A8: Probability

Sheet 2 — MT22

Chapters 3 and 4

1. (a) Let X and Y be independent standard normal random variables. Define R and Θ by $X = R\cos\Theta$, $Y = R\sin\Theta$. Find the joint distribution of R and Θ.

   (b) Let $\mathbf{Z}$ be a vector of independent standard normal random variables $Z_1, Z_2, \ldots, Z_n$. Let A be an orthogonal $n \times n$ matrix. Find the joint distribution of the vector $\mathbf{W}$ with entries $W_1, W_2, \ldots, W_n$ where $\mathbf{W} = A\mathbf{Z}$. Explain the link to part (a).

2. The Gamma(r, λ) distribution has density $f_{r,\lambda}(x) = \dfrac{1}{\Gamma(r)}\lambda^r x^{r-1} e^{-\lambda x}$ on $\mathbb{R}_+$. Here r is called the *shape* parameter and λ is called the *rate* parameter.

   Use moment generating functions to show that the sum of two independent Gamma-distributed random variables with the same rate parameter is also Gamma-distributed. What can you deduce about sums of exponential random variables?

3. A student makes repeated attempts to solve a problem. Suppose the ith attempt takes time X_i, where X_i are i.i.d. exponential random variables with parameter λ. Each attempt is successful with probability p (independently for each attempt, and independently of the durations). Use moment generating functions to show that the distribution of the total time before the problem is solved has an exponential distribution, and find its parameter.

4. (a) Let X, Y and U be independent random variables, where X and Y have moment generating functions $M_X(t)$ and $M_Y(t)$, and where U has the uniform distribution on $[0, 1]$. Find random variables which are functions of X, Y and U and which have the following moment generating functions:

   (i) $M_X(t)M_Y(t)$; (ii) $e^{bt}M_X(at)$; (iii) $\int_0^1 M_X(tu)du$; (iv) $[M_X(t) + M_Y(t)]/2$.

   (b) Using characteristic functions or otherwise, find $\mathbb{E}\cos(tX)$ and $\mathbb{E}\sin(tX)$ when X has exponential distribution with parameter λ.

   (c) Which random variables X have a real-valued characteristic function?

5. Suppose X has Gamma$(2, \lambda)$ distribution, and the conditional distribution of Y given $X = x$ is uniform on $(0, x)$.

   Find the joint density function of X and Y, the marginal density function of Y, and the conditional density function of X given $Y = y$? How would you describe the distribution of X given $Y = y$? Use this to describe the joint distribution of Y and $X - Y$.

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6. Random variables X and Y have joint density $f(x, y)$. Let $Z = Y/X$. Show that Z has density
$$f_Z(z) = \int_{-\infty}^{\infty} |x| f(x, xz) dx.$$
Suppose now that X and Y are independent standard normal random variables. Show that Z has density
$$f_Z(z) = \frac{1}{\pi\,(1 + z^2)}, \quad -\infty < z < \infty.$$

7. The distribution of the heights of husband-wife pairs in a particular population is modelled by a bivariate normal distribution. The mean height of the women is 165cm and the mean height of the men is 175cm. The standard deviation is 6cm for women and 8cm for men. The correlation of height between husbands and wives is 0.5.

   Let X be the height of a typical wife and Y the height of her husband. Show how Y can be represented as a sum of a term which is a multiple of X and a term which is independent of X. Hence or otherwise:

   (a) Given that a woman has height 168cm, find the expected height of her husband.

   (b) Given that a woman has height 168cm, what is the probability that her husband is above average height?

   (c) What is the probability that a randomly chosen man is taller than a randomly chosen woman?

   (d) What is the probability that a randomly chosen man is taller than his wife?

8. (a) Let X and Y be independent standard normal random variables. Use question 1(a) to show that for a constant $c > 0$,
$$\mathbb{P}(X > 0, Y > -cX) = \frac{1}{4} + \frac{\tan^{-1}(c)}{2\pi}.$$

   (b) Two candidates contest a close election. Each of the n voters votes independently with probability 1/2 each way. Fix $\alpha \in (0, 1)$. Show that, for large n, the probability that the candidate leading after αn votes have been counted is the eventual winner is approximately
$$\frac{1}{2} + \frac{\sin^{-1}(\sqrt{\alpha})}{\pi}.$$
   [*Hint: let S_m be the difference between the vote totals of the two candidates when m votes have been counted. What is the approximate distribution of $S_{\alpha n}$ (when appropriately rescaled)? What is the approximate distribution of $S_n - S_{\alpha n}$ (when appropriately rescaled)? What about their joint distribution? Finally, notice* $\sin^{-1}(\sqrt{\alpha}) = \tan^{-1}\sqrt{\alpha/(1-\alpha)}$.]

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Additional problems

9. Let U, V and W be i.i.d. random variables with uniform distribution on $[0,1]$. Find the distribution of $(UV)^W$.

10. Use characteristic functions to prove the identity
$$\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos\left(\frac{t}{2^n}\right).$$
[*Hint: consider the c.f. of a uniform distribution, and of a distribution taking only two values.*]

11. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. Suppose its expectation $\mathbb{E}[X]$ exists and is finite. Consider the *discretisations* $X_n = 2^{-n}\lfloor 2^n X \rfloor$, $n \geq 0$.
   (a) For each $n \geq 0$, show that X_n is a discrete random variable.
   (b) Recall that definitions and properties of expectations have been established separately for discrete and continuous random variables. Working from these definitions, show that the $\mathbb{E}[X_n]$ exists and satisfies $\mathbb{E}[X] - 2^{-n} \leq \mathbb{E}[X_n] \leq \mathbb{E}[X]$ for each $n \geq 0$. Deduce that $\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n]$.

12. Let $X : \Omega \to [0,\infty)$ be any non-negative random variable. Consider the discretisations $X_n = 2^{-n}\lfloor 2^n X \rfloor$, $n \geq 0$. Define
$$\mathbb{E}[X] := \lim_{n\to\infty} \sum_{k=1}^{\infty} k2^{-n}\mathbb{P}(X_n = k2^{-n}),$$
provided that these limits exist and are finite. If $X : \Omega \to \mathbb{R}$ is any (real-valued) random variable, we consider $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$ so that $X = X^+ - X^-$. If both $\mathbb{E}[X^+]$ and $\mathbb{E}[X^-]$ are finite we define $\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-]$.
   (a) Let X and Y be two random variables whose expectations exist in the sense defined above. Show that $X \leq Y$ implies $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
   (b) If X is a discrete or continuous random variable, show that the new definition of $\mathbb{E}[X]$ is consistent with the previous definitions.
   (c) Let X and Y be two random variables. Show that $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ for all $a, b \in \mathbb{R}$, provided that both $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist and are finite.
   (d) Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)dx$ holds for any non-negative random variable.

   [*These limits of series defining $\mathbb{E}[X]$ are not very elegant compared to the more direct definition of Part B Probability, Measure and Martingales, where we will make sense of $\mathbb{E}[X] := \int_\Omega X(\omega)\mathbb{P}(d\omega)$ once the notion of integration against a probability measure $\mathbb{P}$ is available (as a generalisation of the Lebesgue integral of Part A Integration). Integration theory includes powerful theorems that allow shorter proofs of the results of this problem.*]

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