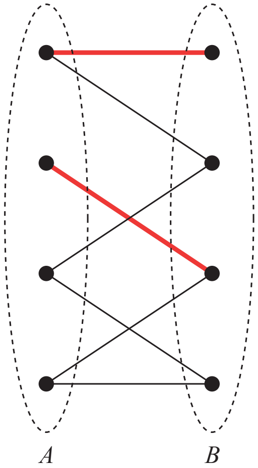

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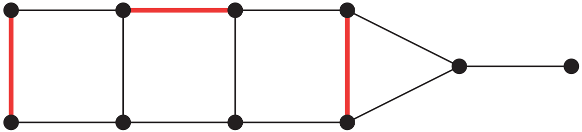

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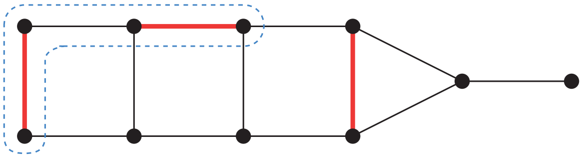

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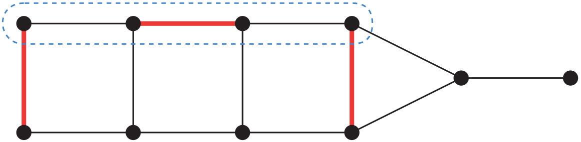

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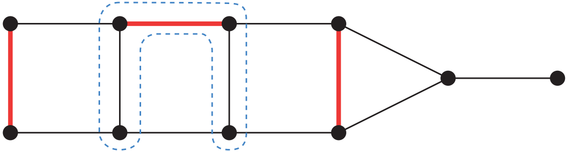

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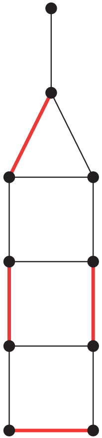

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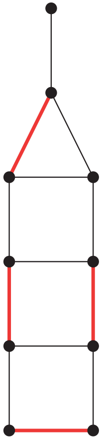

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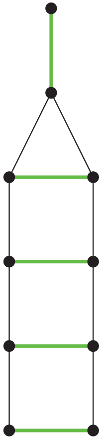

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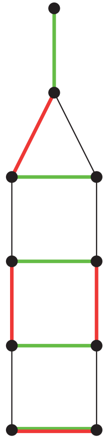

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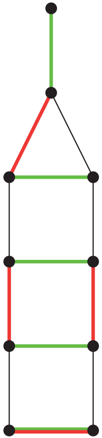

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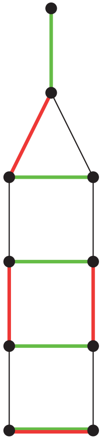
As $|M^*| > |M|$ we can find a path component with more edges of M^* than M: this is an M-augmenting path in G. $\square$



Finding a maximal size matching

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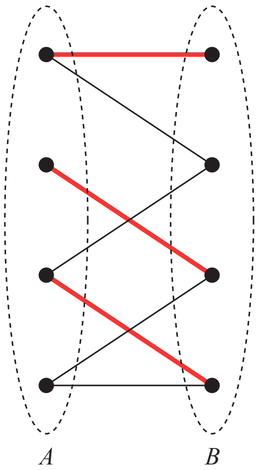
Now suppose that G is bipartite, with parts A and B.



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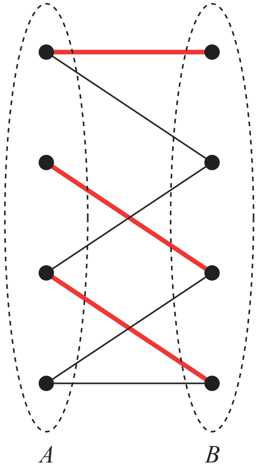

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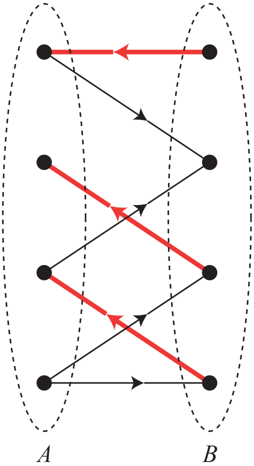
We put directions on $E(G)$, so that all edges in M are one-way from B to A, and all edges not in M are one-way from A to B.



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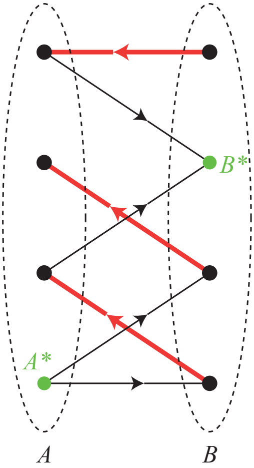

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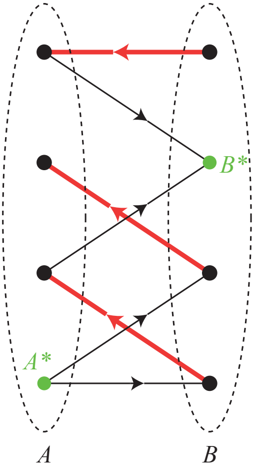
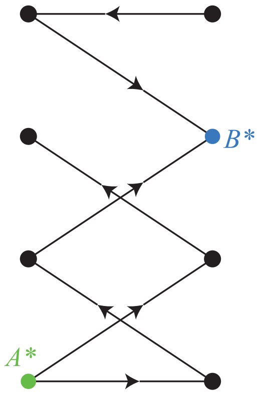
Then an M-augmenting path is equivalent to a directed path from A^* to B^*, i.e. a path that respects directions of edges.



Finding a directed path

Is there a directed path from A^* to B^*?



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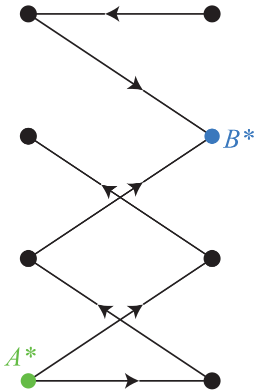

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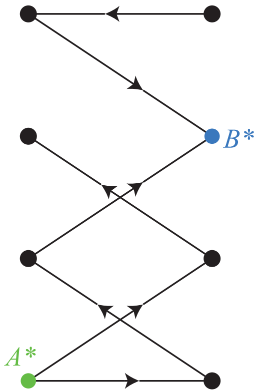

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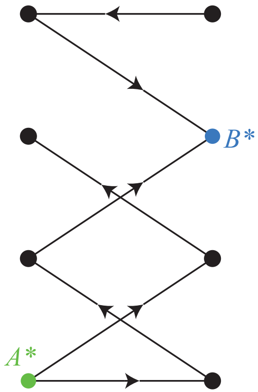

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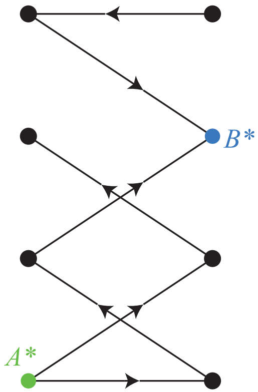

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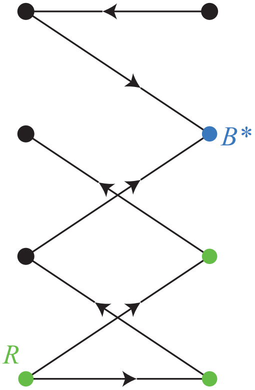

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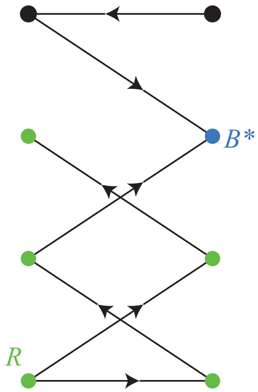

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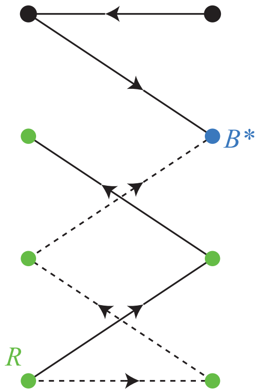

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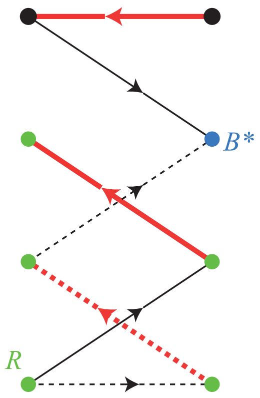

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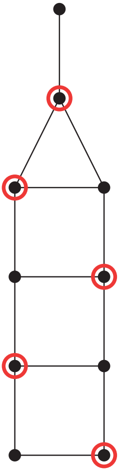

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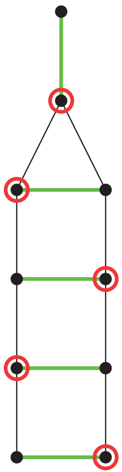

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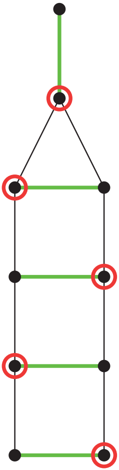
To see this, define an injective map $f : M \to C$, where $f(e)$ is any vertex of $e \cap C$.



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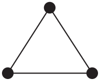

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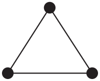

The maximum matching has size 1 but the minimum cover has size 2.

König's Theorem

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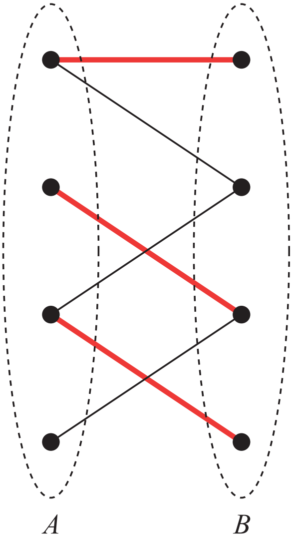
Proof

Let G be a bipartite graph with parts A and B. Let M be a maximum matching in G.



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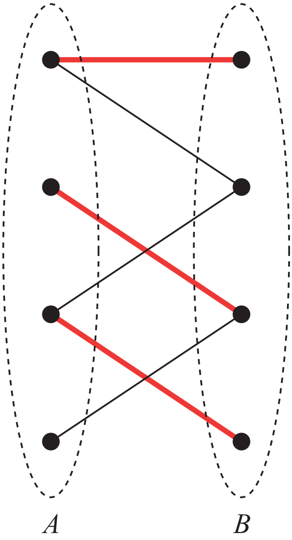

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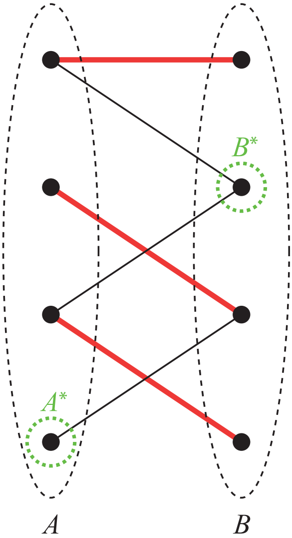

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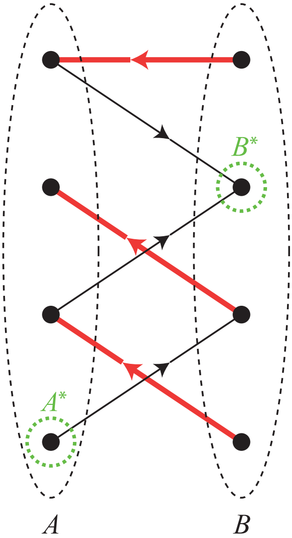

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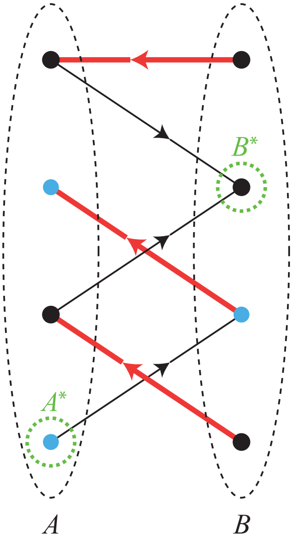

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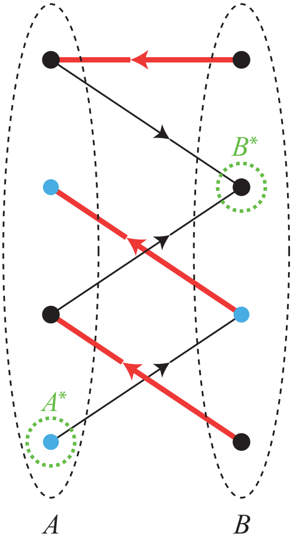

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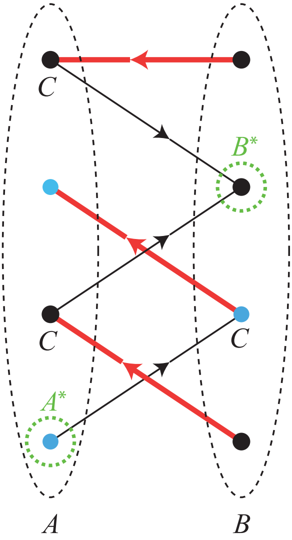

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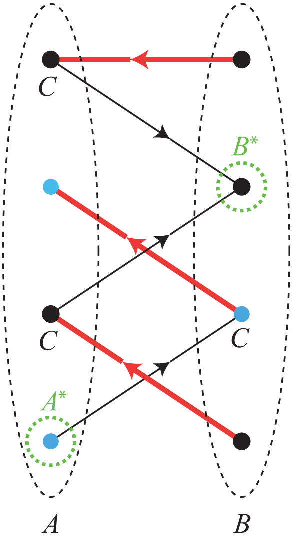

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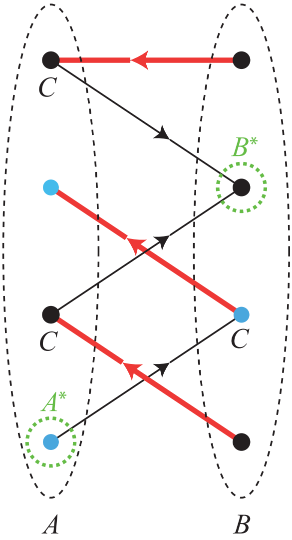

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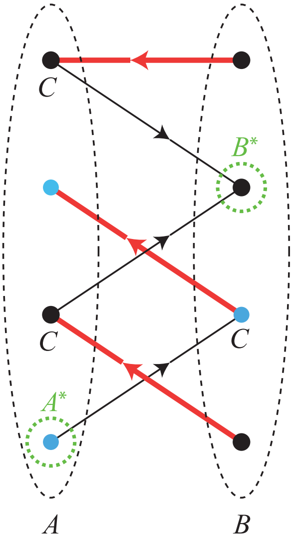

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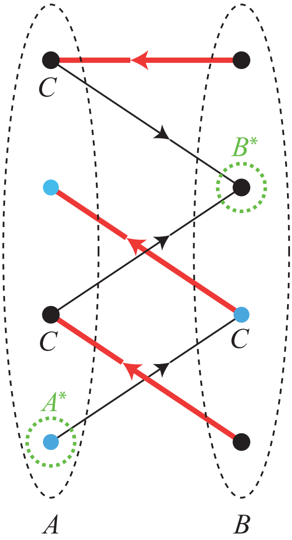

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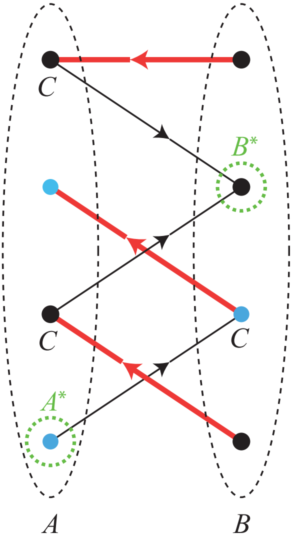

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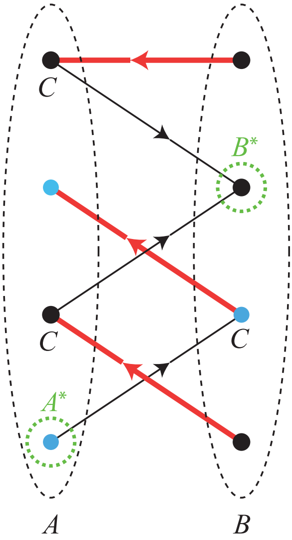

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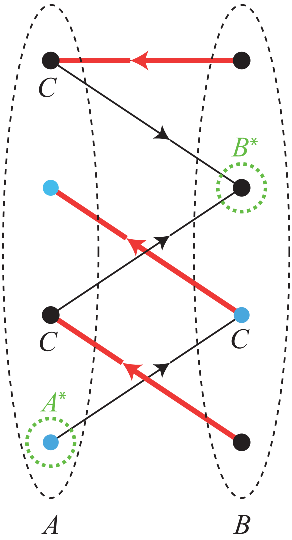

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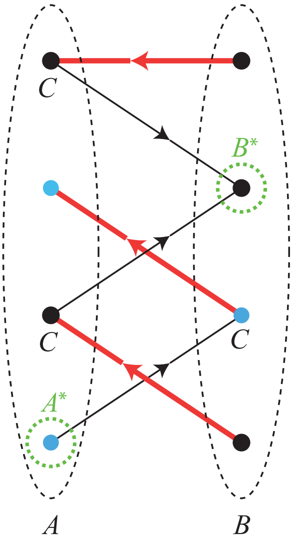

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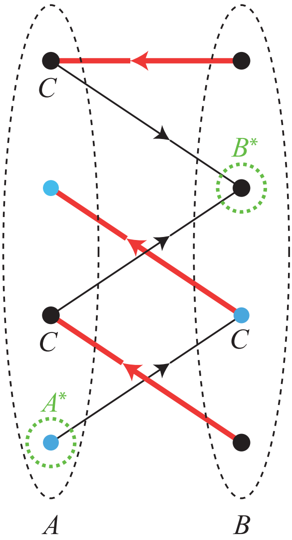

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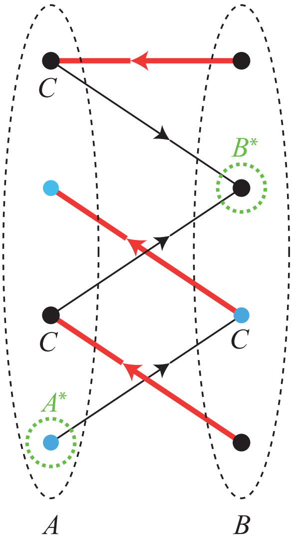

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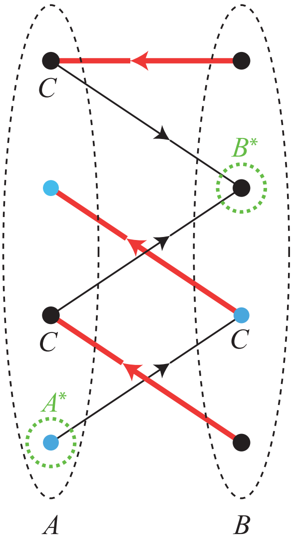

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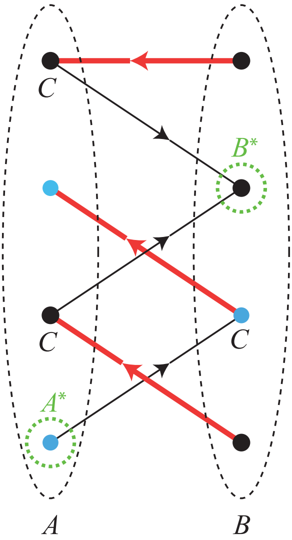

Proof

$C = (A \setminus R) \cup (B \cap R)$.

It remains to show $|C| = |M|$.

It suffices to show that every vertex in C is covered by some edge of M, and that no edge of M covers two vertices of C.

(This will show $|C| \leq |M|$, and we noted previously that $|M| \leq |C|$ is immediate from the definitions.)

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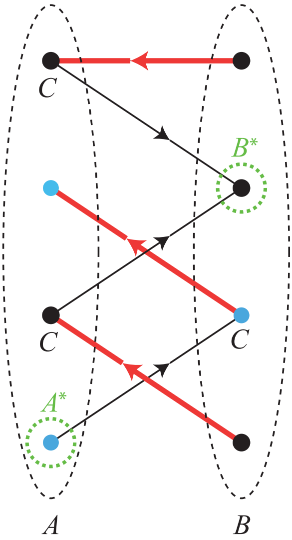
Finally, if $ab \in M$ with $a \in A \setminus R$, $b \in B \cap R$ then we can reach a via b, contradicting $a \notin R$. Thus $|C| = |M|$. □



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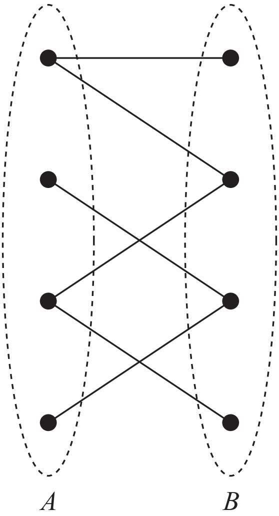

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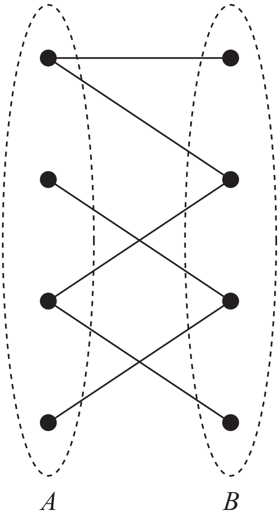

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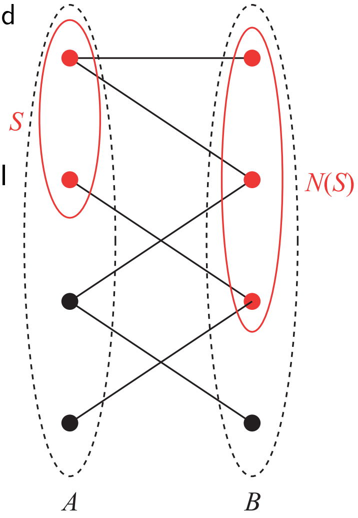

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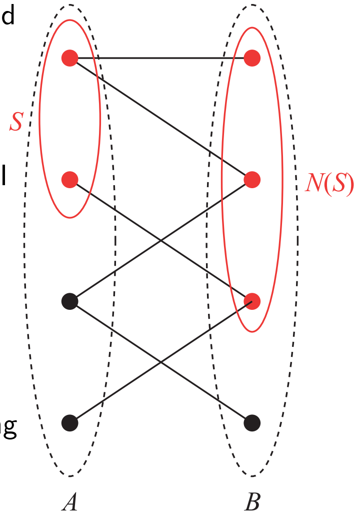

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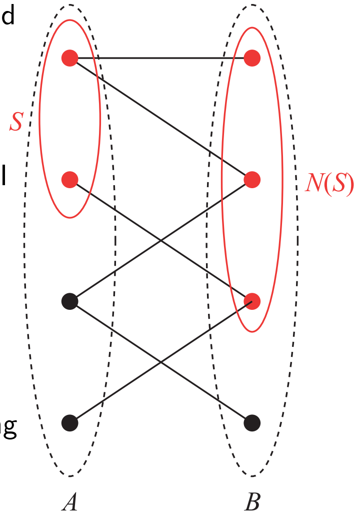
This gives a necessary condition for G to have a matching; it is also sufficient ...



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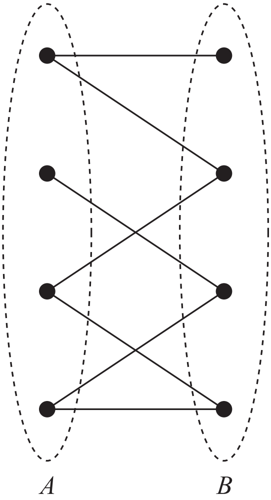

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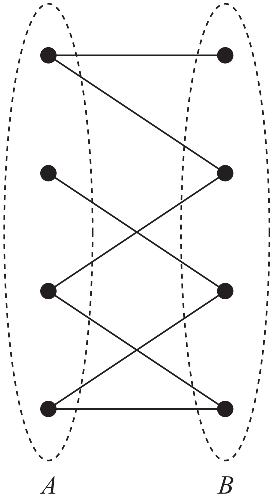

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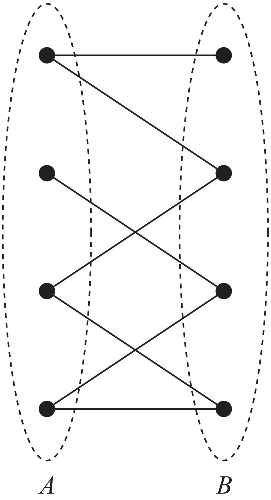

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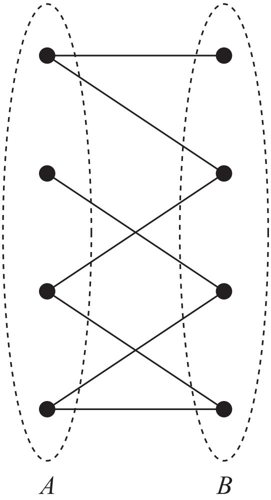

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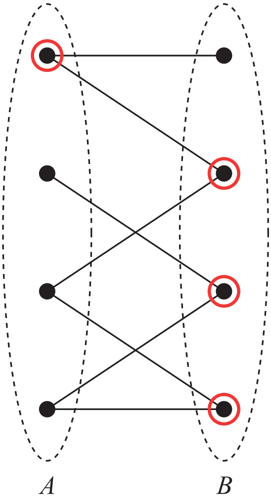

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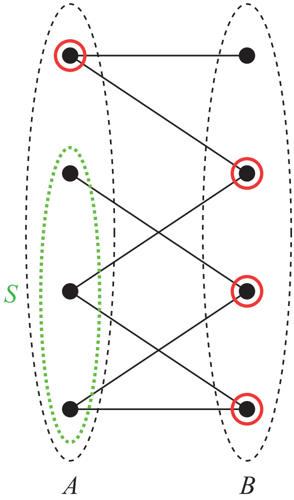

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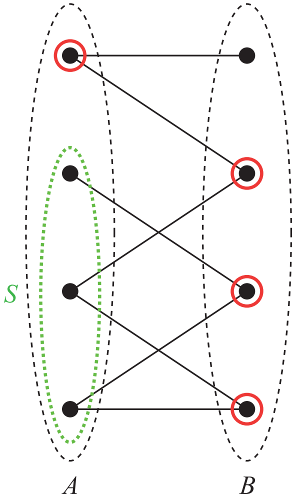

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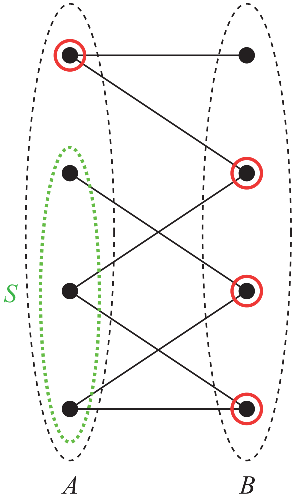
Then $|C| = |A \cap C| + |B \cap C| \geq |A| - |S| + |N(S)| \geq |A|$.



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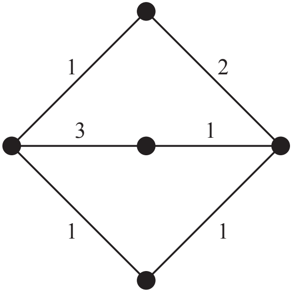
We want to find a shortest postman walk.

Extensions



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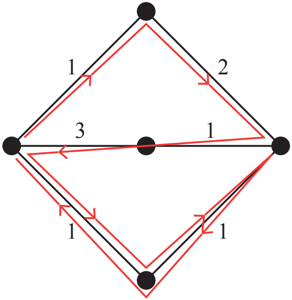

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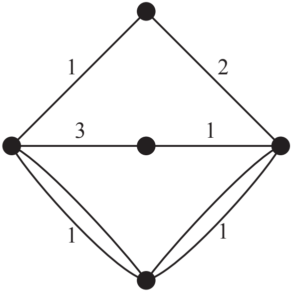

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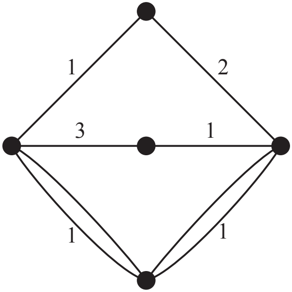
Thus an equivalent reformulation of the Chinese Postman Problem is to find a *minimum weight Eulerian extension* G^* of G, i.e. G^* is obtained from G by copying some edges, so that all degrees in G^* are even, and $c(G^*)$ is as small as possible.



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(An algorithm for this problem was also found by Edmonds, but it is beyond the scope of this course).

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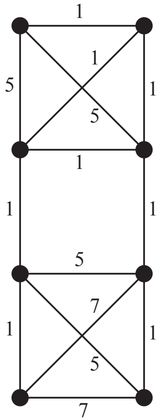

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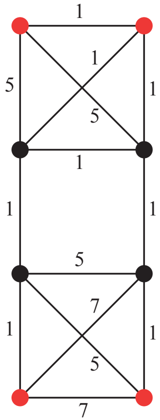

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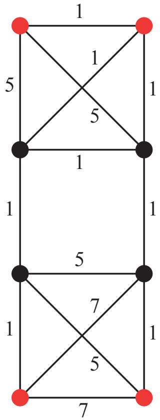

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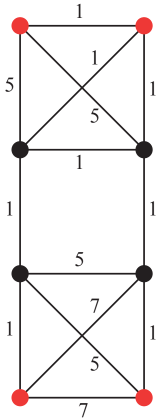

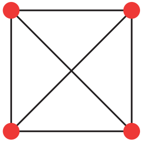

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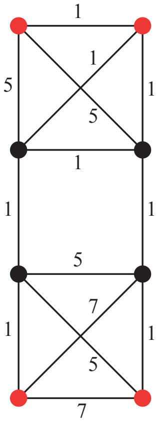

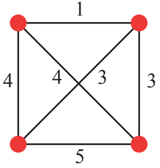

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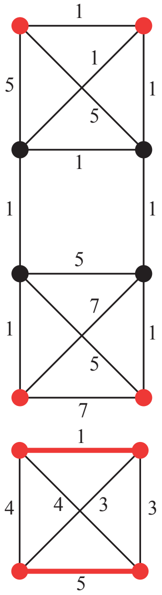

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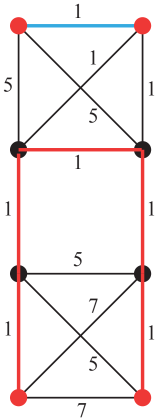

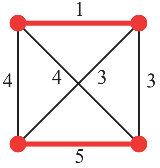

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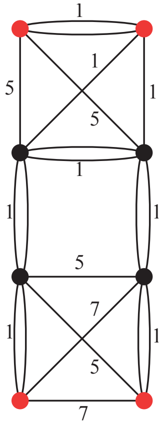

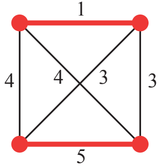

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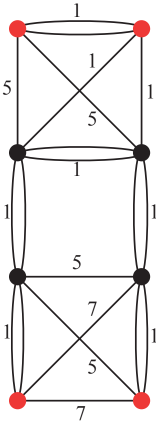

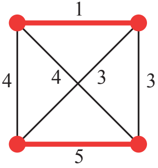

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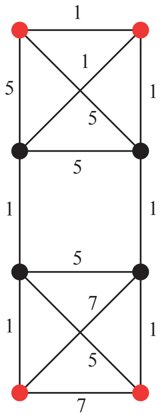

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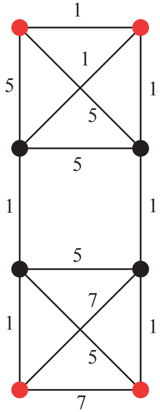

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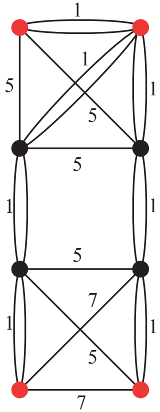

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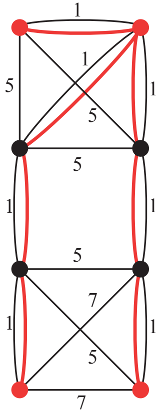

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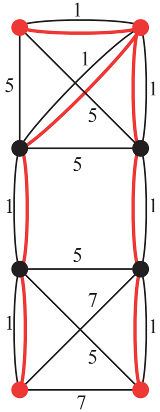

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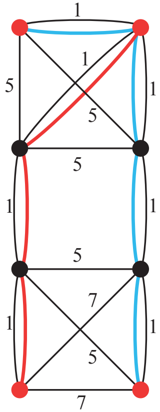

Proof.

Let W^* be a minimum length postman walk. It suffices to show that the algorithm finds a postman walk that is no longer than W^*.

Let G^* be the Eulerian extension of G defined by W^*. Let H be the graph of copied edges: $E(H) = E(G^*) \setminus E(G)$. Note that the set of vertices with odd degree in H is X (i.e. the same set as for G).

We construct a set of paths in H by repeating the following procedure: if the current graph has any vertices of odd degree, apply Lemma 19 to find a path P such that both ends have odd degree, delete the edges of P and repeat.

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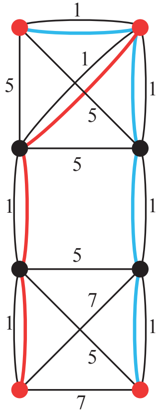

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We construct a set of paths in H by repeating the following procedure: if the current graph has any vertices of odd degree, apply Lemma 19 to find a path P such that both ends have odd degree, delete the edges of P and repeat.
This procedure pairs up the vertices in X so that each pair is connected by a path in H.

Edmonds' algorithm works

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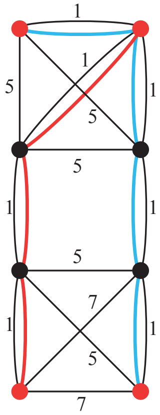
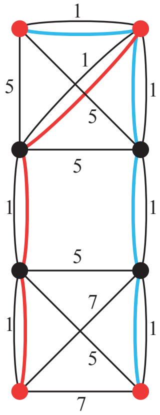


Edmonds' algorithm works

Theorem 20. Edmonds' Algorithm finds a minimum length postman walk.

Let $H' \subseteq H$ be the graph formed by the union of these paths.



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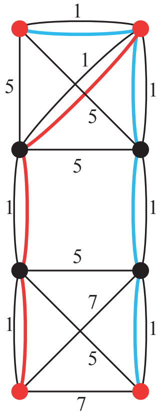
Let $H' \subseteq H$ be the graph formed by the union of these paths.

Let G' be the Eulerian extension of G defined by copying the edges of H'.



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Theorem 20. Edmonds' Algorithm finds a minimum length postman walk.

Let $H' \subseteq H$ be the graph formed by the union of these paths.

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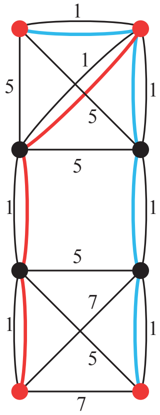
Let W' be an Euler tour in G', interpreted as a postman walk in G. Then $c(W') \leq c(W^*)$.



Edmonds' algorithm works

Theorem 20. Edmonds' Algorithm finds a minimum length postman walk.

Let $H' \subseteq H$ be the graph formed by the union of these paths.

Let G' be the Eulerian extension of G defined by copying the edges of H'.

Let W' be an Euler tour in G', interpreted as a postman walk in G. Then $c(W') \leq c(W^*)$. By definition of the algorithm it finds a postman walk that is no longer than W'. $\square$