4 Vortex motion

4.1 Helmholtz' Principle

So far in this course, we have seen how to calculate the velocity potential caused by placing a vortex at a given location $z = c$ in a flow, by using the method of images or conformal mapping, for example. In practice, however, vortices in a fluid do not stay still: they are convected by the flow, so that c will actually be a function of t.

We recall that the vorticity $\boldsymbol{\omega}$ satisfies the equation

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}. \tag{4.1}$$

In two-dimensional flow, the right-hand side of (4.1) is identically zero, so the vorticity is conserved following the flow. This suggests that a vortex, which we can think of as a line source of vorticity, should convect with the prevailing velocity of the flow in which it sits.

Example 4.1 ***A vortex in a uniform flow***
Suppose a vortex of strength Γ sits at the point $z = c$ in an infinite expanse of fluid flowing uniformly in the x-direction with speed U. We superimpose the complex potentials due to the uniform flow and the vortex to obtain

$$w(z) = Uz - \frac{\mathrm{i}\Gamma}{2\pi} \log(z - c), \tag{4.2}$$

and the resulting velocity components are given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = U - \frac{\mathrm{i}\Gamma}{2\pi(z - c)}. \tag{4.3}$$

The first term on the right-hand side of equation (4.3) *gives the velocity due to the uniform flow; the second term is the flow due to the vortex.*

When we now try to calculate the background flow experienced by the vortex, we encounter a problem, since (4.3) *implies that the velocity is unbounded as $z \to c$. The resolution of this difficulty is to ignore the velocity due to the vortex itself, i.e. the final term in* (4.3)*. If we do so, we conclude that the vortex moves with velocity components given by*

$$u - \mathrm{i}v\big|_{\text{vortex}} = U. \tag{4.4}$$

Hence the vortex propagates with the uniform flow, in agreement with our physical intuition.

Example 4.1 is a very simple illustration of **Helmholtz' Principle**:

$$\text{A vortex moves with the velocity field due to } \textit{everything except itself}. \tag{4.5}$$

To make this principle more explicit, suppose we know the complex potential $w(z)$ for a flow containing a vortex at the point $z = c$. Helmholtz' Principle implies that the vortex should move with velocity components $\mathrm{d}c/\mathrm{d}t = u + \mathrm{i}v|_{\text{vortex}}$ given by the *regular part* of the fluid velocity as $z \to c$. We write this as

$$u - \mathrm{i}v\big|_{\text{vortex}} = \mathrm{reg}_{z\to c}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right). \tag{4.6}$$

For the case of a vortex, we expect the singularity to arise from the vortex itself and so to find the regular part of the velocity, we first subtract this velocity off. Then evaluating the remaining flow (which should now be bounded) at the vortex location $z = c$ gives the vortex's velocity, i.e.

$$u - \mathrm{i}v\big|_{\text{vortex}} = \mathrm{reg}_{z\to c}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) = \lim_{z\to c}\left(\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\mathrm{i}\Gamma}{2\pi(z-c)}\right). \tag{4.7}$$

As a result, $c(t)$ satisfies the differential equation

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}t} = \lim_{z\to c}\left(\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\mathrm{i}\Gamma}{2\pi(z-c)}\right). \tag{4.8}$$

4.2 Examples

Example 4.2 ***A vortex next to a wall***
Suppose fluid occupies the half-plane $\operatorname{Im} z > 0$ *bounded by a rigid impermeable wall along the real-z-axis, and a vortex of strength* Γ *is at the point* $z = c$.

We find the velocity potential by using the method of images, placing a vortex of equal and opposite strength at the image point $z = \bar{c}$*:*

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z-c) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z-\bar{c}). \tag{4.9}$$

The velocity field is given by

$$u - \mathrm{i}v = -\frac{\mathrm{i}\Gamma}{2\pi(z-c)} + \frac{\mathrm{i}\Gamma}{2\pi(z-\bar{c})}, \tag{4.10}$$

and substitution into (4.8) *tells us that the vortex location* $c(t)$ *satisfies*

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}t} = \frac{\mathrm{i}\Gamma}{2\pi(c-\bar{c})}. \tag{4.11}$$

We can easily see that the right-hand side of (4.11) *is purely real, implying that the vortex will move in the real-z-direction, parallel to the wall.*

To flesh this out further, let us write the components of the vortex location as $c = x + \mathrm{i}y$*, so that* (4.11) *becomes*

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\Gamma}{4\pi y}. \tag{4.12}$$

Hence we see that the distance y *of the vortex from the wall remains constant, while it propagates in the x-direction with speed* $\Gamma/4\pi y$*, as shown schematically in* Figure 4.1(i).

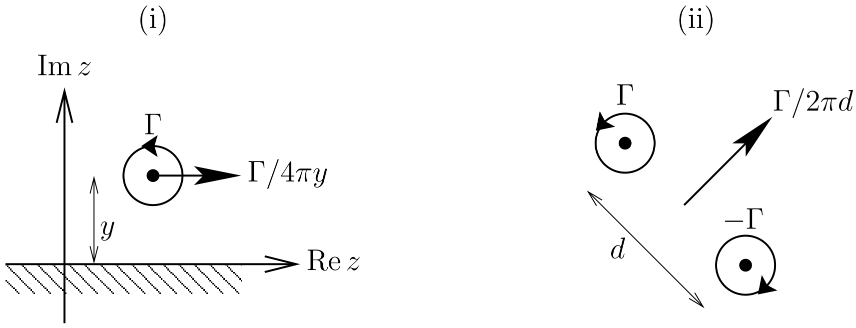

Figure 4.1: (i) Schematic of a vortex propagating parallel to a wall along the real-z-axis. (ii) Schematic of a pair of equal and opposite vortices a distance d apart.

Example 4.2 is a simple model to explain how a *starting vortex* propagates away from an accelerating aerofoil. We note that the complex potential (4.9) also describes the flow due to two equal and opposite vortices a distance $2y$ apart. The result (4.12) implies that a system of equal and opposite vortices a distance d apart will travel at speed $\Gamma/2\pi d$ in a direction perpendicular to the line joining them, as shown schematically in Figure 4.1(ii). Such a system is the two-dimensional equivalent of a *vortex ring*, an example of which is a smoke ring.

Example 4.3 ***Vortex in a quadrant***
Suppose a vortex of strength Γ *is at the point* c *in the quadrant* $0 < \arg z < \pi/2$ *bounded by impermeable walls at* $\arg z = 0, \pi/2$. *This problem was solved previously in* Section 2 *by using the method of images, and the complex potential was found to be*

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z-c) - \frac{\mathrm{i}\Gamma}{2\pi}\log(z+c) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z-\overline{c}) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z+\overline{c}). \tag{4.13}$$

Hence the velocity components are given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{i}\Gamma}{2\pi}\left\{-\frac{1}{z-c} - \frac{1}{z+c} + \frac{1}{z-\overline{c}} + \frac{1}{z+\overline{c}}\right\}. \tag{4.14}$$

The first term in braces corresponds to the vortex itself. When we neglect this term and evaluate the rest at the vortex location $z = c$, *we find that the vortex velocity satisfies*

$$\begin{aligned} \frac{\mathrm{d}\overline{c}}{\mathrm{d}t} &= \frac{\mathrm{i}\Gamma}{2\pi}\left\{-\frac{1}{2c} + \frac{1}{c-\overline{c}} + \frac{1}{c+\overline{c}}\right\} \\ &= \frac{\mathrm{i}\Gamma}{4\pi}\left\{-\frac{x-\mathrm{i}y}{x^2+y^2} - \frac{\mathrm{i}}{y} + \frac{1}{x}\right\}, \end{aligned} \tag{4.15}$$

when we write the vortex position as $c = x + \mathrm{i}y$. *Hence the components* $x(t)$ *and* $y(t)$ *satisfy the coupled differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\Gamma x^2}{4\pi y\,(x^2+y^2)}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\Gamma y^2}{4\pi x\,(x^2+y^2)}. \tag{4.16}$$

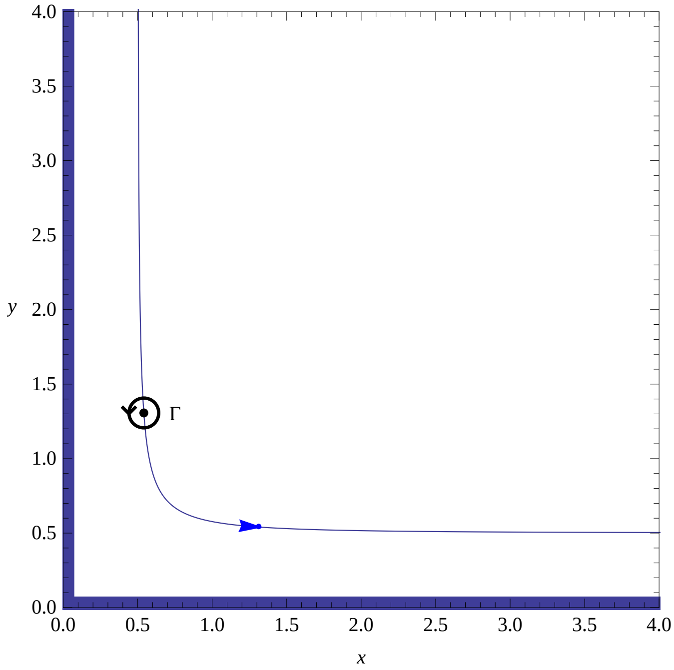

Figure 4.2: The path of a vortex confined to the quadrant $x > 0$, $y > 0$.

By taking the ratio of these equations, we find that the vortex follows a path in the (x, y)*-plane on which*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^3}{x^3}. \tag{4.17}$$

By integrating this equation, we find that

$$\frac{1}{x^2} + \frac{1}{y^2} = \textit{constant}. \tag{4.18}$$

Such a path is plotted in Figure 4.2.

Example 4.4 ***Vortex in a channel***
Suppose fluid fills the channel $-a < \operatorname{Im} z < a$*, in which a vortex of strength* $\Gamma > 0$ *sits at the point* $z = c$*. The channel is mapped to the half-space* $\operatorname{Re}\zeta > 0$ *by the conformal mapping* $\zeta = \mathrm{e}^{\pi z/2a}$ *and the complex potential*

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\mathrm{e}^{\pi z/2a} - \mathrm{e}^{\pi c/2a}\right) + \frac{\mathrm{i}\Gamma}{2\pi}\log\left(\mathrm{e}^{\pi z/2a} + \mathrm{e}^{\pi \bar{c}/2a}\right) \tag{4.19}$$

is then easily found by using the method of images. The velocity components are thus given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{i}\Gamma}{4a}\left\{\frac{1}{\mathrm{e}^{\pi(c-z)/2a} - 1} + \frac{1}{\mathrm{e}^{\pi(\bar{c}-z)/2a} + 1}\right\}. \tag{4.20}$$

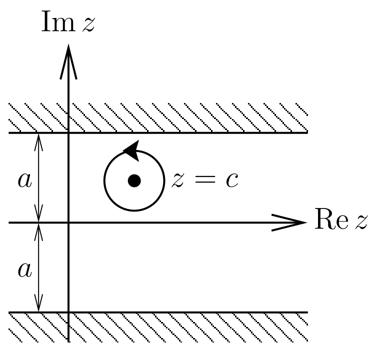

Figure 4.3: Schematic of a vortex in the channel $-a < \operatorname{Im} z < a$.

Now we wish to find the behaviour of the right-hand side of (4.20) *as* $z \to c$. *To this end, we expand the first term in braces to obtain*

$$\begin{aligned} \frac{1}{\mathrm{e}^{\pi(c-z)/2a} - 1} &\sim \left(\frac{\pi(c-z)}{2a} + \frac{\pi^2(c-z)^2}{8a^2} + \mathrm{O}\left((z-c)^3\right) \right)^{-1} \\ &\sim \frac{2a}{\pi(c-z)} \left(1 + \frac{\pi(c-z)}{4a} + \mathrm{O}\left((z-c)^2\right) \right)^{-1} \\ &\sim \frac{2a}{\pi(c-z)} - \frac{1}{2} + \mathrm{O}\left(z-c\right) \quad \text{as } z \to c. \end{aligned} \tag{4.21}$$

Substituting (4.21) *into* (4.20), *we find that*

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}t} = \lim_{z\to c} \left(\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\mathrm{i}\Gamma}{2\pi(z-c)} \right) = \frac{\mathrm{i}\Gamma}{4a} \left(-\frac{1}{2} + \frac{1}{\mathrm{e}^{\pi(\bar{c}-c)/2a} + 1} \right), \tag{4.22}$$

and simplification leads to

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}t} = -\frac{\Gamma}{8a} \tan\left(\frac{\pi\left(c - \bar{c}\right)}{4\mathrm{i}a} \right). \tag{4.23}$$

If we write the components of c *as* $c = x + \mathrm{i}y$, *then we find that* x *and* y *satisfy the differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\Gamma}{8a} \tan\left(\frac{\pi y}{2a} \right), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 0. \tag{4.24}$$

Hence the vortex moves horizontally along the channel, in the negative *x-direction if* $y > 0$ *or the* positive *x-direction if* $y < 0$.

Example 4.4 illustrates how in general it may be necessary to expand $\mathrm{d}w/\mathrm{d}z$ in a Laurent expansion about $z = c$ to evaluate the vortex velocity. A physical interpretation of the result (4.24) is that the vortex moves under the influence of whichever wall is closer. If $y < 0$ (for example), the bottom wall dominates and causes the vortex to propagate to the right, as in Example 4.2. As the vortex approaches the lower wall $y \searrow -a$ and the velocity takes the form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\Gamma}{8a}\cot\left(\frac{\pi}{2a}(y+a)\right) \\ &\sim \frac{\Gamma}{4\pi(y+a)} \quad \text{as } y \searrow -a,\end{aligned} \tag{4.25}$$

which agrees with the result (4.12), $y + a$ now being the distance from the wall.