

A9: Statistics
Sheet 4 — HT 2023
(Lectures 12–16, Notes section 4)

1. Suppose X_1, \dots, X_n are independent, each having a geometric distribution with probability mass function $f(x | \theta) = (1 - \theta)^x \theta$ for $x = 0, 1, \dots$. Suppose that the prior for θ is a $\text{Beta}(a, b)$ density. Find the posterior distribution of θ .
2. Let $\theta > 0$ be an unknown parameter and let $c > 0$ be a known constant. Conditional on θ , suppose X_1, \dots, X_n are independent each with probability density function

$$f(x | \theta) = \theta c^\theta x^{-(\theta+1)}, \quad x \geq c$$

and suppose the prior for θ is a $\text{Gamma}(\alpha, \beta)$ density. Find the posterior distribution of θ .

3. Let $r \geq 1$ be a known integer and let $\theta \in [0, 1]$ be an unknown parameter. The negative binomial distribution with index r and parameter θ has probability mass function

$$f(x | \theta) = \binom{x+r-1}{x} (1-\theta)^x \theta^r \quad \text{for } x = 0, 1, \dots$$

Let θ have a $\text{Beta}(a, b)$ prior density and suppose, given θ , that X_1, \dots, X_n are independent each with the above negative binomial distribution.

- (a) Show that the posterior density is also a Beta density.
- (b) Explain how to construct a $100(1 - \alpha)\%$ equal-tailed credible interval for θ . Will this interval be a highest posterior density interval?

4. Suppose that X has a $N(\theta, \phi)$ distribution, where ϕ is known, Suppose also that the prior distribution for θ is $N(\theta_0, \phi_0)$, where θ_0 and ϕ_0 are known.
- Find the posterior distribution of θ given $X = x$.
 - Show that the posterior mean of θ always lies between the prior mean and the observed value x .
 - Construct a $100(1 - \alpha)\%$ highest posterior density interval for θ .
 - Let $\phi = 2$, $\theta_0 = 0$ and $\phi_0 = 2$.
 - Suppose the observed value is $x = 4$. What are the mean and variance of the resulting posterior distribution? Sketch the prior, likelihood, and posterior on a single set of coordinate axes.
 - Repeat (i) assuming $\phi_0 = 18$. Explain any resulting differences. Which of these two priors would likely have more appeal for a frequentist statistician?
5. Let X be the number of heads when a coin with probability θ of heads is flipped n times.
- When the prior is $\pi(\theta)$, the prior predictive distribution for X (the predictive distribution before observing any data) is given by

$$P(X = k) = \int_0^1 P(X = k|\theta)\pi(\theta) d\theta, \quad k = 0, 1, \dots, n.$$

Find the prior predictive distribution when $\pi(\theta)$ is uniform on $(0, 1)$.
 - Suppose you assign a $\text{Beta}(a, b)$ prior for θ , and then you observe x heads out of n flips. Show that the posterior mean of θ is always lies between your prior mean, $a/(a + b)$, and the observed relative frequency of heads, x/n .
 - Show that, if the prior distribution on θ is uniform, then the posterior variance is always less than the prior variance.
 - Give an example of a $\text{Beta}(a, b)$ prior distribution and values of x, n for which the posterior variance is larger than the prior variance. (Try $x = n = 1$.)
6. A coin, with probability θ of heads, is flipped n times and r heads are observed.
- If the prior for θ is a uniform distribution on $(0, 1)$, what is the probability that the next flip is a head?
 - Can you generalise to the case where θ has a $\text{Beta}(a, b)$ prior and where we wish to find the probability of getting k heads from m further flips?

7. (a) Let $X \sim N(\theta, \sigma_0^2)$, where σ_0^2 is known. Find the Jeffreys' prior for θ .
- (b) Let $X \sim N(\mu_0, \sigma^2)$, where μ_0 is known. Find the Jeffreys' prior for σ .
- (c) Let X be Poisson with parameter λ . Find the Jeffreys' prior for λ . Check that the posterior distribution of λ given $X = x$ is proper, but that the Jeffreys' prior is not.
8. Suppose X is the number of successes in a binomial experiment with n trials and probability of success θ . Either $H_0 : \theta = \frac{1}{2}$ or $H_1 : \theta = \frac{3}{4}$ is true. Show that the posterior probability that H_0 is true is greater than the prior probability for H_0 if and only if

$$x \log 3 < n \log 2.$$

9. Let $X \sim \text{Binomial}(n, \theta)$, where the prior for θ is uniform on $(0, 1)$. Suppose that we wish to compare the hypotheses $H_0 : \theta \leq \frac{1}{2}$ and $H_1 : \theta > \frac{1}{2}$.

What are the prior odds of H_0 relative to H_1 ?

Find an expression for the posterior odds of H_0 relative to H_1 .

If we observe $X = n$, find the Bayes factor B of H_0 relative to H_1 .

Check that $B \rightarrow 0$ as $n \rightarrow \infty$. Why is this expected?

10. Suppose we have a random sample X_1, \dots, X_n from a Poisson distribution with mean θ . Suppose we wish to test the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ and that, under H_1 , the prior distribution $\pi(\theta|H_1)$ for θ is given by

$$\pi(\theta|H_1) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0.$$

Calculate the Bayes factor of H_0 relative to H_1 .

When $n = 6$, $\sum x_i = 19$, $\theta_0 = 2$, find the numerical value of the Bayes factor (i) when $\alpha = 4$ and $\beta = \frac{2}{3}$, and (ii) when $\alpha = 36$ and $\beta = 6$. Compare and interpret the values of the Bayes factor in cases (i) and (ii).