Part A1: Differential Equations I

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*based on notes by Peter Grindrod, Colin Please, Paul Tod, Lionel Mason and others, with modifications by the lecturer

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Introduction

The solution of problems in most parts of applied mathematics and many areas of pure can more often than not be reduced to the problem of solving some differential equations. Indeed, many parts of pure maths were originally motivated by issues arising from differential equations, including large parts of algebra and much of analysis, and Differential equations are a central topic in research in both pure and applied mathematics to this day. From Prelims, and even from school, you know how to solve some differential equations. Indeed most of the study of differential equations in the first year consisted of finding explicit solutions of particular ODEs or PDEs. However, for many differential equations which arise in practice one is unable to give explicit solutions and, for the most part, this course will consider what information one can discover about solutions without actually finding the solution. Does a solution exist? Is it unique? Does it depend continuously on the initial data? How does it behave asymptotically? What is appropriate data?

So, first we will develop techniques for proving Picard's theorem for the existence and uniqueness of solutions of ODEs; then we will look at how phase plane analysis enables us to estimate the long term behaviour of solutions of plane autonomous systems of ODEs. We will then turn to PDEs and show how the method of characteristics reduces the solution of a first order semi-linear PDE to solving a system of non-linear ODEs. Finally we will look at second order semi-linear PDEs: We classify them and investigate how the different types of problem require different types of boundary data if the problem is to be well posed. We then look at how the maximum principle enables us to prove uniqueness and continuous dependence on the initial data for two very special problems: Poisson's equation and the inhomogeneous heat equation – each with suitable data.

Throughout, we shall use the following convenient abbreviations: we shall write

DEs: for differential equations.

ODEs: for ordinary DEs, i.e. differential equations with only ordinary derivatives.

PDEs: for partial DEs, i.e. differential equations with partial derivatives.

The course contains four topics, with a section devoted to each. The chapters are:

1. ODEs and Picard's Theorem (for existence/uniqueness of solutions/continuous dependence on initial data).
2. Plane autonomous systems of ODEs
3. First order semi-linear PDEs: the method of characteristics.
4. Second-order semi-linear PDEs: classification; well posedness; the Maximum Principle and its consequences

Books

The main text is P J Collins *Differential and Integral Equations*, O.U.P. (2006), which can be used for the whole course (Chapters 1-7, 14, 15).

Other good books which cover parts of the course include

W E Boyce and R C DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 7th edition, Wiley (2000).

E Kreyszig, *Advanced Engineering Mathematics, 8th Edition*, Wiley (1999).

G F Carrier and C E Pearson, *Partial Differential Equations – Theory and Technique*, Academic (1988).

J Ockendon, S Howison, A Lacey and A Movchan, *Applied Partial Differential Equations*, Oxford (1999) [a more advanced text].

PART I Ordinary Differential Equations

1 ODEs and Picard's Theorem

1.1 Introduction

An ODE is an equation for $y(x)$ of the form

$$G(x, y, y', y'', \ ..., \ y^{(n)}) = 0.$$

We refer to y as the dependent variable and x as the independent variable. Usually this can be solved for the highest derivative of y and written in the form

$$\frac{d^n y}{dx^n} := y^{(n)}(x) = F(x, y, y', \ ..., \ y^{(n-1)}).$$

Then the *order* of the ODE is n, the order of the highest derivative which appears. Given an ODE, certain obvious questions arise. We could ask:

- Does it have solutions? Can we find them (explicitly or implicitly)? If not, can we at least say something about their qualitative behaviour?
- Given data e.g. the values $y(a)$, $y'(a)$, ... of $y(x)$ and its first $n-1$ derivatives at some initial value a of x, does it have a solution? is it unique? does it depend continuously on the given data?

We shall consider these questions in Part I.

For simplicity, we begin with a first-order ODE with data:

$$y'(x) = f(x, y(x)) \text{ with } y(a) = b. \tag{1.1}$$

This is an *initial value problem* or IVP, since we are given y at an initial, or starting, value of x. You know how to solve a variety of equations like this. You might expect that a solution *exists*, so there is some function that satisfies the equations (even if you cannot find a formula for it) and perhaps you expect, for a given initial values, the solution is *unique* (there is only one function that satisfies the ODE and the initial data) as you may not have encountered the following difficulties.

Warning examples:

Consider this IVP:

$$y' = 3y^{2/3}; \quad y(0) = 0. \tag{1.2}$$

So separate the variables (a prelims technique):

$$\int \frac{dy}{3y^{2/3}} = \int dx,$$

to get $y = (x + A)^3$, so if $y(0) = 0$

(i) There is a solution $y = x^3$;

(ii) But evidently there is another solution: by direct checking $y = 0$ will do;

(iii) In fact we can find that there are infinitely many solutions. Pick a, b with $a \leq 0 \leq b$ and define

$$\begin{aligned} y &= (x-a)^3 \qquad x < a \\ &= 0 \qquad a \leq x < b \\ &= (x-b)^3 \qquad b \leq x \end{aligned}$$

The solution does exist but is not unique (in fact far from it, since we've found infinitely many solutions).

Furthermore, even if a solution of (1.1) exists, it may not exist for all x.

For example consider the IVP

$$y' = y^2; \quad y(0) = 1. \tag{1.3}$$

Using separation of variables we can see this has solution $y = \dfrac{1}{1-x}$; so $y \to \infty$ as $x \to 1$, and the solution only exists on $x < 1$. We will see later that this solution is in fact unique.

So, if we want to have a unique solution to the problem (1.1) we must impose conditions on f, and we cannot necessarily expect to have solutions for all x. This will be the first *existence theorem* which you've encountered. The proof is quite technical, certainly the most technical thing in the course. In particular, you need to remember from Prelims the *Weierstrass M-test* for convergence of a series of functions.

To be precise then, we shall seek a solution of problem (1.1) in a rectangle R about the initial point $(x, y) = (a, b)$, so suppose $R = \{(x, y) : |x - a| \leq h,\ |y - b| \leq k\}$ as in figure 1.1:

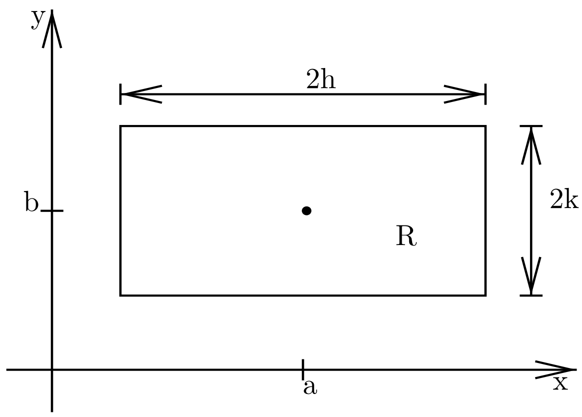


Figure 1.1: The rectangle R

Our first assumption is that $f : R \to \mathbb{R}$ is continuous in R. Note that if y is a solution of (1.1), say on an interval $[a-h, a+h]$, then integrating (1.1) from a to variable $x \in [a-h, a+h]$ yields

$$y(x) - y(a) = [y(t)]_a^x = \int_a^x f(t, y(t))dt$$

for any x so rearranging

$$y(x) = b + \int_a^x f(t, y(t))dt. \tag{1.4}$$

We note that since $f : R \to \mathbb{R}$ and $y : [a-h, a+h] \to \mathbb{R}$ are continuous, also the function $x \mapsto f(x, y(x))$ is a continuous function on $[a-h, a+h]$ so integrable.

Conversely, if $y(t)$ is continuous on $[a-h, a+h]$ and satisfies (1.4), then $y(a) = b$ and by the Fundamental theorem of Calculus we can differentiate (1.4) to get that y is a solution of (1.1). Thus (1.1) and (1.4) are equivalent. We have transformed the differential equation to an *integral equation* - the unknown y is given in terms of an integral rather than a differential. There is a general theory of these, but we only need to deal with the particular case of (1.4). The standard approach is to seek a solution by *iteration* or successive approximation.

1.2 Picard's method of successive approximation

We start with an initial guess and then improve it. The guesses, or successive approximations or iterates, are labeled $y_n(x)$ starting with $y_0(x)$. Take

$$\left.\begin{array}{r} y_0(x) = b \\ y_{n+1}(x) = b + \int_a^x f(t, y_n(t))dt \end{array}\right\} \tag{1.5}$$

That is, we start with the simplest guess, that y equals its initial value, and at each stage substitute the current guess into the right-hand-side of (1.4) to get the next guess. We need to know if this process converges, and if it does whether it converges to a solution of the problem (1.4). Consider the differences between successive approximations:

$$\left.\begin{aligned} e_0(x) &= b \\ e_{n+1}(x) &= y_{n+1}(x) - y_n(x) \end{aligned}\right\} \tag{1.6}$$

and note that

$$y_n(x) = \sum_0^n e_k(x). \tag{1.7}$$

We want y_n to converge, so we want the series $\sum_0^n e_k(x)$ to converge. So we must estimate the differences $e_n(x)$. This will need a condition on f, but here is the key idea.

Notice that

$$e_{n+1}(x) = y_{n+1}(x) - y_n(x) = \int_a^x [f(t, y_n(t)) - f(t, y_{n-1}(t))]dt$$

and recall that *the modulus of the integral of a function is less than or equal to the integral of the modulus* (because the function can be negative). Therefore

$$|\, e_{n+1}(x)| \leq \left| \int_a^x |f(t, y_n(t)) - f(t, y_{n-1}(t))|\, dt \right|. \tag{1.8}$$

(The modulus outside the integral on the right hand side is required to cover the case $x \leq a$.) We want to bound the integrand on the right-hand side in terms of the error $e_n(t) = |y_n(t) - y_{n-1}(t)|$ of the previous step which motivates the following definition:

Definition 1.1. *A function $f(x, y)$ on a rectangle R satisfies a* Lipschitz condition *(with constant L) if $\exists$ real positive L such that*

$$|f(x, u) - f(x, v)| \leq L|u - v| \textit{ for all } (x, u) \in R,\ (x, v) \in R. \tag{1.9}$$

This is a new condition on a function, stronger than being continuous in the second variable but weaker than being differentiable. It turns out to be the right condition to make Picard's theorem, which is the existence theorem we want, work, as it allows us to bound the integrand in (1.8) by

$$|f(t, y_n(t)) - f(t, y_{n-1}(t))| \leq L|e_n(t)|.$$

Important note: One way to ensure that f satisfies a Lipschitz condition on R is the following: Suppose that, on R, f is differentiable with respect to y , with $|f_y(x, y)| \leq K$. Then for any $(x, u) \in R,\ (x, v) \in R$ the mean value theorem (applied to the function $[k - h, k + h] \ni y \mapsto f(x, y)$) gives

$$|f(x, u) - f(x, v)| = |f_y(x, w)(u - v)| \leq K|u - v| \tag{1.10}$$

where w is some intermediate value. So, f clearly satisfies the Lipschitz condition on such intervals.

On the other hand $f(y) = |y|$ is Lipschitz continuous, but is not differentiable at $y = 0$.

1.3 Picard's Theorem

Theorem 1.1. (Picard's existence theorem):

Let $f : R \to \mathbb{R}$ be a function defined on the rectangle $R := \{(x, y) : |x-a| \le h,\ |y-b| \le k\}$ which satisfies
P(i): *(a) f is continuous in R, with bound M (so $|f(x, y)| \le M$) and (b) $Mh \le k$.*
P(ii): *f satisfies a Lipschitz condition in R.*

Then the IVP

$$y'(x) = f(x, y(x)) \text{ with } y(a) = b.$$

has a unique solution $y : [a - h, a + h] \to [b - k, b + k]$.

We will prove the existence of a solution by showing that the iterates y_n defined in (1.5) converge as $n \to \infty$ to a solution y of the IVP and will do this by showing that the series in (1.7) converges. We break the proof into a series of steps:

Claim 1: Each y_n is well defined, continuous and $|y_n(x) - b| \le k$ for $x \in [a - h, a + h]$.

Proof of Claim 1: This is clearly true for $n = 0$, so suppose claim is true for some $n \ge 0$. Then for $t \in [a - h, a + h]$ we have that $(t, y_n(t)) \in R$ so as f is defined and continuous on R and as y_n is continuous we have that $t \mapsto f(t, y_n(t))$ is a continuous function on the interval $[a - h, a + h]$. Thus y_{n+1} is well defined and continuous by properties of integration and by P(i)

$$\begin{aligned} |y_{n+1}(x) - b| &\le \left| \int_a^x |f(t, y_n(t))| dt \right| \\ &\le \left| M \int_a^x dt \right| = M|x - a| \le Mh \le k \end{aligned}$$

for every $x \in [a - h, a + h]$. Thus the claim is true by induction.

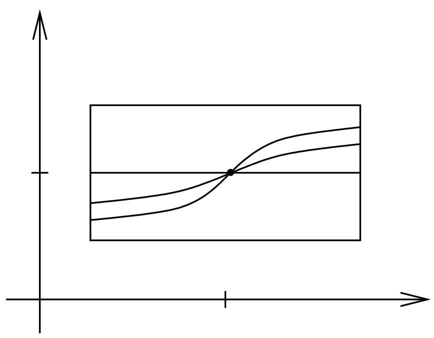


Figure 1.2: successive iterates graphed in R.

We next prove:

Claim 2: For $|x-a| \le h$ and $n \in \mathbb{N}$

$$|e_n(x)| \le \frac{L^{n-1}M}{n!}|x-a|^n \tag{1.11}$$

where L is such that the Lipschitz condition (1.9) holds.

We remark that this claim in particular implies that

$$|e_n(x)| \le \frac{L^{n-1}M}{n!}h^n \text{ for all } |x-a| \le h. \tag{1.12}$$

which will be the estimate that we will use to show that $\sum e_n$ converges uniformly using M-test.

Proof of Claim 2: We recall that the Lipschitz condition P(ii) combined with the fact that the graph of y_n is in the rectangle implies that for all $|t-a| \le h$

$$|f(t, y_n(t)) - f(t, y_{n-1}(t))| \le L|y_n(t) - y_{n-1}(t)| = L|e_n(t)|. \tag{1.13}$$

From (1.8) and (1.13) thus

$$\begin{aligned} |\, e_{n+1}(x)| &\le \left| \int_a^x |f(t, y_n(t)) - f(t, y_{n-1}(t))| dt \right| \\ &\le L \left| \int_a^x |e_n(t)| dt \right|. \end{aligned} \tag{1.14}$$

Now we prove (1.11) by induction:

$$e_1(x) = y_1(x) - b = \int_a^x f(t, b) dt. \tag{1.15}$$

By **P(i)**, f is bounded by M so that

$$|e_1(x)| \le \left| \int_a^x |f(t,b)| dt \right| \le M|x-a|, \tag{1.16}$$

so (1.11) is true for $n = 1$. Now suppose that (1.11) is true for n, then

$$\begin{aligned} |\, e_{n+1}(x)| \ &\le \ L \left| \int_a^x |e_n(t)| dt \right| \\ &\le \ L \left| \int_a^x \frac{L^{n-1}M}{n!} |t-a|^n dt \right| = \frac{L^n M}{(n+1)!} |x-a|^{n+1}, \end{aligned}$$

so that (1.11) is true by induction.

We now use these two claims to prove the *existence* of a solution to the integral equation (1.4) by showing

Claim 3: The iterates $y_n(x) = \sum_{j=0}^{n} e_j(x)$ converge uniformly to a continuous function y_∞ and y_∞ is a solution of the integral equation (1.4).

Proof of Claim 3: The uniform convergence immediately follows from the Weierstrass M-test and (1.12), since $\sum_{n=1}^{\infty} M_n$ for $M_n = \dfrac{L^{n-1}Mh^n}{n!}$ converges and M_n is a upper bound on $|e_n(x)|$ that is independent of x. Thus, $y_n = \sum_0^n e_k$ converges uniformly to a limit y_∞ on $[a-h, a+h]$ and this limit is continuous as it is the uniform limit of the continuous functions y_n.

To see that y_∞ is a solution of (1.1), we would like to take the limit in (1.5) and exchange the limit and the integral to get that

$$\begin{aligned} y_\infty(x) &= \lim_{n\to\infty} y_{n+1}(x) = b + \lim_{n\to\infty} \int_a^x f(t, y_n(t))dt \overset{(*)}{=} b + \int_a^x \lim_{n\to\infty} f(t, y_n(t))dt \\ &= b + \int_a^x f(t, y_\infty(t))dt. \end{aligned} \tag{1.17}$$

The reason that we are allowed to switch limit and integral in (*) is that the integrands $f(t, y_n(t))$ converge uniformly to $f(t, y_\infty(t))$ since the uniform convergence of the y_n and the Lipschitz condition allow us to estimate

$$\sup_{t\in[a-h,a+h]} |f(t, y_n(t)) - f(t, y_\infty(t))| \leq \sup_{t\in[a-h,a+h]} L|y_n(t) - y_\infty(t)| \to 0.$$

We have thus proven the existence of a solution of the integral equation and as remarked previously differentiating the integral equation implies that

$$y_\infty'(x) = f(x, y_\infty(x))$$

and since also $y_\infty(a) = b$, thus y_∞ is a solution of the IVP.

This completes the proof of existence and it remains to show

Claim 4: The solution of (1.1) is unique among all functions $y : [a-h, a+h] \to [b-k, b+k]$.

Proof of Claim 4: Let $y_1(x)$ and $y_2(x)$ be two solutions of (1.1) (for the same f, a and b!) and set $e(x) := y_2(x) - y_1(x)$. We aim to show that $e(x) = 0$ for all x.

As the IVP is equivalent to the integral equation (1.4) we can subtract the two integral equations satisfied by $y_{1,2}$ to see that

$$e(x) = y_2(x) - y_1(x) = \int_a^x (f(t, y_2(t)) - f(t, y_1(t)))dt$$

so using the triangle inequality for integrals and the Lipschitz condition we get

$$\begin{aligned} |\,e(x)| &\leq \left| \int_a^x |f(t, y_2(t)) - f(t, y_1(t))|dt \right| \leq L \left| \int_a^x |y_2(t) - y_1(t)|dt \right| \\ &\leq L \left| \int_a^x |e(t)|dt \right| \end{aligned} \tag{1.18}$$

Now $e(x)$ is continuous on $[a-h, a+h]$ therefore, it is bounded say $|e(x)| \le B$ so

$$|\, e(x)| \le L \left| \int_a^x B dt \right| = LB|x-a|$$

and inducting on n, using (1.18), for each n

$$|\, e(x)| \le BL^n \frac{|x-a|^n}{n!}$$

So that for each n

$$|\, e(x)| \le B \frac{L^n h^n}{n!} \to 0 \text{ as } n \to \infty \text{ and } e(x) = 0.$$

Thus the difference is zero, so the solutions are the same which establishes uniqueness (uniqueness proofs almost always go like this: assume there are two and make their difference vanish).

This completes the proof of Picard's Theorem. □

Note that the proof of uniqueness of the solution only holds among those solutions whose graph lies in R. This is however enough since we have:

Remark: Let f be so that **P(i)** hold. Then the graph of any solution $y(x)$ of (1.1) for $|x-a| \le h$ must lie in R.

Indeed, suppose not. Then, by the continuity of y, there will exist a 'first' x_0, with $|x_0 - a| < h$, where $(x_0, y(x_0))$ is on the boundary of R. That is such that $|x_0 - a| < h$, $|y(x_0) - b| = k$ but $|y(x) - b| < k$ if $|x-a| < |x_0 - a|$, see figure 1.3. But then

$$|y(x_0) - b| \le \left| \int_a^{x_0} |f(s, y(s))| ds \right| \le M|x_0 - a| < Mh = k$$

a contradiction. Equivalently we can argue that since $|f|$ is bounded by M we know that the slope of the graph of any solution of $y'(x) = F(x, y(x))$ is at most M so $y(x)$ cannot increase from $y(a) = b$ to a value larger than $b + Mh$ on an interval of length h.

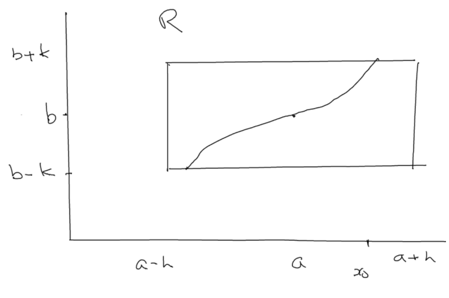


Figure 1.3: x_0 is the first x where the graph meets the boundary of the rectangle

Remark: An alternative (and more standard) way of proving uniqueness of solutions of differential equations is via Gronvall's inequality and this proof will be carried out in section 1.5.

Since the warning example doesn't have a unique solution, something goes wrong for it. As an exercise, show that the warning example fails the Lipschitz condition (in any neighbourhood of the initial point).

The following example also fails the Lipschitz condition in any neighbourhood of the initial point $y = 0$. However, the Lipschitz condition does hold on any rectangle which does not contain any point $(x, 0)$.

Example: Consider the IVP

$$y' = x^2 y^{1/5}, \quad y(0) = b.$$

So we consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 y^{1/5}$ which is clearly continuous. (Note that when we write $y^{1/5}$ here we mean to take the real root: so that if y is negative we will take $-|y|^{1/5}$.)

Case $b = 0$: $f(x, y)$ does not satisfy a Lipschitz condition on any rectangle of the form $R_0 = \{(x, y) : |x| \le h,\ |y| \le k\}$, where $h > 0$ and $k > 0$.

Suppose it does, then there exists a finite constant L such that for all $|x| \le h$ and $|y|, |\tilde{y}| \le k$

$$|x^2||y^{1/5} - \tilde{y}^{1/5}| \le L|y - \tilde{y}|$$

so in particular (choosing $\tilde{y} = 0$ and $x = h$)

$$|h^2||y^{-4/5}| \le L \text{ for every } y \in [-h, h] \setminus \{0\}.$$

But this is a contradiction as $|h^2||y^{-4/5}|$ is unbounded as $y \to 0$ so the function does not satisfy a Lipschitz condition on R_0. So Picard's theorem does not apply if we take $b = 0$. (We saw that f satisfies a Lipschitz condition on any rectangle where its derivative with respect to y exists and is bounded. The problem here is that the derivative of f is unbounded as $y \to 0$ – indeed the derivative does not exist at $y = 0$.)

Case $b > 0$: However, the assumptions of Picard's theorem will be satisfied if we take as initial condition $y(0) = b > 0$, provided we take a rectangle, R_b, given by $R_b = \{|x| \le h,$ $|y - b| \le k\}$ when $0 < k < b$, so that y cannot be zero in this rectangle .

On any such rectangle $f_y(x, y) = \frac{x^2 y^{-4/5}}{5}$ is bounded, so, by (1.10), f satisfies a Lipschitz condition, and **P(ii)** is satisfied.

For **P(i)**: $f(x, y)$ is continuous on R_b and

$$\max_{R_b} |x^2 y^{1/5}| \le h^2 (b + k)^{1/5} =: M,$$

so Picard's theorem applies in a rectangle where $h > 0$ satisfies

$$h^2 (b + k)^{1/5} h \le k.$$

That is

$$h^3 \leq \frac{k}{(b+k)^{1/5}}. \tag{1.19}$$

We can of course solve this problem directly using separation of variables if we wish giving

$$y = \left(4x^3/15 + b^{4/5}\right)^{5/4},$$

so actually the solution exists for all x. Note the solution above is valid for $b = 0$ BUT the trivial solution is also valid. So while we still have existence, uniqueness does not hold.

1.4 Extension of solutions and global existence.

The result in Section 1.5 is a *local* result in that it guarantees existence and uniqueness of a solution on the interval $[a-h, a+h]$, where h satisfies $Mh \leq k$ (though this h need not be the best possible).

As we have seen there are examples of initial value problems where solutions do not exist for all $x \in (-\infty, \infty)$. So we would like to find conditions which guarantee that the solution does exist for all $x \in (-\infty, \infty)$ (or if f is only defined for x in an certain interval then on the whole such interval). One such condition is the *global Lipschitz condition*, where we can find a constant L such that the Lipschitz condition holds for all y. First we will see that if the Lipschitz condition is global in y, but L still depends on the interval $[a-h, a+h]$, then there is existence on all of $[a-h, a+h]$

Suppose we require that $f(x, y)$ is defined and continuous for all $(x, y) \in [a-h, a+h] \times \mathbb{R}$ and instead of (**P(ii)**) we have

(**P(iii)**): $f(x, y)$ satisfies the Lipschitz condition for all real y and all $x \in [a-h, a+h]$.

Then the last condition in claim 1 is not required and hence we don't need to ask that $Mh \leq k$. If we investigate the proof of the Picard existence theorem, we see that M also appears in (1.16). However for claim 2 to hold it is sufficient to take $M = \sup_{x \in [a-h, a+h]} |f(x, b)|$, which exists as $x \mapsto f(x, b)$ is a continuous function on the closed bounded interval $[a-h, a+h]$ (whereas $(x, y) \mapsto f(x, y)$ might be unbounded on the unbounded set $[a-h, a+h] \times \mathbb{R}$) . The rest of the proof applies without change and we hence obtain that the solution exists and is unique $\forall x \in [a-h, a+h]$.

If (**P(iii)**) holds for each $h > 0$ (for some L that is allowed to depend on h), then we can carry out this argument for every $h > 0$ and, by letting $h \to \infty$ deduce that the solution exists on all of $\mathbb{R}$. In this case we say that we have a *global solution*.

Example: If $f(x, y) = p(x)y + q(x)$, where p and q are continuous on $|x-a| \leq h$, then f satisfies (**P(iii)**).

Remark We do not need the interval in (**P(iii)**) to be a balanced interval (ie of the form $[a-h, a+h]$) because we can deal with $x \leq a$ and $x \geq a$ separately. Thus if $a \in [c, d]$ and we require that f is continuous on $[c,d] \times \mathbb{R}$ and that it satisfies the Lipschitz condition on this set, then a solution of (1.1) exists and is unique for all $x \in [c,d]$.

1.5 Gronwall's inequality and continuous dependence on the initial data.

We will now prove Gronwall's inequality which will be used to provide another proof of uniqueness of solutions, but will also be used to show that solutions depend continuously on the initial data.

Theorem 1.2. (Gronwall's inequality) : *Suppose $A \geq 0$ and $b \geq 0$ are constants and v is a non-negative continuous function satisfying*

$$v(x) \leq b + A \left| \int_a^x v(s)ds \right| \tag{1.20}$$

then

$$v(x) \leq b e^{A|x-a|}.$$

(The modulus is needed to take care of the case $x \leq a$.)

Proof: We use an integrating factor.

For $x \geq a$ let $V(x) = \int_a^x v(s)ds$, so that $V'(x) = v(x)$. As $x \geq a$ and $v \geq 0$ also $V(x) \geq 0$ and we have

$$V'(x) \leq b + AV(x).$$

Multiply through by the integrating factor e^{-Ax} so

$$\begin{aligned}
(V'(x) - AV(x))e^{-Ax} &\leq be^{-Ax} \text{ that is} \\
\frac{d}{dx}(V(x)e^{-Ax}) &\leq be^{-Ax}, \text{ so, integrating and noting that } V(a) = 0 \\
V(x)e^{-Ax} &\leq \int_a^x be^{-As}ds = \frac{b}{A}(e^{-Aa} - e^{-Ax}), \text{ so} \\
V(x) &\leq \frac{b}{A}(e^{A(x-a)} - 1).
\end{aligned}$$

Finally, using (1.20)

$$v(x) \leq b + A\int_a^x v(s)ds = b + AV(x) \leq b + A\frac{b}{A}(e^{A(x-a)} - 1) = be^{A(x-a)},$$

as required. Similarly if $x \leq a$. $\square$

Remark: Gronwall's inequality says that v is bounded above by the solution of the integral equation one obtains when there is equality in (1.20). For, if we differentiate

$$v(x) = b + A\int_a^x v(s)ds,$$

we get

$$v'(x) = Av(x), \quad v(a) = b,$$

which has solution $v(x) = be^{A(x-a)}$.

Application: Alternative proof of the uniqueness part of Picard's theorem and continuous dependence on initital data

Suppose that y and z are solutions of the ordinary differential equation $y'(x) = f(x, y(x))$ with $y(a) = b$ and $z(a) = c$, where f satisfies conditions (**P(i)**) and (**P(ii)**). We want to bound the difference between these solutions in terms of the difference $|b - c|$ and show in particular that if $b = c$ then we must have $y(x) = z(x)$ for all x. This gives an alternative (and indeed simpler) proof of the uniqueness part of Picard's theorem.

Setting $v(x) = |y(x) - z(x)|$ (note Gronvall requires v to be non-negative) and using that

$$y(x) - z(x) = b - c + \int_a^x (f(s, y(s)) - f(s, z(s))ds$$

we get from the Lipschitz condition that

$$v(x) \leq |b - c| + L \left| \int_a^x v(s)ds \right|.$$

We can thus apply Gronwall's inequality to get that

$$|y(x) - z(x)| = v(x) \leq |b - c|e^{L|x-a|} \leq |b - c|e^{Lh}. \tag{1.21}$$

In particular, if y and z solve the initial value problem with the same initial value $y(a) = z(a) = b$, then the functions y and z are equal, which proves the uniqueness part of Picard's theorem.

More generally, we have obtained a bound on $|y(x) - z(x)|$ for all x in the interval where both solutions are defined in terms of $|b - c|$.

We say a solution is *continuously dependent on the initial data* on an interval I if we can make $\sup_{x \in I} |y(x) - z(x)|$ as small as we like by taking $|b - c|$ small enough. In other words the error in the solution will be small provided the error in the initial data is small enough. To be precise, in this case, solutions are continuously dependent on the initial data for $x \in [a - h, a + h]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if y and z are as above,

$$|b - c| < \delta \quad \Rightarrow |y(x) - z(x)| \leq \epsilon, \quad \forall x \in [a - h, a + h].$$

This is clearly true from (1.21), because given $\epsilon > 0$, we have $|y(x) - z(x)| \leq \epsilon$ whenever $|b - c| < e^{-Lh}\epsilon$, so we can take $\delta = e^{-Lh}\epsilon$.

1.6 Picard's Theorem via the CMT

We can prove Picard's theorem in a more efficient way, which is really equivalent to our previous method, by using the contraction mapping theorem (CMT). This is a very useful method of proving existence and uniqueness of solutions of nonlinear differential equations and many, many other things besides. The results we need will be discussed in the course on Metric Spaces and Complex Analysis. We will assume the results proved there.

Define $\mathcal{C}_{h,k} = \mathcal{C}([a-h, a+h]; [b-k, b+k])$, the space of continuous functions $y : [a-h, a+h] \to [b-k, b+k]$. As is shown in the Metric Spaces course, for $y, z \in \mathcal{C}_{h,k}$ if we define

$$d(y,z) := ||y-z||_{\sup} := \sup_{x \in [a-h, a+h]} |y(x) - z(x)|$$

then $(\mathcal{C}_{h,k}, d)$ is a complete metric space (we call $||\cdot||_{\sup}$ the "sup norm").

Also we say that a map $T : \mathcal{C}_{h,k} \to \mathcal{C}_{h,k}$ is a contraction if there exists $K < 1$ such that

$$||T(y) - T(z)||_{\sup} \le K||y-z||_{sup},$$

and then we have the CMT, which says:

Theorem 1.3. (Contraction Mapping Theorem) *(Banach) Let X be a complete metric space and let $T : X \to X$ be a contraction. Then there is a unique fixed point $y \in X$, i.e. a unique y such that $Ty = y$.*

To prove Picard's Theorem via the CMT we will first apply this theorem for $X = \mathcal{C}_{\eta,k} = \mathcal{C}([a-\eta, a+\eta]; [b-k, b+k])$ for a small enough $0 < \eta \le h$ that we chose below, which will give that there exists a unique solution for $|x-a| \le \eta$. In a second step we will then discuss how this solution can be extended to all of $[a-h, a+h]$ if $Mh \le k$ by repeating the argument with a new choice of the space X.

We again consider the IVP (1.1)

Theorem 1.4. (Picard's existence theorem.) *Let $f : R \to \mathbb{R}$ be a function defined on the rectangle $R := \{(x,y) : |x-a| \le h,\ |y-b| \le k\}$ which satisfies conditions* **P(i)***(a) and* **P(ii)** *and let $\eta > 0$ be so that $L\eta < 1$ and $M\eta \le k$.*

Then the initial value problem (1.1) has a unique solution for $x \in [a-\eta, a+\eta]$.

Proof.

The strategy is to express (1.1) as a fixed point problem and use the CMT.

As before, we can write the initial value problem as an integral equation

$$y(x) = b + \int_a^x f(s, y(s))ds \tag{1.22}$$

Provided $f(s, y(s))$ is continuous in s, y is a solution of the differential equation if and only if y is a solution of the integral equation.

If we define

$$(Ty)(x) = b + \int_a^x f(s, y(s))ds$$

then we can write (1.22) as a fixed point problem

$$y = Ty.$$

We will work in the complete metric space $\mathcal{C}_{\eta,k} = \mathcal{C}([a-\eta, a+\eta]; [b-k, b+k])$, where we will choose $\eta \le h$ so that $T : \mathcal{C}_{\eta,k} \to \mathcal{C}_{\eta,k}$ and so that T is a contraction. We begin by proving **Claim 1:** If $\eta > 0$ is so that $M\eta \le k$ then $T : \mathcal{C}_{\eta,k} \to \mathcal{C}_{\eta,k}$

Proof. First we note that from the properties of integration, $(Ty)(x) \in \mathcal{C}([a-\eta, a+\eta]; \mathbb{R})$. All that we require is thus to show that $||Ty - b||_{\sup} \le k$ if $||y - b||_{\sup} \le k$.

But

$$\begin{aligned} ||Ty - b||_{\sup} &= \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x f(s, y(s))ds \right| && (1.23) \\ &\le \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x |f(s, y(s))|ds \right| && (1.24) \\ &\le M\eta \le k, && (1.25) \end{aligned}$$

provided $M\eta \le k$. □

Claim 2: If $L\eta < 1$ then T is a contraction (with $K = L\eta$):

Proof. Given $y, z \in C_{\eta,k}$ we can bound

$$\begin{aligned} ||Ty - Tz||_{\sup} &= \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x f(s, y(s)) - f(s, z(s))ds \right| \\ &\le \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x |f(s, y(s)) - f(s, z(s))|ds \right| \\ &\le \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x L|y(s) - z(s)|ds \right| \le L\eta||y - z||_{\sup} \le K||y - z||_{\sup} \end{aligned}$$

where $K := \eta L < 1$ provided $\eta < 1/L$. □

If we hence choose $\eta < \min\{h, k/M, 1/L\}$ then T satisfies the conditions of the CMT and has a unique fixed point, $y(x)$. As explained before, a (continuous) function y solves the integral equation $Ty = y$ if and only if it is continuously differentiable and a solution of the initial value problem, so we have established that the initial value problem has a unique solution on the interval $[a - \eta, a + \eta]$. □

Note that our proof using CMT produces a more restricted range of x values than did our proof on one dimension. The range of η depends on L as well as M and k. However, if $Mh \le k$, actually we only need $\eta \le h$, and we can now extend the range of the solution to all $x \in [a - h, a + h]$, by iteration.

Corollary 1.5. *If $Mh \leq k$, then the initial value problem has a unique solution on the whole interval* $[a-h, a+h]$

As uniqueness follows from Gronvall's Lemma as explained in Section 1.5 we only have to prove that the solution exists on the whole interval $[a-h, a+h]$. There are two very different ways of proving this, one by iterating the above argument to construct a solution on larger and larger intervals, and one by arguing by contradiction.

Proof: We look at $x \geq a$ first. If $h < 1/L$ we are done. (Take $\eta = h$.)

Otherwise we choose $\eta_1 < 1/L$. Then, from Theorem 1.4, there exists a unique solution, y_0 say, on $[a, a+\eta_1]$.

Now choose $\eta_2 = \min\{2\eta_1, h\}$, and look for a solution, y_1 say, on $[a+\eta_1, a+\eta_2]$, of the ODE with initial data $y_1(a+\eta_1) = y_0(a+\eta_1)$.

Now define

$$\begin{array}{l} y(x) = y_0(x), x \in [a, a+\eta_1] \\ y(x) = y_1(x), x \in [a+\eta_1, a+\eta_2] \end{array}$$

To construct y_1: As in Theorem 1.4, but we now work in the space $X_1 := \mathcal{C}([a+\eta_1, a+\eta_2]; [b-k, b+k])$, and take (for $a+\eta_1 \leq x \leq a+\eta_2$)

$$\begin{aligned} (T_1 y)(x) &= y_0(a+\eta_1) + \int_{a+\eta_1}^{x} f(s, y(s))ds \\ &= b + \int_{a}^{a+\eta_1} f(s, y_0(s))ds + \int_{a+\eta_1}^{x} f(s, y(s))ds. \end{aligned} \tag{1.26}$$

So $T_1 : X_1 \to X_1$ because from (1.26)

$$\begin{aligned} ||T_1 y - b||_{\sup} \leq M\eta_1 + M(x-(a+\eta_1) = M(x-a) \leq M\eta_2 \\ \leq Mh \leq k. \end{aligned}$$

Also T_1 is a contraction as the proof of claim 2 only requires that the length of the interval we work on, which for T_1 is $\eta_2 - \eta_1$, is less than $1/L$. Thus we obtain the existence of a unique solution on $[a, a+\eta_2]$. Repeating this argument, both in positive and negative direction, we continue to be able to extend the solution and after finitely many steps have reached the endpoint $a+h$ of the original interval, since we can carry out each step except the very last one (where we will be able to choose $\eta_j = h$ since we'll have $h - \eta_{j-1} < \frac{1}{L}$) with the same 'stepsize' $\eta_k - \eta_{k-1} = \eta_1$.

Alternative proof of Corollary 1.5 (via contradiction):

As we know that the solution $y(x)$ exists *at least* on $[a-\eta, a+\eta]$ we can consider the *maximal subset* I of $[a-h, a+h]$ on which the solution $y(x)$ of (1.1) exists. We want to show that I is indeed all of $[a-h, a+h]$. To see this we first note that I must be a *closed* interval, for if $y(x)$ solves (1.1) on some $(c, d) \subset [a-h, a+h]$ then we can extend y to the closed interval simply by setting $y(d) := \lim_{x \nearrow d} y(x)$ and $y(c) := \lim_{x \searrow c} y(x)$.

Note that this is possible as $Mh \leq k$ ensures that $|y(x) - b| \leq k$ and since we have a uniform bound on $|y'(x)| \leq M$ and hence uniform continuity of y.

So $I = [c, d]$ and to show the claim we need to exclude the possibility that $c > a - h$ or $d < a + h$. So suppose that $d < a + h$. Then we know that $\tilde{b} = y(d)$ satisfies $|b - \tilde{b}| \leq M(d - a) < k$. Choosing $\tilde{k} = k - M(d - a)$ and $\tilde{h} \in (0, a + h - d)$ small enough so that $M\tilde{h} \leq \tilde{k}$ we can apply Theorem 1.4 to get the existence of a solution $\tilde{y}(x)$ on an interval $[d - \tilde{\eta}, d + \tilde{\eta}]$ for some $\tilde{\eta} > 0$ of our ODE , now with initial value $\tilde{y}(d) = y(d)$. But then we can use $\tilde{y}$ to extend y to a larger interval which gives a contradiction.

Global Existence: If f is continuous for all $x \in [a - h, a + h]$, and all y and satisfies a global Lipschitz condition (i.e. condition **P(iii)** on $[a - h, a + h] \times \mathbb{R}$), then we instead work in the spaces $\mathcal{C}_h = \mathcal{C}([a - h, a + h]; \mathbb{R})$, respectively $\mathcal{C}_\eta = \mathcal{C}([a - \eta, a + \eta]; \mathbb{R})$. As before, claim 1 then no longer requires the condition $Mh \leq k$ and we obtain in a first step that a solution exists on $[a - \eta, a + \eta]$ for $\eta < \frac{1}{L}$. We can then carry out either or the two arguments above to see that this solution indeed exists on all of $[a - h, a + h]$.

Comparison of the two methods of the proof of Picard:

(1)The proof using the CMT is shorter and simpler than the direct proof because much of the work has been done in proving the CMT and once we have chosen a suitable space, we have only to check that the conditions apply. Also, the CMT automatically gives uniqueness of solutions, which has to be proved separately in the direct method. Furthermore CMT can be used in more general situations to prove existence of solutions.

2) By Theorem 1.4.3 of the Prelims Analysis II lecture notes, a sequence of continuous functions y_n converges in the sup norm if and only if it is uniformly convergent. Thus convergence in the sup norm is equivalent to uniform convergence. Furthermore in the CMT the fixed point is given by the limit in $\mathcal{C}_{h,k}$ of $y_n = Ty_{n-1}$, with y_0 any point in the space. So if we take $y_0(x) = b$ (for all x) the fixed point is given by the uniform limit of the successive approximations as in the direct proof.

(3) One feature of the proof using the CMT was that it produced a less delicate result, in that the range of x for which it applied is more restricted. (Though it was easy to extend the range using iteration or a contradiction arguement.) This sometimes happens when we use abstract results rather than direct computations, because the direct computations can be more delicate. We can see why this happens in this case if we investigate the direct proof. In the direct proof we are working pointwise and each time we apply the inductive step using (1.14) we integrate $(x - a)^n$, and thus end up dividing by $n!$, so we have a series which converges for all $x \in [a - h, a + h]$. But in the CMT we are working in $\mathcal{C}([a - h, a + h])$, so on each integration we take the supremum which, of course, does not depend on x, and thus we integrate a constant, so the $n!$ is absent.

Remark: There are many other fixed point theorems, and other abstract results, which can be used to prove existence of solutions of more general equations involving derivatives (including partial derivatives) and integrals. These powerful theorems generally require some general theory of Banach and Hilbert spaces (see the B4 courses) and a knowledge of suitable spaces (eg Sobolev spaces) in which to apply them (see the part C courses on functional analysic methods for PDEs and fixed point methods for non-linear PDEs). The above proof can in particular be adjusted to prove the existence of solutions of partial differential equations, such as heat equations with non-linearities.

1.7 Picard's Theorem for systems and higher order ODEs via the CMT

We now want to look at existence and uniqueness of solutions of systems of ODEs. As well as being of interest in itself, this will be useful in particular for proving the existence of solutions of equations with higher order derivatives. We consider a pair of first order ODEs, for the functions y_1 and y_2.

$$y_1'(x) = f_1(x, y_1(x), y_2(x)) \tag{1.27}$$

$$y_2'(x) = f_2(x, y_1(x), y_2(x)) \tag{1.28}$$

$$\text{with initial data} \quad y_1(a) = b_1, \quad y_2(a) = b_2. \tag{1.29}$$

We can introduce vector notation

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

So we can write equations (1.27)–(1.29) as

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x)), \tag{1.30}$$

$$\underline{y}(a) = \underline{b}, \tag{1.31}$$

Now we want to prove Picard's Theorem for such systems of ODEs. Our previous proof using the CMT will extend in a very natural way.

We need a 'distance' in $\mathbb{R}^2$. In the Metric Spaces course the various norms l^1, l^2 (the Euclidean distance) and l^∞ on $\mathbb{R}^n$ were defined. We could use any of these (or any other norm on R^n), but we will make the fairly arbitrary choice to use the l^1 norm, $||\underline{y}||_1 = |y_1| + |y_2|$. In place of the rectangle R we will use the subset $S = \{(x, \underline{y}) \in \mathbb{R}^3 : |x - a| \leq h, \ \underline{y} \in B_k(\underline{b})\}$, where $B_k(\underline{b})$ is the closed disc in $\mathbb{R}^2$ centred on $\underline{b}$, radius k with respect to the l^1 norm. That is $B_k(\underline{b}) = \{\underline{y} \in \mathbb{R}^2 : ||\underline{y} - \underline{b}||_1 \leq k\}$ (ie $B_k(\underline{b}) = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - b_1| + |y_2 - b_2| \leq k\}$.

We will suppose that

(**H(i)**) $f_1(x, y_1, y_2)$ and $f_2(x, y_1, y_2)$ are continuous on S, and are hence bounded (because f_1 and f_2 are continuous functions on the closed bounded set S), say $|f_1(x, y)| + |f_2(x, y)| \leq M$ on S.

(**H(ii)**) $f_1(x, y_1, y_2)$ and $f_2(x, y_1, y_2)$ are Lipschitz continuous with respect to (y_1, y_2) on S. That is, there exist L_1 and L_2 such that for $x \in [a - h, a + h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$\begin{aligned} |f_1(x, u_1, u_2) - f_1(x, v_1, v_2)| &\leq L_1(|u_1 - v_1| + |u_2 - v_2|) \text{ and} \\ |f_2(x, u_1, u_2) - f_2(x, v_1, v_2)| &\leq L_2(|u_1 - v_1| + |u_2 - v_2|). \end{aligned}$$

It is easy to see that these conditions are equivalent to the following:

(H(i))′ $\underline{f}(x, \underline{y})$ is continuous on S, and bounded by M, say (that is $||\underline{f}(x, \underline{y})||_1 \le M$). [$M$ must exist because $\underline{f}$ is a continuous function on the closed bounded set S.]

(H(ii))′ $\underline{f}(x, \underline{y})$ is Lipschitz continuous with respect to $\underline{y}$ on S. That is, there exists L such that for $x \in [a-h, a+h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$||\underline{f}(x, \underline{u}) - \underline{f}(x, \underline{v})||_1 \le L||u - v||_1.$$

Note that we can take $L = L_1 + L_2$.

We now get the following version of Picard's existence theorem

Theorem 1.6. (Picard's existence theorem for systems.) *Let $f_1, f_2 : S \to \mathbb{R}$ be functions for which the conditions (**H(i)**) and (**H(ii)**) (or (**H(i)**)′ and (**H(ii)**)′) hold true for the set $S = [a-h, a+h] \times B_k(\underline{b}) \subset \mathbb{R}^3$. Then there exists $0 < \eta \le h$, such that the initial value problem (1.31) has a unique solution for $x \in [a-\eta, a+\eta]$*

Our previous proof using the CMT will extend to this case if we work in the complete metric space $\mathcal{C}_\eta := \mathcal{C}([a-\eta, a+\eta]; B_k(\underline{b}))$, the space of continuous functions mapping from $[a-\eta, a+\eta]$ to $B_k(\underline{b})$ with norm (or distance) on $\mathcal{C}_\eta$ defined by

$$||\underline{y}||_{\sup} = \sup_{x\in[a-\eta,a+\eta]} ||\underline{y}(x)||_1 \left(:= \sup_{x\in[a-\eta,a+\eta]} (|y_1(x)| + |y_2(x)|). \right)$$

As before, we can write the initial value problem as an integral equation

$$\underline{y}(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

where by the integral we mean that we integrate componentwise. Provided $\underline{f}(s, \underline{y}(s))$ is continuous in s, $\underline{y}$ is a solution of the differential equation if and only if $\underline{y}$ is a solution of the integral equation.

If we define

$$(T\underline{y})(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

then we can write this as a fixed point problem

$$\underline{y} = T\underline{y}.$$

As before we can now work in the complete metric space $\mathcal{C}_\eta$, to show that, provided we choose $\eta < \min\{h, k/M, 1/L\}$, then $T : \mathcal{C}_\eta \to \mathcal{C}_\eta$ and is a contraction (see problem sheet).

Again we can extend the range of the solution to all $x \in [a-h, a+h]$, by iteration.

Corollary 1.7. *If $Mh \le k$ then there is a unique solution for all $x \in [a-h, a+h]$*

Again if the functions are globally Lipschitz with respect to (y_1, y_2), then the solution is global.

This all extends easily to systems of n equations.

Picard for Higher Order ODEs

With Picard extended to first-order systems, it is a small step to extend it to a single, higher order ODE. For simplicity, we consider just an IVP for *linear* second-order ODEs (which will be considered in more detail in DEs2):

$$y'' + p(x)y' + q(x)y = r(x)$$

with initial data

$$y(a) = b \quad y'(a) = c,$$

and p, q, r continuous for $|x - a| \le h$.

To reduce this to a first-order system, introduce $z = y'$ and write

$$y' = z := f_1(x, y, z)$$

$$z' = -pz - qy + r := f_2(x, y, z)$$

with data $y(a) = b, \quad z(a) = c$. This is precisely in the form to which the previous section applies, and it's easy to check that the global Lipschitz condition is satisfied, so we get:

Theorem 1.8. (Picard for second-order linear ODEs)

With the assumptions as above, the solution exists for $|x - a| \le h$, and is unique.

Clearly this method can be extended to the IVP for an n-th order linear ODE. In particular, this justifies our belief that an n-th order ODE needs n pieces of data to fix a unique solution.

1.8 Summary

So we have looked at existence and uniqueness of solutions of various initial value problems. We found that Lipschitz continuity gives existence and uniqueness and that uniqueness can fail without the Lipschitz continuity. Even with Lipschitz continuity, the existence is often local, though a global Lipschitz condition on an interval containing the initial point will give existence and uniqueness of solutions on that interval.

There were two different methods of proof of existence and uniqueness. First we did a direct proof using successive approximations. Then we saw that using the CMT simplifies the proof, because the hard work has already been done in the proof of the CMT and proving completeness of $\mathcal{C}$. This proof readily extends to treat systems of ODEs (though here again we could have used successive approximation). A disadvantage of using the CMT was that it gave the result only for a restricted range of x, though it was easy to extend using iteration. This can be a feature of proofs using abstract results, because sometimes some of the detail is lost in the abstraction.

We also derived Gronwall's inequality, and used it to show that solutions depend continuously on the initial data. It also gave another proof of uniqueness.

2 Plane autonomous systems of ODEs

The definition: a *plane autonomous system of ODEs* is a pair of ODEs of the form;

$$\frac{dx}{dt} = X(x, y) \qquad (2.1)$$
$$\frac{dy}{dt} = Y(x, y)$$

Here "autonomous" means there is no t-dependence in X or Y, and "plane" means there are just two equations, so we can draw pictures in the (x, y) - plane, which will then be called the *phase plane*.

Given initial values $x(0) = a,\ y(0) = b$, expect that there exists a unique solution and this solution which will define *a trajectory* or *phase path* in the phase plane. It is convenient, though not necessary, to think of t as time, and the trajectory as the curve in the plane (including orientation) that is traced out by a moving particle. We put an arrow on the trajectory giving the direction of increasing t. We will denote $\dot{x} = \frac{dx}{dt}$ etc. We will assume throughout that X and Y Lipschitz continuous in x and y (on every bounded subset of $\mathbb{R}^2$) as this will allow us to apply Picard's theorem to obtain important properties of solutions for these plane autonomous system and of the corresponding trajectories.

Important observations

- If $(x(t), y(t))$ is a solution of (2.1) then for any fixed number $t_0 \in \mathbb{R}$ also
$$\tilde{x}(t) := x(t + t_0), \tilde{y}(t) := y(t + t_0)$$
solve (1.5) and they trace out the same trajectories.
- Through every point (x_0, y_0) there exists a UNIQUE trajectory. In particular, different trajectories can NEVER intersect, though they might asymptote to the same point (x^*, y^*) as $t \to \infty$ or as $t \to -\infty$ (and any such point must be a critical point, see below)

The first point immediately follows when we insert $(\tilde{x}(t), \tilde{y}(t))$ into the equations as this gives
$$\dot{\tilde{x}}(t) = \dot{x}(t + t_0) = X(x(t + t_0),\ y(t + t_0)) = X(\tilde{x}(t),\ \tilde{y}(t))$$
and
$$\dot{\tilde{y}}(t) = Y(\tilde{x}(t),\ \tilde{y}(t)).$$

Note that this does not work if the system is not autonomous (i.e. if either X or Y also depend on t).

The second point is an important consequence of Picard's theorem (and holds true as we assume X, Y Lipschitz): First of all Picard guarantees the existence of a solution

$(x(t), y(t))$ with $x(0) = x_0$ and $y(0) = y_0$ and hence there is a trajectory through the point. If $(\tilde{x}(t), \tilde{y}(t))$ is any other solution that traces out a trajectory through (x_0, y_0) then there is a t_0 so that $(\tilde{x}(t_0), \tilde{y}(t_0)) = (x_0, y_0)$. Looking at $u(t) := \tilde{x}(t - t_0)$, $v(t) := y(t - t_0)$ we get a new solution of (2.1) which has the same initial values as the original $(x(t), y(t))$, namely $(u(0), v(0)) = (x_0, y_0) = (x(0), y(0))$. By the uniqueness part of Picard we thus know that these two solutions $(u(t), v(t))$ and $(x(t), y(t))$ must be the same, so $(\tilde{x}, \tilde{y})$ is nothing else than a time-shift of the original solution so must trace out the same trajectory.

2.1 Critical points and closed trajectories

A *critical point* is a point (x_0, y_0) in the phase plane where $X(x_0, y_0) = Y(x_0, y_0) = 0$. So a critical point is a particular (very special) trajectory corresponding to solutions $(x(t), y(t))$ of (2.1) that are constant in time.

There may be trajectories in the phase plane which are closed i.e. which return to the same point. Provided they don't just correspond to constant solutions and so are simply given by a single point, these correspond to **periodic solutions** of (2.1) as may be seen as follows:

Suppose the trajectory is closed so that for some finite value t_0 of t, $(x(t_0), y(t_0)) = (x(0), y(0))$, while $(x(t), y(t)) \neq (x(0), y(0))$ for $0 < t < t_0$. Define $\bar{x}(t) = x(t + t_0)$, $\bar{y}(t) = y(t + t_0)$. Then as before we see that $(\bar{x}(t), \bar{y}(t))$ is another solution of (2.1) with $\bar{x}(0) = x(t_0) = x(0)$; $\bar{y}(0) = y(t_0) = y(0)$. Now by uniqueness of solution (given Lipschitz again).

$$x(t + t_0) = \bar{x}(t) = x(t)$$

$$y(t + t_0) = \bar{y}(t) = y(t),$$

but this is now true for all t, so a **closed trajectory** corresponds to a **periodic solution** of (2.1) with period t_0. The converse is trivial. $\square$

Note in particular that this means that a trajectory cannot intersect itself, but might close up to a closed curve (without self-intersections).

2.1.1 An example

Consider the harmonic oscillator equation

$$\ddot{x} = -\omega^2 x.$$

Turn this into a plane autonomous system by introducing y as follows:

$$\left.\begin{array}{rll} & \dot{x} = y & = X(x, y) \\ so & \dot{y} = -\omega^2 x & = Y(x, y). \end{array}\right\} \tag{2.2}$$

(Clearly this trick often works for second-order ODEs arising from Newton's equations.) The only critical point is $(0,0)$, but note that

$$\frac{d}{dt}(\omega^2 x^2 + y^2) = 2\omega^2 x\dot{x} + 2y\dot{y} = 0$$

so $\omega^2 x^2 + y^2 =$ constant. (which, from Prelims dynamics, we know to be proportional to the total energy). For a given value of the constant this is the equation of an ellipse, so we can draw all the trajectories in the phase plane as a set of nested (concentric) ellipses:

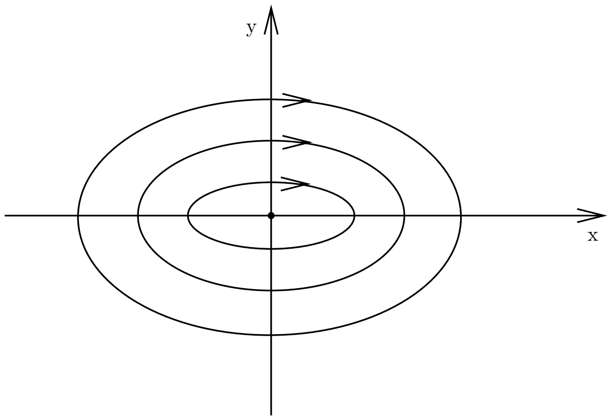


Figure 2.1: The phase diagram for the harmonic oscillator; to put the arrows on the trajectories, notice that $\dot{x} > 0$ if $y > 0$.

The picture in the phase plane is called the *phase diagram* (or *phase portrait*) and from that we see that all trajectories are closed, so all solutions are periodic (as we already know, from Prelims).

2.2 Stability and linearisation

We want to learn how to sketch the trajectories in the phase plane in general and to do this we first consider their *stability.* Intuitively we say a critical point (a,b) is *stable* if near (a,b) the trajectories have all their points close to (a,b) for all t greater than some t_0. We make the formal definition:

Definition A critical point (a,b) is *stable* if given $\epsilon > 0$ there exists $\delta > 0$ and t_0 such that for any solution $(x(t), y(t))$ of (2.1) for which $\sqrt{(x(t_0)-a)^2 + (y(t_0)-b)^2} < \delta$

$$\sqrt{(x(t)-a)^2 + (y(t)-b)^2} < \epsilon, \quad \forall t > t_0.$$

A critical point is *unstable* if it is not stable.

(Here we have used the Euclidean distance. We could use other norms such as l^1 or l^∞)

A common way to analyse the stability of a critical point is to linearise about the point and assume that the stability is the same as for the linearised equation. There are rigorous ways of showing when this is true. We will assume it is valid, pointing out the cases where it is likely fail. Linearising will also enable us to classify the critical points according to what the trajectories look like near the critical point .

So suppose $P = (a, b)$ is a critical point for (2.1), so

$$X(a, b) = 0 = Y(a, b). \tag{2.3}$$

Now $x = a$, $y = b$ is a solution of (2.1). We linearise by setting

$$x = a + \zeta(t);\ \ y = b + \eta(t)$$

where ζ and η are thought of as small. From (2.1), and Taylor's theorem

$$\dot{x} = \dot{\zeta} = X(a + \zeta, b + \eta) = X(a, b) + \zeta X_x|_p + \eta X_y|_p + \text{ h.o.}$$

$$\dot{y} = \dot{\eta} = Y(a, b) + \zeta Y_x|_p + \eta Y_y|_p + \text{ h.o.}$$

where 'h.o.' means quadratic and higher order terms in ζ and η. Now use (2.3) and neglect higher order terms to find

$$\left.\begin{aligned} \begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix}^{\cdot} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} X_x|_p & X_y|_p \\ Y_x|_p & Y_y|_p \end{pmatrix} \end{aligned}\right\} \tag{2.4}$$

Call this (constant) matrix M and set $\underline{Z}(t) = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$ then (2.4) becomes

$$\dot{\underline{Z}} = M\,\underline{Z}. \tag{2.5}$$

We can solve (2.5) with eigen-vectors and eigen-values as follows: $\underline{Z}_0 e^{\lambda t}$ is a solution, with constant vector $\underline{Z}_0$ and constant scalar λ if

$$\lambda \underline{Z}_0 = M\ \underline{Z}_0,$$

i.e. $\underline{Z}_0$ is an eigen-vector of M with eigen-value λ. We are considering just 2×2-matrices, with eigen-values say λ_1 and λ_2 so the general solution if $\lambda_1 \neq \lambda_2$ is

$$\underline{Z}(t) = c_1 \underline{Z}_1 e^{\lambda_1 t} + c_2 \underline{Z}_2 e^{\lambda_2 t}, \tag{2.6}$$

for constants c_i. Recall λ_1, λ_2 may be real, in which case the c_i and the $\underline{Z}_i$ are real, or a complex conjugate pair, in which case the c_i and the $\underline{Z}_i$ are too.

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ say, we need to take more care. The Cayley-Hamilton Theorem (see Algebra I) implies that $(M - \lambda I)^2 = 0$ since the characteristic polynomial is $c_M(x) = (x - \lambda)^2$, so either $M - \lambda I = 0$ or $M - \lambda I \neq 0$. We have a dichotomy:

(i) if $M - \lambda I = 0$ then $M = \lambda I$ and the solution is

$$\underline{Z}(t) = \underline{C}e^{\lambda t} \tag{2.7}$$

for any constant vector $\underline{C}$.

(ii) if $M - \lambda I \neq 0$ then there exists a constant vector $\underline{Z}_1$ with

$$\underline{Z}_0 := (M - \lambda I)\underline{Z}_1 \neq 0$$

but

$$(M - \lambda I)\underline{Z}_0 = (M - \lambda I)^2\underline{Z}_1 = 0.$$

(So $\underline{Z}_0$ is the one linearly independent eigenvector of M.) One now checks that the solution of (2.5) is

$$(c_1\underline{Z}_1 + (c_0 + c_1 t)\underline{Z}_0)e^{\lambda t}. \tag{2.8}$$

Now we can use (2.6) and (2.8) to classify critical points.

2.3 Classification of critical points

We shall assume that neither eigenvalue of the matrix M is zero, which is the requirement that the critical point be *non-degenerate*. A proper discussion of this point would take us outside the course but roughly speaking if a critical point is *degenerate* then we need to keep more terms in the Taylor expansion leading to (2.4), and the problem is much harder.

Case 1. $0 < \lambda_1 < \lambda_2$ (both real of course)

From (2.6), as $t \to -\infty$, $\underline{Z}(t) \to 0$, and $\underline{Z}(t) \sim c_1\underline{Z}_1 e^{\lambda_1 t}$ unless $c_1 = 0$ in which case $\underline{Z}(t) \sim c_2\underline{Z}_2 e^{\lambda_2 t}$, while as $t \to +\infty$, $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_2$, unless $c_2 = 0$ when $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_1$

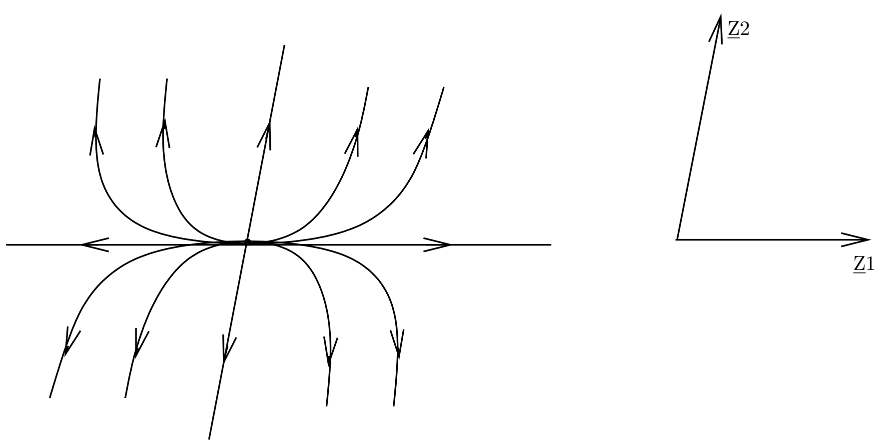


Figure 2.2: An unstable node.

These trajectories converge on the critical point into the past, but go off to infinity in the future. A critical point with these properties is called an **unstable node**.

Case 2: $\lambda_1 < \lambda_2 < 0$ (both real)

This is as above but with $t \to -t$ and the roles of $\underline{Z}_1$, $\underline{Z}_2$ switched. The trajectories converge on the critical point into the future and come in from infinity in the past.

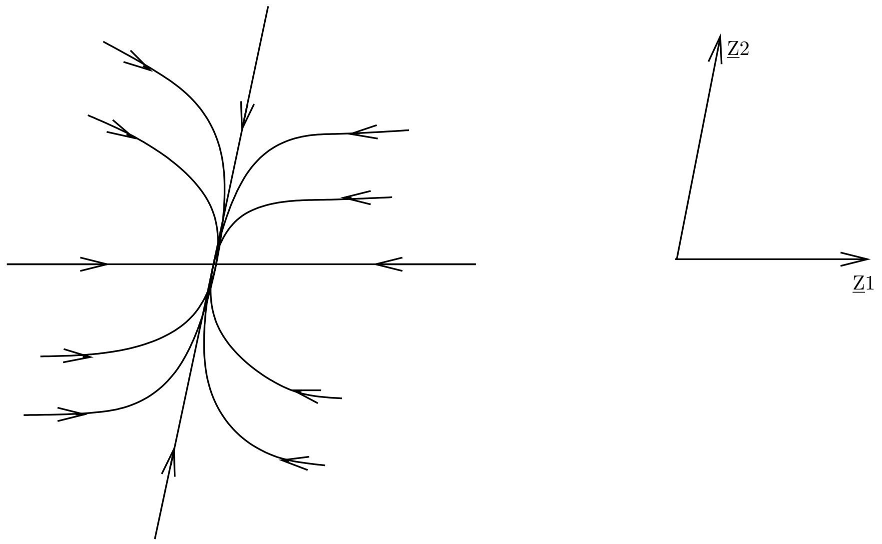


Figure 2.3: A stable node.

This is a **stable node**.

Case 3: $\lambda_1 = \lambda_2 = \lambda$. If the solution of the linearised equation is given by (2.7) (case (i)) we have a **star** , while if the solution is given by (2.8) (case (ii)) there is an **inflected node** . In both cases the critical point is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

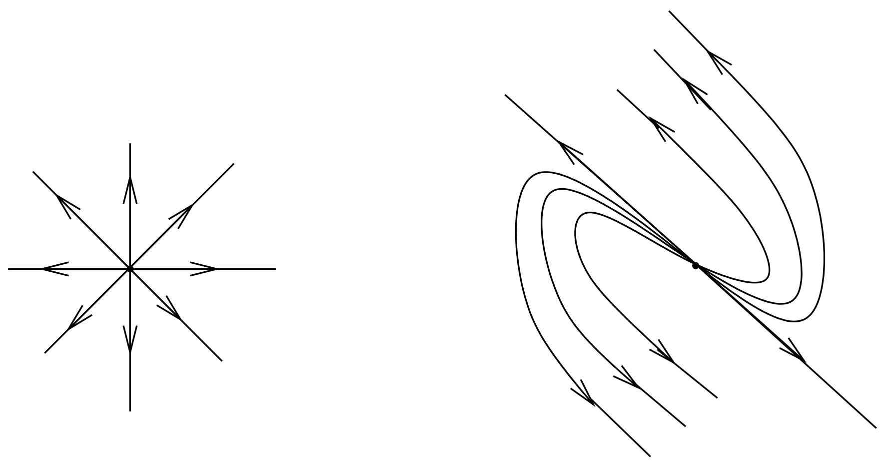


Figure 2.4: Unstable star case (i) and unstable inflected node case (ii)

Case 4: $\lambda_1 < 0 < \lambda_2$ (both real)

$$\begin{array}{llll} \text{If } c_1 = 0 \text{ then} & \underline{Z}(t) \to \infty & \text{along } \underline{Z}_2 & \text{as } t \to \infty \\ & \to 0 & \text{along } \underline{Z}_2 & \text{as } t \to -\infty. \\ \text{If } c_2 = 0 \text{ then} & \underline{Z}(t) \to 0 & \text{along } \underline{Z}_1 & \text{as } t \to \infty \\ & \to \infty & \text{along } \underline{Z}_1 & \text{as } t \to -\infty. \end{array}$$

If $c_1, c_2 \neq 0$ then $\underline{Z}(t) \to \infty$ along $\underline{Z}_2$ as $t \to \infty$ and along $\underline{Z}_1$ as $t \to -\infty$.

Most trajectories come in approximately parallel to $\pm\underline{Z}_1$ and go out becoming asymptotic to $\pm\underline{Z}_2$.

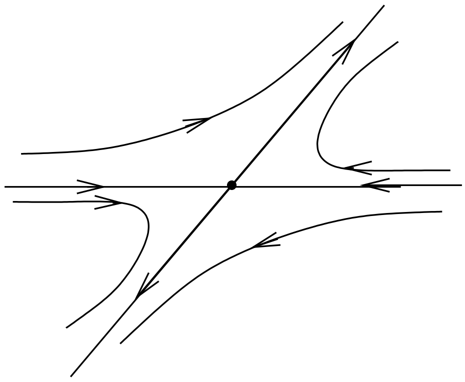


Figure 2.5: A saddle.

This is a **saddle** (to motivate the name, think of the trajectories as contour lines on a map; then two opposite directions from the critical point are uphill and the two orthogonal directions are downhill).

If the eigen-values are a complex conjugate pair we may write

$$\lambda_1 = \mu - i\nu, \quad \lambda_2 = \mu + i\nu \quad \mu, \nu \in \mathbb{R},$$

and the classification continues in terms of μ and ν.

In (2.6) the $c_i Z_i$ are a conjugate pair so if we put $c_1 = re^{i\theta}$, $\underline{Z}_1 = (1, ke^{i\phi})^T$, then

$$c_1\underline{Z}_1 = \begin{pmatrix} re^{i\theta} \\ rke^{i(\phi+\theta)} \end{pmatrix}$$

so that

$$\underline{Z}(t) = e^{\mu t}\begin{pmatrix} 2r\cos(\nu t - \theta) \\ 2rk\cos(\nu t - (\phi+\theta)) \end{pmatrix}.$$

Case 5: $\mu = 0$

Then $\underline{Z}(t)$ is periodic.

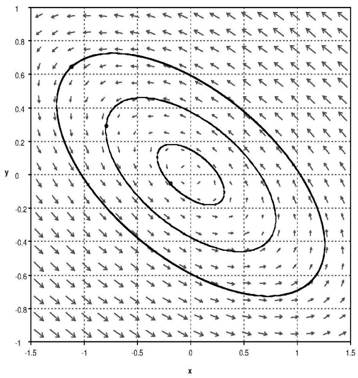


Figure 2.6: An anticlockwise centre ($B < 0$); $X = -x - 3y,\ Y = x + y$

This case is called a **centre**, and is stable. The sense of the trajectories, clockwise or anticlockwise, depends on the sign of B; $B > 0$ is clockwise (take $\zeta = 0$ and $\eta > 0$, then $\dot{\zeta} = B\eta > 0$).
To see that this centre is stable: Take $t_0 = 0$. Consider the path whose maxmum distance from the critical point, (a, b), is $\epsilon > 0$. Let $\delta > 0$ be the minimum distance of this path from (a, b). Then
$\sqrt{(x(0)-a)^2 + (y(0)-b)^2} \leq \delta$ implies $\sqrt{(x(t)-a)^2 + (y(t)-b)^2} \leq \epsilon$, for all $t \geq 0$.

Case 6: $\mu \neq 0$

This is just like case 5, but with the extra factor $e^{\mu t}$, which is monotonic in time. We have another dichotomy:

(i) $\mu > 0$ then $|\underline{Z}(t)| \to \infty$ as $t \to \infty$ so the trajectory spirals out, into the future. This is called an **unstable spiral**.

(ii) $\mu < 0$ this is the previous with time reversed so it spirals in, and is called a **stable spiral**.

In case 6, as in case 5, the sense of the spiral is dictated by the sign of B.

[*An alternative method of looking at case 5 and 6:*

Case 5: $\mu = 0$

so $\lambda_1 = -i\nu$ *and* $\lambda_i^2 = -\nu^2 < 0$*; but, as both the trace and determinant of a matrix are invariant under* $P^{-1}M\,P$ *transformations, in terms of the matrix* M *of (2.4),* trace $\mathrm{M} = \mathrm{A} + \mathrm{D} = \lambda_1 + \lambda_2 = \mathrm{i}\nu - \mathrm{i}\nu = 0$ *so* det $\mathrm{M} = \mathrm{AD} - \mathrm{BC} = -\mathrm{A}^2 - \mathrm{BC} = \lambda_1\lambda_2 = \nu^2 > 0$

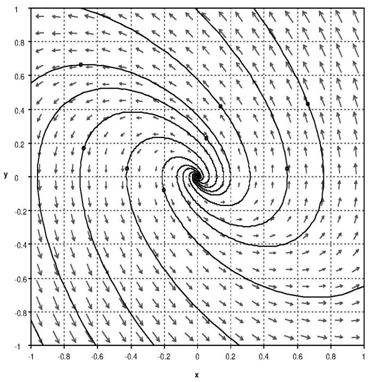


Figure 2.7: A antilockwise unstable spiral; X=-y, Y=x+y. Reverse the arrows for a stable spiral .

Equation (2.4) becomes

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{2.9}$$

As an exercise, show that now $-C\zeta^2 + 2A\zeta\eta + B\eta^2$ *is constant in time. We know that* B, C *have opposite signs with* $(-BC) > A^2$ *so this is the equation of an ellipse.*

This case is called a **centre***.*

Case 6: $\mu \neq 0$

So, in (2.6), we must have $\underline{Z}_1 = \bar{\underline{Z}}_2$ *and* $c_1 = \bar{c}_2$ *and*

$$\underline{Z}(t) = e^{\mu t}\left[c_1 \underline{Z}_1 e^{-i\nu t} + \bar{c}_1 \bar{\underline{Z}}_1 e^{i\nu t}\right],$$

which is just like case 5, but with the extra factor $e^{\mu t}$*, which is monotonic in time. So:*

(i) $\mu > 0$ *then* $|\underline{Z}(t)| \to \infty$ *as* $t \to \infty$ *so the trajectory spirals out, into the future. An* **unstable spiral***.*

(ii) $\mu < 0$ *this is the previous with time reversed so it spirals in, a* **stable spiral***.]*

Important observation:

Both the trace and determinant of a matrix are invariant under $P^{-1}M\,P$ transformations, so in terms of the matrix M of (2.4), trace M $=$ A $+$ D $=$ $\lambda_1 + \lambda_2$ and det M $=$ AD $-$ BC $= \lambda_1\lambda_2$

Thus: if $A+D>0$ then we have one of the cases 1, 4 or 6(i), all of which are **unstable** (but if $A+D<0$ the critical point can be stable or unstable). Further det $M=\lambda_1\lambda_2$. So when the eigenvalues are real the sign of det M tells us whether the signs of the eigenvalues are the same or different. The determinant is always positive in the case of complex eigenvalues.

Relationship to non-linear problem: One hopes that the linearistion will have the same type of critical point as the original system. In general if the linearisation has a node, saddle point or spiral, then so does the original system, but proving this is beyond the scope of this course. However, a centre in the linearisation does not imply a centre in the nonlinear system. This is not surprising when one reflects that a centre in the linear system arises when $\mathrm{Re}\,\lambda = 0$ so the perturbation involved when one returns to the nonlinear system, however small, can change this property.

Analysing the critical points and their local behaviour is important in determining the general behaviour of trajectories of an ODE system. Connecting the various critical points together requires care. It helps to remember that trajectories can never intersect and that while different trajectories can asymptote to the same point (x_0, y_0) this can only be the case if (x_0, y_0) is a critical point. Also that the signs of $X(x_0, y_0)$ and $Y(x_0, y_0)$ give the signs of $dx/dt(t)$ and $dy/dt(t)$ respectively of solutions of (2.1) at the time where they pass through this point.

We note that a trajectory can only become horizontal in a point (x_0, y_0) if $Y(x_0, x_0) = 0$ as this means that the corresponding solution of (2.1) has velocity $dy/dt = 0$ in the moment where it passes through that point.

Similarly, the only points (x_0, y_0) in the plane where trajectories can become vertical are points where $X(x_0, y_0) = 0$.

To draw a phase diagram it hence helps to draw the "nullclines", which are the curves in the plane on which $X(x, y) = 0$ respectively $Y(x, y) = 0$.

Such nullclines obviously cross at critical points. To find the nullclines sketch the curves $X(x, y) = 0$ and the curves $Y(x, y) = 0$. In particular in any region bounded by nullclines the trajectories must have a single sign for dx/dt and for dy/dt. Hence a simple examination of the expressions for X and Y in any region will determine if all the arrows in that region are "up and to the left", "up and to the right", "down and to the left" or "down and to the right".

2.3.1 An example

Find and classify the critical points for the system

$$\begin{aligned}\dot{x} &= x - y = X(x,y) \\ \dot{y} &= 1 - xy = Y(x,y)\end{aligned} \tag{2.10}$$

Solution: for the critical points, from $X = 0$ deduce $x = y$, therefore from $Y = 0$ deduce $x^2 = 1$, and we have either $(1,1)$ or $(-1,-1)$.

For the classification, calculate

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix},$$

and evaluate at the critical points:

$$\text{at } (1,1) : M = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} : \ \lambda^2 - 2 = 0 : \ \lambda = \pm\sqrt{2}$$

this is a **saddle**. The corresponding eigenvectors are:

$$\lambda_1 = -\sqrt{2} \quad \underline{Z}_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \text{ direction in}$$

$$\lambda_2 = \sqrt{2} \quad \underline{Z}_2 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \text{ direction out}$$

$$\text{at } (-1,1) : \ M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \lambda^2 - 2\lambda + 2 = 0 : \ \lambda = 1 \pm i.$$

this is an **unstable spiral**; $B < 0$, so its described anticlockwise.

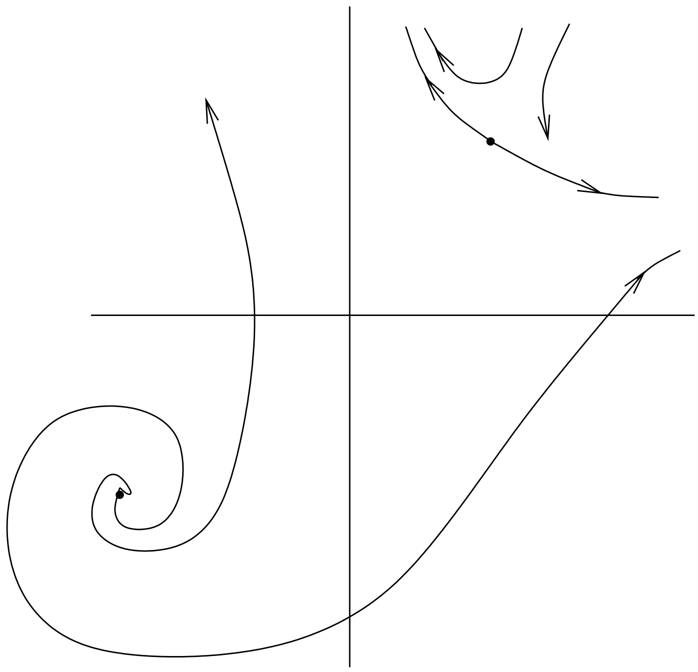


Figure 2.8: The phase diagram of (2.10)

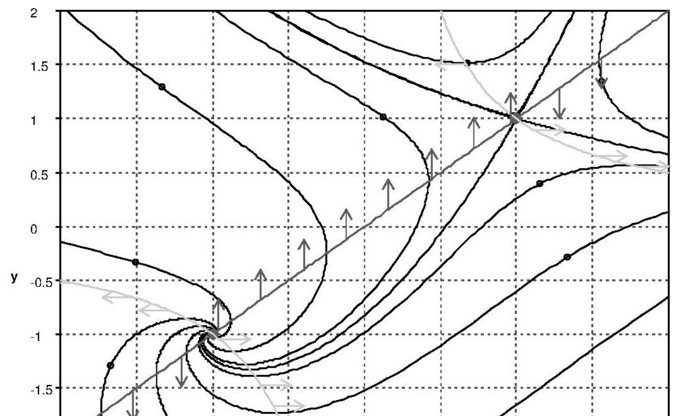


Figure 2.9: The phase plane diagram of (2.10) showing the nullclines $y = x$ and $xy = 1$.

2.3.2 Further example: the damped pendulum

Another example from mechanics: a simple plane pendulum with a damping force proportional to the angular velocity. We shall use the analysis of plane autonomous systems to understand the motion.

Take θ to be the angle with the downward vertical, then Newton's equation is

$$ml\ddot{\theta} = -mg\sin\theta - mkl\dot{\theta},$$

where m is the mass of the bob, l is the length of the string, g is the acceleration due to gravity and k is a (real, positive) constant determining the friction. We cast this as a plane autonomous system in the usual way: set $x = \theta$ and $y = \dot{x} = \dot{\theta}$ so

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l}\sin x - ky \end{aligned}$$

For simplicity below, we'll also assume that $k^2 < \frac{4g}{l}$, so that the damping isn't too large.

To sketch the phase diagram, we first find and classify the critical points. The critical points satisfy $y = 0 = \sin x$, so are located at $(x, y) = (N\pi, 0)$. Then

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x & -k \end{pmatrix}$$

The classification depends on whether N is even or odd:

$$\text{for } x = 2n\pi \quad M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -k \end{pmatrix}$$

which gives a stable spiral (clockwise);

$$\text{for } x = (2n+1)\pi \quad M = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -k \end{pmatrix}$$

which gives a saddle.
We now have enough information to sketch the phase diagram (note that $\dot{x}$ is positive or negative according as y is).

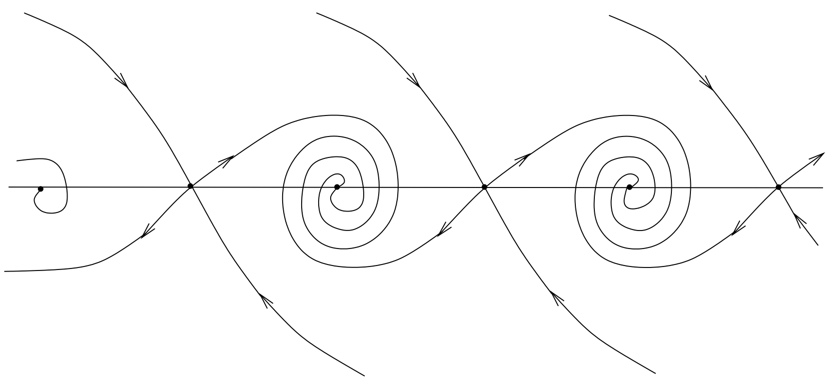


Figure 2.10: The phase diagram of the damped pendulum

2.3.3 An important example: The Lotka–Volterra predator-prey equations

This is a simplified mathematical model of a *predator-prey* system. Think of variables x standing for the population of prey, and y for the population of predators, both functions of t for time. As time passes, x increases as the prey breed, but decreases as the predators predate; likewise y increases by predation but decreases if too many predators compete. We assume that x and y are governed by the following plane autonomous system:

$$\begin{aligned} \dot{x} &= \alpha x - \gamma xy \\ \dot{y} &= -\beta y + \delta xy, \end{aligned} \tag{2.11}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are positive real constants. Because of the interpretation as populations, we only care about $x \geq 0,\ y \geq 0$ but we shall consider the whole plane for simplicity. Again, the aim is to use the analysis of plane autonomous systems to lead us to the phase diagram and an understanding of the dynamics.

For the critical points first, set

$$X := x(\alpha - \gamma y) = 0$$

$$Y := y(-\beta + \delta x) = 0.$$

There are two solutions, $(0,0)$ and $(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma})$. For the matrix:

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \delta y & -\beta + \delta x \end{pmatrix}$$

so first

$$\text{at } (0,0): \ M = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$$

which gives a saddle, where, it is easy to see, the out-direction is the x-axis and the in-direction is the y-axis. Next

$$\text{at } \left(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma}\right): \ M = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\gamma} & 0 \end{pmatrix} : \lambda^2 + \alpha\beta = 0$$

which gives a centre, described anticlockwise since $B < 0$.

We have found and classified the critical points. Before sketching the phase diagram, it is worth noting, from (2.11), that the axes are particular trajectories, and trajectories can only cross at critical points (as noted before).

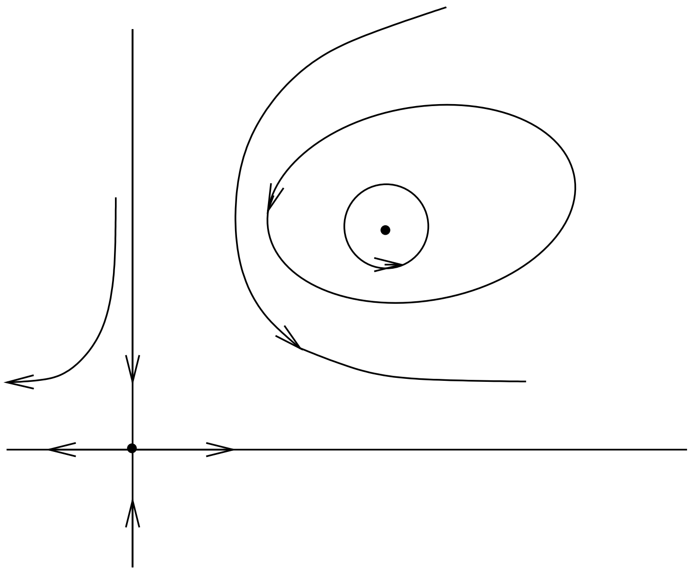


Figure 2.11: The phase diagram for the Lotka–Volterra system

Therefore any trajectory which is ever in the first quadrant is confined to the first quadrant, and no trajectory can enter the first quadrant from outside. Since there is a centre in the first quadrant, it looks as though all trajectories in the first quadrant may be **periodic**. This is true, and can be seen by the following argument: form the ratio

$$\frac{\dot{y}}{\dot{x}} = \frac{y(-\beta + \alpha x)}{x(\alpha - \gamma y)} = \frac{dy}{dx}.$$

and separate

$$\frac{(\alpha - \gamma y)}{y} dy - \frac{(-\beta + \delta x)}{x} dx = 0;$$

now integrate

$$\beta \log x - \delta x + \alpha \log y - \gamma y = C. \tag{2.12}$$

for a constant C. For different values of C, (2.12) is the equation of the trajectory or equivalently the trajectories are the level sets of the function $f(x, y) = \beta \log x - \delta x + \alpha \log y - \gamma y$.

One can see graphically that these level sets $\{(x, y) : f(x, y) = C\}$ cannot spiral to a closed curve or to the critical point $(\frac{\beta}{\delta}, \frac{\alpha}{\gamma})$ (the latter can be seen as $(\frac{\beta}{\delta}, \frac{\alpha}{\gamma})$ is the unique maximum of f so cannot be approached by another level set, to rigorously prove that level sets cannot spiral towards closed curves one could argue using the analyticity of f but this goes beyond the remit of this course).

This only leaves the possibilities that the level sets are either closed curves or that they are curves that either escape to infinity or approach one of the axes. The second possibility is excluded as f tends to minus infinity on the axes and at infinity while the function f is given by a fixed number C on the level set.

Having excluded all other possibilities we hence deduce that the level sets of f, and hence the trajectories, are all closed curves and hence that all solutions of (2.11) are periodic.

This useful technique can be applied to other examples.

2.3.4 Another example from population dynamics.

This is a simple model for two species in competition. Suppose that, when suitably scaled, the population on an island of rabbits ($x \geq 0$) and sheep ($y \geq 0$) satisfies the plane autonomous system:

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y). \tag{2.13}$$

(The populations are in competition for resources so each has a negative effect on the other)

If we analyse this system we find that the critical points are $(0,0),\ (3,0),\ (0,2),\ (1,1)$.

Then at $(0,0)$:

$$M = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

which has eigenvalues 3 and 2, with eigenvectors are (1,0), and (0,1) and is an unstable node.

At $(3,0)$:

$$M = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

which has eigenvalues -3 and -1, with eigenvectors (1,0), and (-3,1) and is a stable node.

At $(0,2)$:

$$M = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

which has eigenvalues -1 and -2, with eigenvectors (-1,2), and (0,1) and is a stable node.

At $(1,1)$:

$$M = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

which has eigenvalues $-1 - \sqrt{2}$ and $-1 + \sqrt{2}$, with eigenvectors $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$. and is a saddle point.

Again, as x and y represent populations we require that any trajectory which starts out in the first quadrant will remain there. As in the previous example this is indeed the case as the axes are particular trajectories.

Looking at the phase diagram we can see that, in the long term, depending on the initial data, either the rabbits or the sheep will survive.

Other values of the coefficients will give different outcomes - see problem sheet 2.

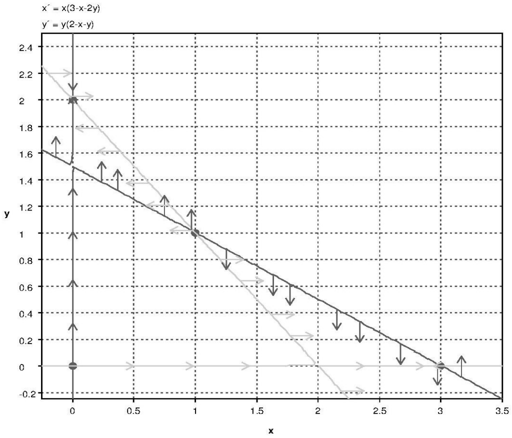


Figure 2.12: The nullclines for the equations (2.13).

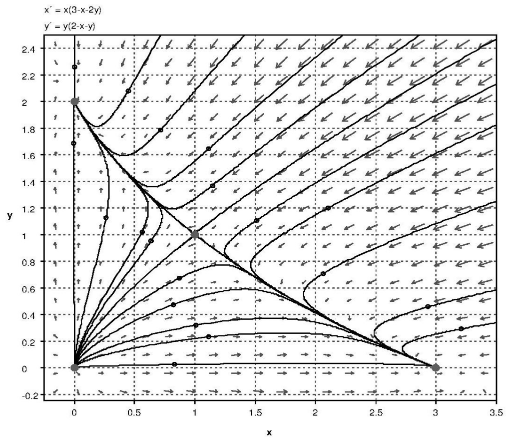


Figure 2.13: Phase diagram for the equations (2.13) for competitive species - no nullclines.

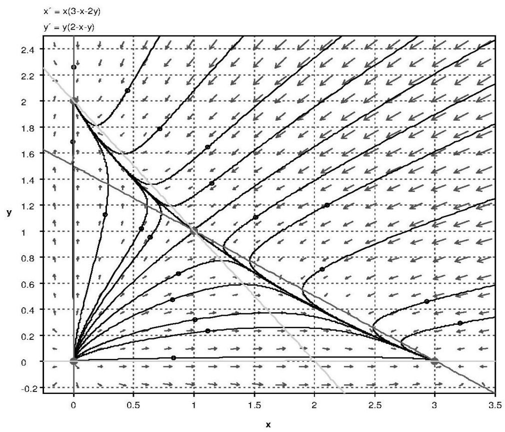


Figure 2.14: The phase diagram for the equations (2.13) for competitive species - with the nullclines Rabbits or sheep survive, depending on the initial data.

2.3.5 Another important example: limit cycles

Consider the plane autonomous system:

$$\dot{x} = (1 - (x^2 + y^2)^{\frac{1}{2}})x - y$$

$$\dot{y} = (1 - (x^2 + y^2)^{\frac{1}{2}})y + x.$$

Put $x^2 + y^2 = r^2$ then

$$X = x(1 - r) - y$$

$$Y = y(1 - r) + x$$

and one sees that only critical point is $(0, 0)$. One could go through the classification for this to find that it is an unstable spiral (exercise!).

Alternatively, in this case, we can analyse the full nonlinear system. We shall transform to polar coordinates. The simplest way to do this is as follows: first

$$r\dot{r} = x\dot{x} + y\dot{y} = x[x(1 - r) - y] + y[y(1 - r) + x]$$

$$= r^2(1 - r)$$

or

$$\dot{r} = r(1 - r).$$

Then, with

$$y = r \sin \theta,$$

we find

$$\dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta} = y(1 - r) + x,$$

which gives $\dot{\theta}$, so the system becomes

$$\left.\begin{aligned} \dot{\theta} &= 1 \\ \dot{r} &= r(1 - r). \end{aligned}\right\}$$

Unlike the system in its previous form, we can solve this. First

$$\theta = t + \text{ const},$$

and then

$$\int dt = \int \frac{dr}{r(1 - r)} = \int \left(\frac{1}{r} + \frac{1}{1 - r} \right) dr$$

so

$$\log \frac{r}{|1 - r|} = t + \text{ const}$$

i.e.

$$\frac{r}{1 - r} = Ae^t.$$

Solve for r and change the constant:

$$r = \frac{1}{1 + Be^{-t}} = \frac{1}{1 + (\frac{1}{r_0} - 1)e^{-t}}$$

where $r(0) = r_0$.

Note that as $t \to \infty$, $r \to 1$, while as $t \to -\infty$ either $r \to 0$ if $r_0 < 1$ or $r \to \infty$ at some finite t if $r_0 > 1$.

Now it is clear that the origin is an unstable spiral, and that the trajectories spiral out of it anticlockwise. We can also see that $r = 1$ is a closed trajectory and that all other trajectories (except the fixed point at the origin) tend to it; we call such a closed trajectory a **limit cycle**. It is **stable** because the other trajectories converge on it. (For an example of an unstable limit cycle we could consider the same system but with t changed to $-t$.)

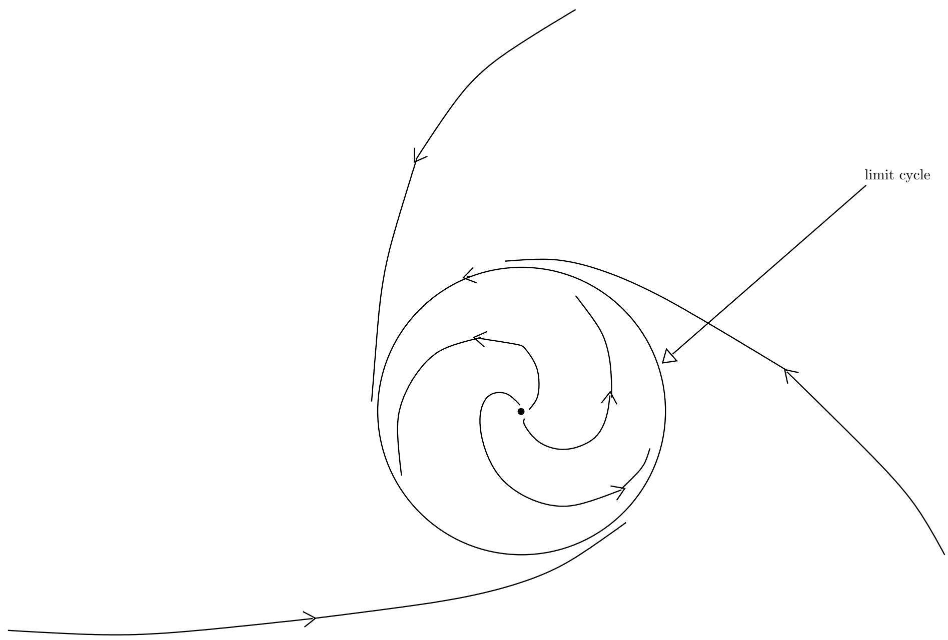


Figure 2.15: Phase diagram with a limit cycle

Another system with a limit cycle arises from the *Van der Pol equation*:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

where ϵ is a positive real constant. If $\epsilon = 0$ this is the harmonic oscillator again. If $\epsilon \neq 0$ then the usual trick produces a plane autonomous system:

$$\dot{x} = y$$

$$\dot{y} = -\epsilon(x^2 - 1)y - x.$$

The only critical point is $(0, 0)$ and it's an unstable spiral for $\epsilon > 0$ (exercise!).

Claim: Its beyond us to show this, but this system has a unique limit cycle, which is stable. There are some good illustrations for this in e.g. Boyce and di Prima (pp 496–500 of the 5th edition).

2.4 The Bendixson–Dulac Theorem

It's important to be able to detect periodic solutions, but it can be tricky. We end this section with a discussion of a test that can rule them out.

Theorem 2.1. (Bendixson–Dulac) *Consider the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, with $X, Y \in C^1$. If there exists a function $\varphi(x, y) \in C^1$ with*

$$\rho := \frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y) > 0$$

in a simply connected region R then there can be no nontrivial closed trajectories lying entirely in R.

Proof. (By nontrivial, I mean I want the trajectory must have an inside i.e. it isn't just a fixed point.) So suppose C is a closed trajectory lying entirely in R and let D be the disc (which also lies entirely in R, as R is simply connected) whose boundary is C. We apply Green's theorem in the plane. Consider the integral

$$\begin{aligned}\int\int_D \rho \, dxdy &= \int\int_D \left[\frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y)\right] \, dxdy \\ &= \oint_C -\varphi Y \, dx + \varphi X \, dy \\ &= \oint_C -\varphi \left(-\dot{y}dx + \dot{x}dy\right).\end{aligned}$$

But on C, $dx = \dot{x}dt$, $dy = \dot{y}dt$ so this is zero, which contradicts positivity of ρ, so there can be no such C.

2.4.1 Corollary.

If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

has fixed sign in a simply connected region R, then there are no nontrivial closed trajectories lying entirely in R.

This is just the previous but with φ const — in an example, always try this first!

2.4.2 Examples

(i) the damped pendulum (section 2.3.2)

$$\dot{x} = y$$
$$\dot{y} = -\frac{g}{l}\sin x - ky$$

has no periodic solutions.
Here

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -k \; < 0;$$

now use the corollary.

(ii)

$$\ddot{x} + f(x)\dot{x} + x = 0$$

has no periodic solutions in a simply connected region where f has a fixed sign.

By the usual trick we get the system

$$\dot{x} = y$$
$$\dot{y} = -yf(x) - x$$

then

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -f(x)$$

and we use the corollary.

(iii) The system

$$\dot{x} = y$$
$$\dot{y} = -x - y + x^2 + y^2$$

has no periodic solutions.

The corollary doesn't help so try the general case:

$$\rho := (\varphi X)_x + (\varphi Y)_y = \varphi(-1 + 2y) + X\varphi_x + Y\varphi_y.$$

Now guess: $\varphi_y = 0$ then

$$\rho = \varphi(-1+2y) + y\varphi_x = -\varphi + y(\varphi_x + 2\varphi)$$

so if we take $\varphi = -e^{-2x}$ the coefficient of y (which can take either sign) is zero and $\rho = 2e^{-2x} > 0$ and we are done.

PART II Partial Differential Equations.

3 First order semi-linear PDEs: method of characteristics

3.1 The problem

In this chapter, we consider *first-order* PDEs

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z(x,y)) \tag{3.1}$$

for an unknown function $z = z(x,y)$.

The PDE is said to be *semi-linear* as it is linear in the highest order partial derivatives, with the coefficients of the highest order partial derivatives depending only on x and y. If P and Q depend also on z the PDE is said to be *quasi-linear*. Everything that we discuss in this chapter is valid for semi-linear equations, though some parts can also be applied to quasilinear PDEs (see Remarks 3.1 and 3.2 below).

We will assume throughout this section that, in the region specified, $P(x,y)$ and $Q(x,y)$ are Lipschitz continuous in x and y and $R(x,y,z)$ is continuous and Lipschitz continuous in z. This will be enough to apply Picard's theorem to ensure that the characteristic equations have a unique solution through each point (see Proposition 3.1 below).

We want to find a unique solution to (3.1) given suitable data and determine its *domain of definition*. This is the region in the (x,y)-plane in which the solution is uniquely determined by the data. It turns out to depend on both the equation and the data.

The solution of (3.1) will be a function

$$z = f(x,y)$$

and to construct f it will be useful to consider the graph of this function, i.e. the surface $\Sigma := \{(x,y,z) : z = f(x,y)\}$. We shall refer to this as the *solution surface*.

The idea of the method of characteristics is to try to generate this solution surface, initially as a parametrised surface, and then from it obtain the desired solution $z = f(x,y)$ of (3.1).

Recall that if a surface Σ is given by the graph of a function f then the vectors $(1,0,\partial_x f)$ and $(0,1,\partial_y f)$ are tangent to Σ. We can thus generate a vector $\mathbf{n}$ that is normal to Σ via

$$\mathbf{n} = (1,0,\partial_x f) \wedge (0,1,\partial_y f) = (-\partial_x f, -\partial_y f, 1).$$

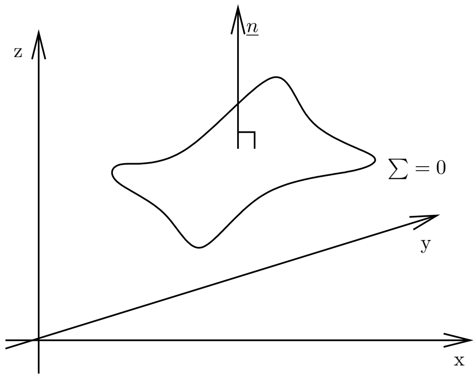


Figure 3.1: The solution surface

Hence we see that f is a solution of the PDE

$$P(x,y)\partial_x f(x,y) + Q(x,y)\partial_y f(x,y) = R(x,y,f(x,y)) \tag{3.2}$$

if and only if the the vector $\mathbf{t} = (P, Q, R)$ satisfies

$$\mathbf{t} \cdot \mathbf{n} = -P\partial_x f - Q\partial_y f + R = 0,$$

for $\mathbf{n}$ as above. I.e. f is a solution if $\mathbf{t}$ is perpendicular to the normal vector $\mathbf{n}$ of $\Sigma = \text{graph}(f)$ or equivalently if $\mathbf{t}$ is tangential to Σ.

We can hence reformulate our PDE for the function f as the geometric condition on the solution surface $\Sigma = \text{graph}(f)$ that $\mathbf{t}(x,y,z) = (P,Q,R)(x,y,z)$ is tangential to Σ at every point $(x,y,z) \in \Sigma$.

To solve the PDE (3.1) we hence want to construct such a surface Σ, initially as a parametrised surface, and then determine the function f by writing this surface as a graph, i.e. by solving for $z = z(x,y)$ (if possible).

We usually consider the PDE (3.1) together with data, which asks that the unknown function $z(x,y)$ is given by a prescribed function $g(x,y)$ on a curve $\gamma_0 = (\gamma_1, \gamma_2)$ in the xy-plane. This is equivalent to asking that the solution surface Σ contains the *initial curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), g(\gamma_1(s), \gamma_2(s)))$.

To construct our solution surface, we will start with the given initial curve, then determine curves, so called characteristics, that start on the initial curve and that move in direction of the vector (P, Q, R). The solution surface Σ will then be generated by all of these curves.

3.2 The big idea: characteristics

We look for a curve Γ whose tangent is $\mathbf{t}$. If $\Gamma = (x(t), y(t), z(t))$ in terms of a parameter t this means

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= Q(x, y)\\ \dot{z} &= R(x, y, z)\end{aligned} \tag{3.3}$$

These are the *characteristic equations* and the curve Γ is a *characteristic curve* or just *a characteristic.* Given a characteristic $(x(t), y(t), z(t))$, we call the curve $(x(t), y(t))$, which lies below it in the (x, y)-plane, the *characteristic projection* or *characteristic trace.*

The next result shows that characteristics exist, and gives the crucial property of them:

Proposition 3.1. *Suppose that $P(x, y)$ and $Q(x, y)$ are Lipschitz continuous in x and y and $R(x, y, z)$ is continuous and satisfies a Lipschitz condition in z. Then*

(a) Through every point $(x_0, y_0) \in \mathbb{R}^2$ there is a unique characteristic projection.

(b) Through every $(x_0, y_0, z_0) \in \mathbb{R}^3$ there is a unique characteristic.

(c) If f is a solution of the PDE (3.1) *and Γ is a characteristic through a point p that is contained in the solution surface $\Sigma = graph(f)$ then the whole characteristic is contained in Σ.*

We note in particular that characteristic projections can never intersect (this is very much a feature of semilinear equations and does not hold true for quasilinear PDEs, compare remark 3.1 below.).

Proof. (a) Since P and Q do not depend on z the first two equations $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ of (4.15) are a plane autonomous system (with Lipschitz functions $X = P, Y = Q$). From the previous chapter we know that Picard guarantees a unique trajectory through every point and as these trajectories are simply the characteristics projections we obtain (a).

(b) Part (a) already provides a unique solution $(x(t), y(t))$, t in some interval I_1, to the first two equations of (4.15) with $x(0) = y(0) = (x_0, y_0)$.

Given this $(x(t), y(t))$ we can set $F(t, z) := R(x(t), y(t), z)$ and view the third equation as an ODE $\dot{z}(t) = F(t, z(t))$ for z. Since R satisfies a Lipschitz condition with respect to z we obtain from Picard's theorem that there is a unique solution $z(t)$ with $z(0) = z_0$. Hence we find a unique characteristic through (x_0, y_0, z_0).

(c) Let f be a solution of the PDE and let $\Sigma = \text{graph}(f) = \{(x, y, f(x, y))\}$ be the corresponding solution surfaces. Suppose that $(x(t), y(t), z(t))$ is a characteristic through a point $p \in \Sigma$. Shifting time we can assume that $p = (x, y, z)(0)$ and hence

$z(0) = f(x(0), y(0))$. We now want to prove that the whole curve stays in Σ, i.e. that

$$w(t) := z(t) - f(x(t), y(t))$$

remains equal to zero.

To do this, we want to show that $|w(t)|$ satisfies the conditions of Gronwall's inequality (1.27) with $b = 0$. Namely we want to show that $|w(t)| \leq L|\int_0^t |w(s)|ds|$, L the constant from the Lipschitz condition $|R(x,y,z) - R(x,y,\tilde{z})| \leq L|z - \tilde{z}|$.

To see this we differentiate $w(t) = z(t) - f(x(t), y(t))$ and use first the characteristic equations and then that f solves the PDE (3.2) to get

$$\begin{aligned}\dot{w} &= \dot{z} - (\partial_x f(x,y)\dot{x} + \partial_y f(x,y)\dot{y}) = R(x,y,z) - (\partial_x f(x,y)P(x,y)\partial_y f(x,y)Q(x,y)) \\ &= R(x,y,z) - R(x,y,f(x,y)).\end{aligned} \tag{3.4}$$

As $w(0) = 0$ we hence get

$$|w(t)| = |\int_0^t \dot{w}(s)ds| \leq |\int_0^t |R(x,y,z) - R(x,y,f(x,y))|ds| \leq L|\int_0^t |w(s)|ds| \tag{3.5}$$

and we can apply Gronvall to see that $|w(t)| \leq 0 \cdot e^{L|t|}$, i.e. that $w(t) = z(t) - f(x(t), y(t))$ remains zero for all t.

□

Remark 3.1. *If we consider instead quasilinear PDEs*

$$P(x,y,z(x,y))\partial_x z(x,y) + Q(x,y,z(x,y))\partial_y z(x,y) = R(x,y,z(x,y)) \tag{3.6}$$

then statement (a) is false. If two characteristics $\Gamma(t) = (x,y,z)(t)$ and $\tilde{\Gamma}(t) = (\tilde{x},\tilde{y},\tilde{z})(t)$ pass through points (x_0,y_0,z_0) and $(x_0,y_0,\tilde{z}_0)$ with the same x and y coordinates, but with $z_0 \neq \tilde{z}_0$ then we cannot expect that the projections of these characteristics agree, as the ODEs satisfied by (x,y) and $(\tilde{x},\tilde{y})$ also contain a dependence on the functions $z(t)$ and $\tilde{z}(t)$. For quasilinear PDEs it is hence possible that characteristic projections intersect, while the above Proposition excludes this possibility for semilinear PDEs.

Conversely, statements (b) and (c) of the Proposition remain valid also for quasilinear PDEs (provided P, Q, R are Lipschitz wrt all variables x, y, z) and the proofs can be easily adapted to this setting.

It is hence possible to solve also quasilinear PDEs with the method of characteristics, but one needs to be more careful, in particular when determining the domain of definition.

3.2.1 Examples of characteristics

We need to gain proficiency in calculating characteristics and calculate the characteristics for the followig PDEs:

Example (a):

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z. \tag{3.7}$$

From (3.3) we write down the characteristic equations and solve them giving

$$\begin{aligned} \dot{x} &= P = x; \qquad & x &= Ae^t \\ \dot{y} &= Q = y; \qquad & y &= Be^t \\ \dot{z} &= R = z; \qquad & z &= Ce^t \end{aligned}$$

with $A,\ B,\ C$ constants (trivial to solve). The characteristics are hence half-lines from the origin (not including the origin) in $\mathbb{R}^3$ and the characteristic projections are half-lines from the origin (not including the origin) in the xy plane.

Example (b):

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z. \tag{3.8}$$

The characteristic equations and their solutions are

$$\begin{aligned} \dot{x} &= y; \qquad & x(t) &= Bt + \frac{t^2}{2} + A \\ \dot{y} &= 1; \qquad & y(t) &= B + t \\ \dot{z} &= z; \qquad & z(t) &= Ce^t \end{aligned} \tag{3.9}$$

with $A,\ B,\ C$ constants. To solve this system, pass over the first, solve the second, then come back to the first and third. (I am adopting a convention to introduce the constants A, B, C in the first, second and third of the characteristic equations respectively.)

In general solving the characteristic equations needs experience and luck; there isn't a general algorithm.

3.3 The Cauchy problem

A Cauchy Problem for a PDE is the combination of the PDE together with boundary data that, in principle, will give a unique solution, at least locally. We will look for suitable data and determine the domain on which the solution is uniquely determined.

Suppose we are prescribing the solution z of (3.1) along a curve $\gamma_0 = (\gamma_1, \gamma_2)$ (called the *data curve*) in the (x, y)-plane, i.e. ask that $z(x, y) = g(x, y)$ for a given function g for all points on the data curve. Setting $\gamma_3 = g(\gamma_1, \gamma_2)$ this produces a curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in space (called the *initial curve*) which needs to be in our solution surface Σ.

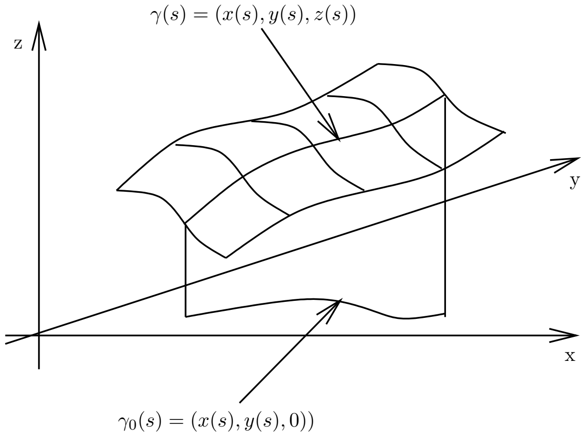


Figure 3.2: Geometry of the Cauchy problem.

To determine Σ we first parametrise the curve γ over some interval I, so consider $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ for $s \in I$. Here we will always assume that the components of γ are continuously differentiable (though will discuss the possibility that $gamma_3'$ is discontinuous later in section 3.7). Then, to solve (3.1), we construct the solution surface Σ by taking the characteristics through the points of $\gamma(s)$ (because Proposition 3.1(b) tells us that the solution surface is generated by these characteristics) . Thus the method of solution, the *method of characteristics*, is

(i) Parametrise γ over some interval I.

(ii) Determine the solutions $(x(t,s), y(t,s), z(t,s))$ of the characteristic equations

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \\ \dot{z} &= R(x,y,z) \end{aligned} \tag{3.10}$$

with initial data $x(0,s) = \gamma_1(s);\; y(0,s) = \gamma_2(s);\; z(0,s) = \gamma_3(s),\; s \in I$.

(iii) This yields the solution surface Σ in parametric form, i.e. $\Sigma = \{(x(t,s), y(t,s), z(t,s))\}$ where s ranges over the interval I over which we parametrised the initial curve γ, while for each s we consider the maximal set of t's for which we can solve all three characteristic equations.

(iv) Having a parametric form of Σ we then want to eliminate the parameters s, t and write Σ as a graph to read off the solution. This is a question we will explore below, and there is a restriction on the data for the method to work, also to be found later.

3.4 Examples

(a) Solve

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z,$$

with $z(x,0) = x$ for $1 \le x \le 2$.

We introduce a parameter s for the data, say $\gamma(s) = (s, 0, s)$, for $1 \le s \le 2$, and then solve the characteristic equations (done in section 3.2.1) with this as data at $t = 0$

$$x = Bt + \frac{t^2}{2} + A; \quad x(0,s) = A = s$$
$$y = B + t; \quad y(0,s) = B = 0$$
$$z = Ce^t; \quad z(0,s) = C = s$$

So, $C = s$, $B = 0$, $A = s$ and the parametric form of the solution is

$$\left.\begin{array}{l} x = s + \frac{1}{2}t^2 \\ y = t \\ z = se^t \end{array}\right\} \tag{3.11}$$

for $1 \le s \le 2$ and $t \in \mathbb{R}$.

(b) (From Ockendon et al) Solve

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x+y)z \tag{3.12}$$

with $z = 1$ on the segment of the circle $(x-2)^2 + y^2 = 2$, $y \ge 0$.

So we can take $\gamma(s) = (2 - \sqrt{2}\cos s, \sqrt{2}\sin s, 1)$, $s \in [0, \pi]$, and solve the characteristic equations:

$$\frac{\partial x}{\partial t} = P = x; \quad x = Ae^t; \quad A = (2 - \sqrt{2}\cos s)$$
$$\frac{\partial y}{\partial t} = Q = y; \quad y = Be^t; \quad B = \sqrt{2}\sin s$$
$$\frac{\partial z}{\partial t} = R = (x+y)z = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t)z.$$

We can integrate the final equation to get

$$\log|z| = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t) + C; \quad C = -(2 - \sqrt{2}\cos s + \sqrt{2}\sin s).$$

So the parametric form of the solution is

$$\left.\begin{array}{l} x = (2 - \sqrt{2}\cos s)e^t \\ y = (\sqrt{2}\sin s)e^t \\ z = \exp[((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)(e^t - 1)] \end{array}\right\} \tag{3.13}$$

3.5 Domain of definition

Where is the solution determined uniquely by the data? This is the domain of definition and is the region in the (x, y)-plane where the solution surface is uniquely determined and is given explicitly as $z = f(x, y)$.

The solution surface is swept out by the characteristics through the initial curve, so the solution will be defined in the region swept out by the projections of the characteristics through the initial curve provided these characteristic projections only intersect the data curve once (otherwise the problem can be overdetermined, compare section 3.6 below).

In particular if the initial curve is bounded, and if we don't have to deal with the problem of characteristic projections intersecting the data curve multiple times then the domain of definition will simply be bounded by the projections of the characteristics through the end points of the initial curve.

Example (a) from section 3.4 Here the solution surface is swept out by the characteristics through γ, so has edges given by the characteristics through the ends of γ, which are at $s = 1$ and $s = 2$.

The characteristics are $(x, y, z)(s, t) = (s + \frac{1}{2}t^2, t, se^t)$ for $s \in [1, 2]$ and $t \in \mathbb{R}$ and we can solve for z to get

$$z(x, y) = (x - \frac{1}{2}y^2)e^y.$$

The characteristic projections are given by $x = s + \frac{1}{2}y^2$ and none of them intersect the data curve more than once.

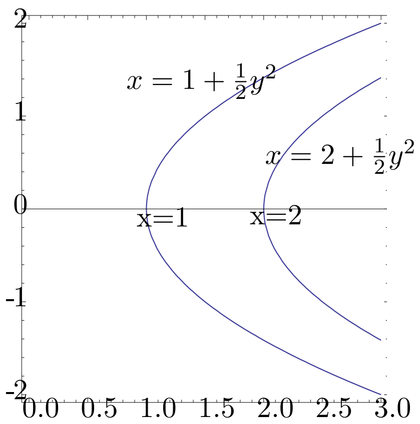


Figure 3.3: The domain of definition for this problem

Hence the domain of definition Ω is the region between the characteristic projections

$x = 1 + \frac{1}{2}y^2$ at $s = 1$ and $x = 2 + \frac{1}{2}y^2$ at $s = 2$, i.e.

$$\Omega = \{(x, y) : 1 + \frac{1}{2}y^2 \leq x \leq 2 + \frac{1}{2}y^2\} \tag{3.14}$$

Blow up:

The method of characteristics reduces the PDE (3.1) to a system of ODEs. As we have already seen nonlinear ODEs can give rise to solutions which blow up, so the same must be true of non linear PDEs, even if those that are semi-linear.

If we have characteristics $t \mapsto (x, y, z)(s, t)$ for which the z component tends to $\pm\infty$ as t approaches a finite time $T_{max}(s)$ (or $T_{min}(s)$), while the (x, y) components of the characteristics remain regular beyond $T_{max}(s)$, then the corresponding solution $z = f(x, y)$ must become singular as (x, y) approach $(x(T_{max}), y(T_{max})$.

In situations like this the domain of definition Ω is still generated by the characteristic projections, but we need to be aware that we are only allowed to consider $(x, y)(s, t)$ for t so that $z(s, t)$ is well defined, i.e. only for $t < T_{max}(s)$. One part of the boundary of the domain of definition Ω is then given by the curve $\{(x, y)(s, T_{max}(s))\}$, $s \in I$, at which the solution f blows-up.

As a simple example which illustrates this behaviour you can consider the equation

$$x\partial_x z + y\partial_y z = -z^2$$

with prescribed data of $z(x, y) = \alpha \in \mathbb{R}$ for on $\{(x, y) : x + y = 1, \quad x \in [0, 1]\}$.

3.6 Cauchy data:

Once we are given the surface Σ that is spanned by the characteristics we then want to solve for z as a function of x and y. To do so we need to be able to eliminate t and s in favour of x and y, at least in principle. For this, recall from Prelims the definition of the Jacobian:

$$J(s, t) = \frac{\partial(x, y)}{\partial(t, s)} = \det \begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}. \tag{3.15}$$

Now if

$$x = x(t, s), \text{ and } y = y(t, s)$$

are continuously differentiable functions of t and s in a neighbourhood of a point, then a sufficient condition to be able to find unique continuously differentiable functions

$$t = t(x, y) \text{ and } s = s(x, y)$$

in some neighbourhood of the point, is that J be non-zero at the point. We can then substitute into $z = z(t, s)$ to get

$$z = z(t(x, y), s(x, y)) = f(x, y),$$

a continuously differentiable function of x and y as required. This comes from a result known as the Inverse Function Theorem that you can see in AOS Multidimensional Analysis and Geometry which says that if the matrix $\begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}$ is invertible at a point (s,t), i.e. if the Jacobian is not zero, then we can invert the function $(x,y)(s,t)$ at least near this point, and the inverse function $(s,t)(x,y)$ are again differentiable. Here we will have to take it on trust, but it is the two dimensional equivalent of the one dimensional result you saw in Analysis in Prelims - where you saw that a function $g:\mathbb{R}\to\mathbb{R}$ has a differentiable inverse near $x=a$ if $g'(a)\neq 0$.

If we require that $J(s,0)\neq 0$ on the initial curve γ_0 then we hence get that the problem has a unique solution at least close to the initial curve. As

$$J(s,0)=\det\begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}=\det\begin{pmatrix} P & Q \\ x_s & y_s \end{pmatrix}=Py_s-Qx_s \tag{3.16}$$

for P,Q evaluated at points $x(s,0)=y(s,0)$ on the data curve $\gamma_0=(\gamma_1(s),\gamma_2(s))$ we hence say that the data is *Cauchy data* if

$$P(x,y)y_s-Q(x,y)x_s\neq 0 \text{ on the data curve} \tag{3.17}$$

i.e. if

$$P(\gamma_1(s),\gamma_2(s))\gamma_2'(s)-Q(\gamma_1(s),\gamma_2(s)))\gamma_1'(s)\neq 0$$

for all s in the interval over which we parameterise the data curve γ_0.

Geometrically the condition that the data is Cauchy corresponds to asking that the tangent vector $\gamma_0'(s)=(x_s(s,0),y_s(s,0))$ along the data curve and the tangent vector $(P,Q)=(\dot{x},\dot{y})=(x_t(s,0),y_t(s,0))$ along the characteristic projection through the same point are never parallel. I.e. the data is Cauchy if there are no characteristic projections that meet the data curve tangentially.

As characteristic projections of semilinear PDEs cannot intersect we can use this to detect whether there are any characteristic projections which intersect the data curve more than once and if so, at what points we need to split the data curve to obtain well posed problems.

Remark 3.2. *For the more general quasilinear PDEs, we can still obtain a solution in a small neighbourhood of the data curve if the data is Cauchy using this method. However, for quasilinear PDEs characteristic projections can intersect, so to determined the domain of definition we need to determine how to restrict the range of (s,t) so that characteristic projections don't intersect. To detect whether this can happen we can again consider whether $J(s,t)\neq 0$, now not only for $t=0$ but more more generally.*

Example (b) from section 3.4

Here

$$J=\det\begin{pmatrix} (2-\sqrt{2}\cos s)e^t & (\sqrt{2}\sin s)e^t \\ (\sqrt{2}\sin s)e^t & (\sqrt{2}\cos s)e^t \end{pmatrix}=2e^{2t}(1-\sqrt{2}\cos s),$$

so $J(s,0) = (1-\sqrt{2}\cos s)$ vanishes when $s = \frac{\pi}{4}$. Note that this corresponds to the characteristic projection $y = x$, which touches the data curve at $(1,1)$.

If we were hence to consider the PDE with data prescribed on the full data curve $\gamma(s)$, $s \in [0,\pi]$ then there will be problems as each characteristic projection meets the data curve in two points, so the initial data is likely to be inconsistent.

So we will restrict the data curve to either $s \in [0,\pi/4)$ or to $s \in (\pi/4,\pi]$. This means that we end up with data that is Cauchy and so with a data curve that is so that characteristic projections only intersect once. We hence get a well defined (rather than an overdetermined) problem and the domain of definition will again be traced out by the characteristic projections that intersect (the corresponding part of) the data curve.

In the first case the data curve starts at $s = 0$, so the solution surface will have an edge given by the characteristic through the end of γ at $s = 0$ and the corresponding characteristic projection is $y = 0$, which forms an edge of the domain of definition. The domain of definition is then swept out by the projections of the characteristics through $\gamma(s)$ for $s \in [0,\pi/4)$ so the other boundary curve is the characteristic projection $x = y$ at $s = \pi/4$ and the domain of definition is $0 \leq y < x$. Similarly, we can also determine the domain of definition if we use the other part of the data curve, and in this situation we obtain the same set (this is not always the case and here comes from the original data curve being so the every characteristic projection that intersect the first part $s \in [0,\pi/4)$ of the data curve also intersects the second part and visa versa).

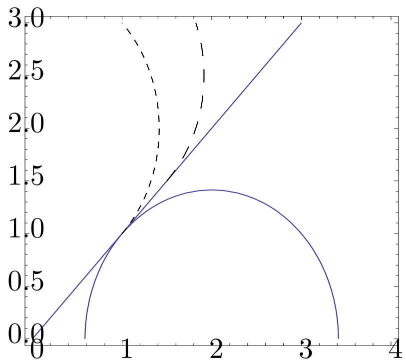


Figure 3.4: Different data curves

Note that if we instead had a data curve which turned to the left of $y = x$ after following the circle upto $s = \pi/4$ such as one of the the dashed curves in the diagram then we would still detect that $J(s,0) = 0$ at $s = \pi/4$ but we would not have the problem that characteristic projections intersect the data curve in multiple points, hence it is likely there would only be problems on the characteristic projection $y = x$.

Remark: The extreme case of the data failing to be Cauchy is if the data curve is a characteristic projection, i.e. if we can parametrise the initial curve as $\gamma(s) = (\gamma_1, \gamma_2, \gamma_3)(s)$ so that $\gamma_{1,2}$ satisfy the characteristic equations $\gamma_1' = P$ and $\gamma_2' = Q$.

Then, if the initial curve is a characteristic, i.e. if also the 3rd charachteristic equation $\gamma_3' = R$ is satisfied, then there will be an infinity of solutions through γ, while otherwise there will be no solution.

For, if γ is a characteristic, then let C be any curve through γ whose projection is nowhere tangent to a characteristic projection. Then there is a solution surface through C. But this was any C so there is an infinity of solutions. On the other hand, if the data curve is a characteristic projection but the initial curve isn't a characteristic then we can have no solution. Indeed if there was a function f that solves the PDE for which γ is in the solution surface $\Sigma = \text{graph}(f)$ then we'd need to have $\gamma_3 = f(\gamma_1, \gamma_2)$ so by chain rule

$$\gamma_3' = f_x\gamma_1' + f_y\gamma_2' = Pf_x + Qf_y = R$$

where the second equality holds as we assumed that (γ_1, γ_2) is a characteristic projection. But this would mean that γ is indeed a characteristic, contradiction.

3.7 Discontinuities in the first derivatives

The characteristic projections have another property. They are the only curves across which the solution surface can have discontinuities in the first derivatives.

For, suppose that $z(x, y)$ is a solution of the PDE (3.1) which is continuously differentiable away from a curve $\alpha_0 = (\alpha_1, \alpha_2)$ in the xy-plane, continuous across α_0 but for which there are discontinuities in the first order partial derivatives as we cross α_0.

We use the superscript $\pm$ to denote the solution on either side of γ and denote the jumps in the partial derivative by $[z_x]_-^+ = z_x^+ - z_x^-$ and $[z_y]_-^+ = z_y^+ - z_y^-$.

As z is continuous across α_0 we know that $z^+(\alpha_1(s), \alpha_2(s)) - z^-(\alpha_1(s), \alpha_2(s)) = 0$. Differentiating this gives gives that

$$[z_x]_-^+\alpha_1' + [z_y]_-^+\alpha_2' = 0 \tag{3.18}$$

At the same time, both z^+ and z^- are solutions of the PDE so

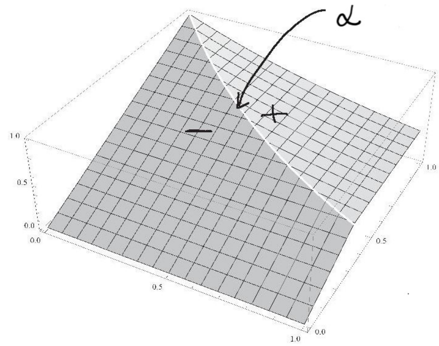
$$Pz_x^+ + Qz_y^+ = R(x, y, z^+) \text{ and } Pz_x^+ + Qz_y^+ = R(x, y, z^-). \tag{3.19}$$

As $z^+ = z^-$ on α the right hand sides agree, so subtracting these equations gives

$$0 = P[z_x]_-^+ + Q[z_y]_-^+ \text{ on } \alpha_0. \tag{3.20}$$

The vector $j = ([z_x]_,^+[z_y]_-^+)$ of the jumps in first derivatives hence solves the homogeneous system of linear equations $\begin{pmatrix} P & Q \\ \alpha_1' & \alpha_2' \end{pmatrix} \cdot j = 0$. So for there to be a non-zero jump, this matrix must be singular, i.e. the rows must be linearly dependent.

As the first row gives the tangent to a characteristic projection, while the second row is the tangent to the curve α_0 across which the derivatives jump, we must have that α_0 is a characteristic projection. So the only curves in the xy-plane across which the first order derivatives of a solution can jump are characteristic projections. In the picture of the solution surface we see this jump in derivative along the curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), z(\alpha_1(s), \alpha_2(s)))$, see figure below.



3.8 General Solution

Another problem we could have considered, is what is the most general solution of (3.1)? Just as we expect the most general solution of an ODE to have n arbitrary constants, so we expect the most general solution of a PDE of order n to have n arbitrary functions.

For example: The first order PDE $\frac{\partial z}{\partial x}(x, y) = 0$, has the most general solution $z = f(y)$ where f is an arbitrary function.

4 Second order semi-linear PDEs

4.1 Classification

In this section, we are interested in second-order PDEs of the following form:

$$\underbrace{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}}_{\text{principal part}} = f(x,y,u,u_x,u_y). \tag{4.1}$$

This PDE is said to be *linear* if f is linear in u, u_x, u_y, otherwise it is said to *semi-linear*. (If the coefficients a, b, c also depend on u, u_x, u_y it is said to be *quasi-linear*. We will consider only semi-linear equations.) You have seen the following examples in Prelims:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{Laplace's equation} \\ u_{xx} - u_{yy} &= 0 \quad \text{wave equation if } y = ct \\ u_{xx} - u_y &= 0 \quad \text{heat equation if } y = t/\kappa. \end{aligned}$$

Equations that are linear (in the dependent variable) have solutions that can be combined by linear superposition (taking linear combinations). In general PDEs that are nonlinear (for example where f, above, depends nonlinearly on u or its derivatives) do not have solutions that are superposable.

We will assume throughout that functions are suitably differentiable.

4.1.1 The idea:

In this section, the key idea is to change coordinates so as to simplify the principal part. So we make the change of variables

$$(x,y) \to (\varphi(x,y), \psi(x,y));$$

with non vanishing Jacobian (basically this ensures that the map is locally invertible):

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0.$$

We will abuse the notation a little and write (the solution) u as either a function of (x,y) or (φ,ψ) as required.

For the change in the partials, we calculate

$$u_x = u_\varphi\varphi_x + u_\psi\psi_x; \quad u_y = u_\varphi\varphi_y + u_\psi\psi_y$$

then

$$\left.\begin{aligned} u_{xx} &= u_{\varphi\varphi}\varphi_x^2 + 2u_{\varphi\psi}\varphi_x\psi_x + u_{\psi\psi}\psi_x^2 + u_\varphi\varphi_{xx} + u_\psi\psi_{xx} \\ u_{xy} &= u_{\varphi\varphi}\varphi_x\varphi_y + u_{\varphi\psi}(\varphi_x\psi_y + \psi_x\varphi_y) + u_{\psi\psi}\psi_x\psi_y + u_\varphi\varphi_{xy} + u_\psi\psi_{xy} \\ u_{yy} &= u_{\varphi\varphi}\varphi_y^2 + 2u_{\varphi\psi}\varphi_y\psi_y + u_{\psi\psi}\psi_y^2 + u_\varphi\varphi_{yy} + u_\psi\psi_{yy} \end{aligned}\right\} \tag{4.2}$$

so that (4.1) becomes

$$A(\varphi,\psi)u_{\varphi\varphi} + 2B(\varphi,\psi)u_{\varphi\psi} + C(\varphi,\psi)u_{\psi\psi} = F(\varphi,\psi,u,u_\varphi u_\psi) \tag{4.3}$$

with

$$\left.\begin{aligned} A &= a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 \\ B &= a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y \\ C &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \end{aligned}\right\} \tag{4.4}$$

(Beware, F will include lower order derivatives from (4.2).) In a matrix notation (4.4) is (check!)

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{pmatrix}$$

so that, taking determinants,

$$(AC - B^2) = (ac - b^2)(\varphi_x\psi_y - \psi_x\varphi_y)^2 = (ac - b^2)\left(\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right)^2. \tag{4.5}$$

(We could obtain (4.5) directly from (4.4) but the matrix notation makes the computation simpler.) Now (4.5) leads to a classification of second-order linear PDEs:

4.1.2 The Classification

Second-order linear PDEs are classified into three types as follows:

1. $ac < b^2$ hyperbolic: e.g. wave equation;
2. $ac > b^2$ elliptic: e.g. Laplace equation;
3. $ac = b^2$ parabolic: e.g. heat equation.

So, by (4.5) the class of the equation is invariant under transformations with non-vanishing Jacobian.

We shall look at the classification in terms of the quadratic polynomial

$$a(x,y)\lambda^2 - 2b(x,y)\lambda + c(x,y) = 0. \tag{4.6}$$

Note: We will assume that $a \neq 0$, in the domain under consideration. If $a = 0$ but $c \neq 0$, we can swap the roles of x and y.

Case 1: hyperbolic type

So $ac < b^2$ and the quadratic has distinct real roots λ_1, λ_2, say. So

$$a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\frac{dy}{dx} + c(x,y) = 0. \tag{4.7}$$

is equivalent to

$$\frac{dy}{dx} = \lambda_1(x,y), \qquad \frac{dy}{dx} = \lambda_2(x,y). \tag{4.8}$$

Suppose these equations have solutions $\varphi(x,y)$=constant, $\psi(x,y)$ =constant, respectively. Set as change of variables

$$\varphi = \varphi(x,y), \quad \psi = \psi(x,y).$$

Then, on $\varphi(x,y)$=constant,

$$\varphi_x + \varphi_y \frac{dy}{dx} = 0$$

so that

$$\lambda_1 \varphi_y = -\varphi_x$$

and thus

$$A(\varphi,\psi) = a(x,y)(\varphi_x)^2 + 2b(x,y)\varphi_x\varphi_y + c(x,y)(\varphi_y)^2 = 0,$$

and analogously $C(\varphi,\psi) = 0$. But $\lambda_1 \neq \lambda_2$, so $\frac{\varphi_x}{\varphi_y} \neq \frac{\psi_x}{\psi_y}$, and from (4.5) $B \neq 0$. Divide (4.3) by B to obtain the equation in the form

$$u_{\varphi\psi} = G(\varphi,\psi,u,u_\varphi,u_\psi). \tag{4.9}$$

This is the *normal form* (or *canonical form*) for a hyperbolic equation; the equation (4.7) is the *characteristic equation*; φ, ψ are *characteristic variables*; curves on which φ or ψ are constant are *characteristic curves.* We can often solve (4.9) explicitly.

Examples:

(a)

$$u_{xx} - u_{yy} = 0.$$

We already know how to solve this, but let us apply the method. So

$$a = 1,\ b = 0,\ c = -1, \text{and } \lambda^2 - 1 = 0.$$

We can take

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

and solve (4.8)

$$y'(x) = 1 \quad y'(x) = -1$$

to get

$$\varphi = x - y \quad \psi = x + y.$$

(There is clearly lots of choice at this stage.) The equation has become

$$u_{\varphi\psi} = 0,$$

which we solve at once by

$$u = f(\varphi) + g(\psi),$$

a solution known from Prelims. So the characteristic curves of the wave equation are $x + ct$=const and $x - ct$ =const.

(b) An example with data: solve

$$xu_{xx} - (x+y)u_{xy} + yu_{yy} + \frac{(x+y)}{(y-x)}(u_x - u_y) = 0, \text{ for } y \neq x$$

with

$$u = \frac{1}{2}(x-1)^2, \quad u_y = 0 \quad \text{on} \quad y = 1.$$

Problem is hyperbolic provided $x \neq y$ (check). The quadratic (4.6) is

$$x\lambda^2 + (x+y)\lambda + y = 0$$

$$= (\lambda + 1)(x\lambda + y);$$

so choose

$$\lambda_1 = -1 \quad \lambda_2 = -\frac{y}{x}$$

and solve

$$y'(x) = -1; \quad y'(x) = -\frac{y}{x},$$

by $x + y =$ const; xy =const, so put

$$\varphi = x + y; \quad \psi = xy.$$

Calculate

$$u_x = u_\varphi + yu_\psi$$

$$u_y = u_\varphi + xu_\psi$$

so that

$$u_{xx} = u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}$$

$$u_{xy} = u_{\varphi\varphi} + xu_{\varphi\psi} + yu_{\varphi\psi} + xyu_{\psi\psi} + u_\psi$$

$$u_{yy} = u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}.$$

(It is always better to calculate the derivatives directly, rather than trying to remember formulae.)

Now the PDE becomes

$$0 = x[u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}]$$
$$-(x+y)[u_{\varphi\varphi} + (x+y)u_{\varphi\psi}xyu_{\psi\psi} + u_{\psi}]$$
$$+y[u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}]$$
$$+(x+y)u_{\psi}$$
$$= (4xy - (x+y)^2)u_{\varphi\psi}$$

so

$$u_{\varphi\psi} = 0$$

and the solution is

$$u = f(\varphi) + g(\psi) = f(x+y) + g(xy).$$

To impose the data, calculate

$$u_y = f'(x+y) + xg'(xy)$$

so on $y = 1$,

$$u = f(x+1) + g(x) = \frac{1}{2}(x-1)^2$$
$$u_y = f'(x+1) + xg'(x) = 0.$$

Differentiate the first:

$$f'(x+1) + g'(x) = x - 1$$

and solve simultaneously with the second:

$$g'(x) = -1,$$

and integrate to find

$$g(x) = -x + c.$$

Substitute back in $u(x,1)$:

$$f(x+1) = \frac{1}{2}(x-1)^2 + x - c = \frac{1}{2}(x+1)^2 - x - c,$$

so

$$f(x) = \frac{1}{2}x^2 - x + 1 - c.$$

Finally

$$u = f(x+y) + g(xy) = \frac{1}{2}(x+y)^2 - (x+y) + 1 - xy.$$

Case 2: elliptic type

Now $ac > b^2$ so (4.6) has a complex conjugate pair of roots, and the integral curves of

$$y'(x) = \lambda(x, y); \quad y'(x) = \bar{\lambda}(x, y)$$

are in complex conjugate pairs, $\varphi(x, y)$ =const; $\psi(x, y) = \bar{\varphi}(x, y)$ =const. Then $A = C = 0$, $B \neq 0$ and the equation becomes

$$u_{\varphi\bar{\varphi}} = G(\varphi, \bar{\varphi}, u, u_\varphi, u_{\bar{\varphi}}).$$

Introduce new variables, ζ, η, given by $\varphi = \zeta + i\eta$, $\bar{\varphi} = \zeta - i\eta$, to obtain the normal form for an elliptic equation (check):

$$u_{\zeta\zeta} + u_{\eta\eta} = H(\zeta, \eta, u, u_\zeta, u_\eta), \tag{4.10}$$

which closely resembles Laplace's equation.

Example: Classify and reduce to normal form the PDE

$$y u_{xx} + u_{yy} = 0, \text{ for } y > 0.$$

$ac - b^2 = y$ so the equation is elliptic when $y > 0$.

The characteristic equation is

$$y(y')^2 + 1 = 0$$

that is

$$y^{1/2} y' = \pm i,$$

so integrating

$$2y^{3/2} \mp 3ix = \text{const.}$$

So take as variables $\zeta = 2y^{3/2}$; $\eta = 3x$. Making the substitution, we find that

$$3\zeta(u_{\zeta\zeta} + u_{\eta\eta}) + u_\eta = 0,$$

but $\zeta \neq 0$ so the normal form is

$$u_{\zeta\zeta} + u_{\eta\eta} = -\frac{u_\zeta}{3\zeta}.$$

Case 3: parabolic type

Now $ac = b^2$ so (4.6) has a repeated root $\lambda(x, y)$. Solve $y'(x) = \lambda(x, y)$ for one new coordinate φ, and pick any ψ with

$$\varphi_x \psi_y - \varphi_y \psi_x \neq 0 \tag{4.11}$$

as the other, then $A = 0$ so $B^2 = AC = 0$. But $C \neq 0$, as ψ =const is not a characteristic curve by (4.11), so we get the normal form for a parabolic equation:

$$u_{\psi\psi} = G(u, \varphi, \psi, u_\varphi, u_\psi),$$

which closely resembles the heat equation.

Example:

Classify and reduce to normal form the equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0 \text{ for } x > 0. \tag{4.12}$$

The relevant quadratic is

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 = (x\lambda - y)^2$$

which has equal roots, so this equation is *parabolic*; $\lambda = \frac{y}{x}$ so solve

$$\frac{dy}{dx} = \frac{y}{x}, \text{ to get, for example, } \varphi = \frac{y}{x}$$

and take, for example, $\psi = x$. Calculate

$$u_x = -\frac{y}{x^2}u_\varphi + u_\psi$$

$$u_y = \frac{1}{x}u_\varphi$$

so that

$$u_{xx} = \frac{y^2}{x^4}u_{\varphi\varphi} - 2\frac{y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + 2\frac{y}{x^3}u_\varphi$$

$$u_{xy} = -\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi$$

$$u_{yy} = \frac{1}{x^2}u_{\varphi\varphi}.$$

The equation becomes

$$x^2\left[\frac{y^2}{x^4}u_{\varphi\varphi} + \frac{2y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + \frac{2y}{x^3}u_\varphi\right] + \\ + 2xy\left[-\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi\right] + y^2\left[\frac{1}{x^2}u_{\varphi\varphi}\right] = x^2 u_{\psi\psi} = 0 \tag{4.13}$$

so the normal form is

$$u_{\psi\psi} = 0$$

with general solution $u = F(\varphi) + \psi G(\varphi)$. In terms of the original variables this is:

$$u = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right). \tag{4.14}$$

NB Very often, a question like this will be phrased in the form 'Classify and reduce to normal form the equation (4.12) and show that the general solution can be written as (4.14)'. Therefore candidates for φ and ψ are proposed by the question itself.

A warning example:

The type can change e.g. classify the equation

$$u_{xx} + yu_{yy} = 0.$$

Then

$$\lambda^2 + y = 0, \quad \lambda^2 = -y,$$

and this is:

- elliptic in $y > 0$,
- parabolic at $y = 0$,
- hyperbolic in $y < 0$.

4.2 Characteristics:

The characteristics of second order semi-linear PDEs have analogous properties to the characteristic *projections* of first order semi-linear PDEs. (Note the difference in terminology.)

Firstly, if there are discontinuities in the second derivatives of a solution across a given curve, then that curve must be a characteristic curve. To see this, suppose that the curve Γ, given parametrically by $(x(s), y(s))$, is a curve across which there are discontinuities in the second derivatives of the solution. Let u^+_{xx} etc denote values on one side of Γ and u^-_{xx} denote values on the other side of Γ. Then differentiating $u_x(x(s), y(s))$ and $u_y(x(s), y(s))$ along Γ, and noting that u, u_x, u_y are continuous across the curve,

$$\begin{aligned}
\frac{du_x}{ds} &= \frac{dx}{ds}u^{\pm}_{xx} + \frac{dy}{ds}u^{\pm}_{xy} \\
\frac{du_y}{ds} &= \frac{dx}{ds}u^{\pm}_{yx} + \frac{dy}{ds}u^{\pm}_{yy} \\
\text{and also } f(x, y, u, u_x, u_y) &= a(x, y)u^{\pm}_{xx} + 2b(x, y)u^{\pm}_{xy} + c(x, y)u^{\pm}_{yy}.
\end{aligned}$$

Subtracting the 'minus' equation from the 'plus' equation

$$\begin{aligned}
0 &= \frac{dx}{ds}[u_{xx}]^+_- + \frac{dy}{ds}[u_{xy}]^+_- \\
0 &= \frac{dx}{ds}[u_{yx}]^+_- + \frac{dy}{ds}[u_{yy}]^+_- \\
0 &= a(x, y)[u_{xx}]^+_- + 2b(x, y)[u_{xy}]^+_- + c(x, y)[u_{yy}]^+_-,
\end{aligned}$$

where $[u_{xx}]_-^+ = u_{xx}^+ - u_{xx}^-$ denotes the jump in u_{xx} across Γ, etc. If there are to be discontinuities in the second derivatives, then this set of equations in the jumps must have a nonzero solution, so that the determinant of the coefficients must be zero. Thus

$$a(x,y)\left(\frac{dy}{ds}\right)^2 - 2b(x,y)\frac{dx}{ds}\frac{dy}{ds} + c(x,y)\left(\frac{dx}{ds}\right)^2 = 0, \tag{4.15}$$

so Γ is a characteristic.

Furthermore, under suitable smoothness conditions, the Cauchy problem for a second order semi-linear PDE, where u and $\frac{\partial u}{\partial n}$ are given along a curve Γ, will have a unique local solution provided Γ is nowhere tangent to a characteristic curve. This result is beyond the scope of this course and will be investigated further in the Part B course, Applied PDEs. It can be seen that it is necessary that Γ is not a characteristic curve, as if u exists then it must have unique second order partial derivatives along Γ and exactly as above this can only be true when Γ is not a characteristic curve.

Remark: Our previous work carried the implicit assumption that $a \neq 0$. Note that (4.15) gives a method of calculating the characteristic curves if $a = 0$. In particular if $a = 0$ and $c = 0$ then the characteristic curves are $x =$const, $y =$const.

4.3 Type and data: well posed problems

We want to say something about the notion of *well posedness* and its connection with type. Our examples are mostly based on knowledge acquired in Prelims.

A problem, consisting of a PDE with data, is said to be *well posed* if the solution:

- exists
- is unique
- depends *continuously* on the data.

Recall that, in Section 1.5, we said that a solution of a DE is *continuously dependent on the data* if the error in the solution is small provided the error in the initial data is small enough.. We then gave a precise definition for ODEs. We now want to extend this definition to PDEs. Data can be given in different ways, so to be precise we will consider a problem where $u(x,y)$ is the solution of a certain PDE in a bounded subset of the plane D, with u given on some curve Γ. Then we will say that the solution depends continuously on the data if :

$\forall \epsilon > 0\ \exists \delta > 0$ such that if u_i, $i = 1, 2$, are solutions with $u_i = f_i$ on Γ then

$$\sup_{\Gamma} |f_1 - f_2| < \delta \quad \Rightarrow \quad \sup_{D} |u_1 - u_2| < \epsilon.$$

The definition extends in a fairly obvious way to other types of data. (Note that there are plenty of other 'distances ' we could use in place of taking the sup, but that is what we will use here.)

Of the three requirements for well posedness it is existence which is the hardest to obtain. In Prelims solutions were found for a number of linear problems, either by making a change of variables and then integrating, or by using separation of variables and Fourier series. Anything more than this is beyond the scope of this course. Uniqueness of solution was also proved for a number of linear problems (even when you didn't know if the solution existed). Proving uniqueness and continuous dependence on the data is much easier for linear problems as we can then start out by looking at the difference between two solutions, which will then be the solution of some suitable problem. Later we will look at the linear equations Poisson's equation and the heat equation and state and prove the maximum principle, which will enable us to prove uniqueness and continuous dependence for suitable boundary data.

But first we need to consider what data might be appropriate. In Prelims you considered three particular PDEs, each with a different type of data which arose from the particular physical problem they modelled. These are summarised in the table below. It turns out that there are mathematical as well as physical reasons why each problem had a different type of data. So first we will look at some of these problems and consider which may be well posed.

Some Models from Prelims:

	PDE	Models	Boundary conditions	
Wave Equation: Hyperbolic	$c^2u_{xx} - u_{tt} = 0$	Waves on a string; u gives displacement	u, $\frac{\partial u}{\partial t}$ given $t = 0$ $\frac{\partial u}{\partial t}$ given $t = 0$; plus end point condition - say $u = 0$ at ends	(IBVP) (Infinite string) (IVP) (Finite string)
Laplace's Equation: Elliptic	$u_{xx} + u_{yy} = 0$, $(x, y) \in D$	Potential Theory or Steady Heat; u gives potential or temperature	u or $\frac{\partial u}{\partial n}$ given on boundary of D	(BVP)
Heat Equation: Parabolic	$u_t = \kappa u_{xx}$	Heat; u gives temperature	u given at $t = 0$; plus endpoint conditions - u or $\frac{\partial u}{\partial x}$ given at ends	(IBVP) (Temperature in finite bar)

Some examples from Prelims: (We will assume that the data is smooth enough for the following to hold.)

(a) Hyperbolic equation: The IVP and IBVP (initial-boundary-value problem) for the wave equation

$$u_{xx} - u_{yy} = 0 \quad (ct = y).$$

For the IVP (modelling an infinite string, where u is the displacement), we know the solution is

$$u = \frac{1}{2}\left[f(x+y) + f(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} g(s)ds, \tag{4.16}$$

where the data are $u(x, 0) = f(x)$ and $u_y(x, 0) = g(x)$, $-\infty < x < \infty$. This is d'Alembert's solution of the IVP: it exists, and is unique and, intuitively at least, a small change in φ, ψ gives a small change in u (see problem sheet for proof). So this problem is well-posed.

For the IBVP (modelling a finite string length L, fixed at each end) consider the data:

$$u(x, 0) = f(x), \quad u_y(x, 0) = g(x) \quad 0 < x < L$$
$$u(0, y) = 0 = u(L, y).$$

So the boundaries are at $x = 0, L$. This IBVP is solved using separation of variables and Fourier series to get a solution

$$u = \sum_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right)$$

with a_n, b_n given in terms of f and g. Uniqueness was shown in Prelims, so this is the unique solution. If we now appeal to intuition for continuous dependence on the data this problem is well-posed.

(b) Elliptic equation:The BVP for the Laplace/Poisson's equation (modelling steady state heat, for example, where u is the temperature)

$$u_{xx} + u_{yy} = 0.$$

Do this first with data at the sides of a square, so $0 \le x, y \le a$ with

$$u(0, y) = u(a, y) = u(x, 0) = 0;\ u(x, a) = f(x).$$

Consider separable solutions $u_n = \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$, then

$$u = \sum_n a_n \frac{\sinh\frac{n\pi y}{a}}{\sinh(n\pi)} \sin\frac{n\pi x}{a}$$

and

$$f(x) = \sum a_n \, \sin \frac{n\pi x}{a}$$

which determines the solution as a Fourier series.

Now a different BVP, with data at the circumference of the unit circle:

$$\text{on } r = 1, \; u = f(\theta)$$

and in polars

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

The separable solutions are

$$\begin{cases} \left(Ar^n + \frac{B}{r^n}\right)(C \cos n\theta + D \sin n\theta) \\ A + B \log r, \quad n = 0 \end{cases}$$

Regularity at $r = 0$ implies

$$u = \frac{1}{2} a_0 + \sum_1^\infty r^n (a_n \cos n\theta + b_n \sin n\theta)$$

and the boundary value at $r = 1$ requires

$$\frac{1}{2} a_0 + \sum (a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$$

which again is solved by Fourier methods.

Uniqueness was proved in Prelims so in each of these cases we have the unique solution. It is plausible, but beyond our scope, to show that there is existence of solution in general. Later we will prove that there is continuous dependence on the data. So this problem is well posed

(c) Parabolic equation: The IBVP for the heat equation (modelling heat flow in a bar length L, with the ends held at zero temperature, u is temperature.)

$$u_{xx} = u_y$$

on the semi-infinite strip where $y = t > 0$ and $0 < x < L$, and data

$$u(x, 0) = f(x); \; u(0, y) = 0 = u(L, y).$$

The relevant separable solutions are

$$u_n = \sin \frac{n\pi x}{L} e^{-\frac{n^2 \pi^2 t}{L^2}}$$

so that

$$u = \sum_n a_n \sin \frac{n\pi x}{L} e^{-\frac{n^2 \pi^2}{L^2} t}$$

and the initial value requires

$$f(x) = \sum a_n \sin \frac{n\pi x}{L}$$

which is solved by Fourier methods. The solution exists *provided the series for u converges* which it will do for positive t. However, note that for negative t the exponentials grow rapidly with n and there is no reason to expect existence.

Uniqueness for this problem was done in Prelims, so again this is the unique solution for positive t. Later we will prove continuous dependence on the data. Thus the problem is well posed *forward in time*.

(d) What then is not well-posed? We give a few examples:

- BVPs for hyperbolic
  e.g. $u_{xx} - u_{yy} = 0$ on the unit square with data

  $$u(0,y) = u(1,y) = u(x,0) = 0;\ u(x,1) = f(x).$$

  Recall this data gave a well-posed problem for the Laplace equation, but here if $f = 0$, then $\sin n\pi x \sin n\pi y$ will do, for any n, while it can be proved that there is no solution at all if $f \neq 0$ (try the Fourier series to see what goes wrong).
- IVPs for elliptic
  e.g. $u_{xx} + u_{yy} = 0$ in the horizontal strip $0 \leq y \leq Y$, $-\infty < x < \infty$, with data

  $$u(x,0) = 0,\ u_y(x,0) = f(x).$$

  We know (from the problem sheet) that this data gives continuous dependence on the data for the wave equation. But not for Laplace's equation. For if $f(x) = \frac{1}{n}\sin nx$ it can be seen that $u(x,y) = \frac{1}{n^2}\sinh ny \sin nx$. But $\sup|\frac{1}{n^2}\sinh ny \sin nx| \to \infty$ as $n \to \infty$, whereas $1/n \sin nx \to 0$. Thus small changes in the initial data can lead to large changes in the solution. [More precisely: Suppose that there is continuous dependence on the initial data about the zero solution. That is: $\forall \epsilon > 0$, $\exists \delta > 0$ such that

  $$\sup_{x\in\mathbb{R}} |f(x) - 0| < \delta \Rightarrow \sup_{x\in\mathbb{R}, 0\leq y\leq Y} |u(x,y) - 0| < \epsilon.$$

  But taking $\epsilon = 1$, say, there exists N such that for all $n > N$, $\sup|\frac{1}{n^2}\sinh ny \sin nx| > 1$, but for any $\delta > 0$ we can choose $n > N$ such that $|\frac{1}{n}\sin nx| < \delta$, giving a contradiction.

- IBVP for elliptic
  e.g. $u_{xx} + u_{yy} = 0$ on the semi-infinite strip $0 \leq x \leq 1$, $y \geq 0$, with data

  $$u(0,y) = u(1,y) = 0,\ u(x,0) = 1,\ u_y(x,0) = 0.$$

This data gives a well-posed problem for the wave equation. If we try for separable solutions here, we have $u_n = \sin n\pi x \cosh n\pi y$ so

$$u = \sum_n a_n \sin n\pi x \cosh n\pi y.$$

Initial conditions need

$$1 = \sum a_n \sin n\pi x$$

whence

$$\begin{aligned} a_n &= 0 \quad n \text{ even} \\ &= \frac{4}{n\pi} \quad n \text{ odd}, \end{aligned}$$

and then

$$u\left(\frac{1}{2}, y\right) = \sum_n \frac{4}{(2n+1)\pi}(-1)^n \cosh(2n+1)\pi y,$$

which does not converge for any $y > 0$ (because the "cosh" terms grow rapidly with n) - there is no solution (strictly speaking, we've only shown that there is no solution of the form considered; we need more).

- The BVP for the heat equation is not well-posed, but we won't show that.

Again, it is beyond our scope to prove it in this course, but these different behaviours are universal for the different types of second-order, linear PDEs. In tabulated form, which problems are well-posed?

	IVP	IBVP	BVP
Hyperbolic	yes	yes	no
Elliptic	no	no	yes
Parabolic	yes	yes	no

where the 'yes' for parabolic equations are only valid forward in time.

4.4 The Maximum Principle

4.4.1 Poisson's equation

The normal form for second-order elliptic PDEs is

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \tag{4.17}$$

The operator on the left-hand side is referred to as the Laplacian, for which the symbols $\nabla^2 u$ and Δu are often used as shorthand. Poisson's equation is a special case of (4.17), in which f depends only on x and y. We have already seen that appropriate boundary data for (4.17) is to give just one boundary condition on u everywhere on a closed curve.

We will consider the Dirichlet problem where u is given on the boundary of D:

$$\begin{aligned} u_{xx} + u_{yy} &= f(x, y) \quad \text{in } D && (4.18) \\ u &= g(x, y) \quad \text{on } \partial D. && (4.19) \end{aligned}$$

It was shown in Prelims, using the divergence theorem, that if a solution exists, then it is unique. Using the maximum principle we will give another proof of this and also show that there is continuous dependence on the data. The solution does exist, but apart from the particular cases considered in Prelims that is beyond the scope of this course. Existence of solutions of general elliptic problems will be considered in the Part C courses Functional Analytic Methods for PDEs and Fixed point methods for nonlinear PDEs.

Theorem 4.1. *(The Maximum principle for the Laplacian) Suppose u satisfies*

$$\Delta u := u_{xx} + u_{yy} \geq 0 \quad (x, y) \in D, \tag{4.20}$$

everywhere within a bounded domain D. Then u attains its maximum value on ∂D.

Remark: This Theorem of course applies to the Poisson equation where we ask that $u_{xx} + u_{yy}$ is given by a prescribed function f, but is equally applicable to get information for non-linear problems, such as solutions of the equation $u_{xx} + u_{yy} = u^2$ for which we know that the right hand side has a given sign.

Remark: As u is a continuous function on the set $\bar{D}$ which is a closed and bounded subset of $\mathbb{R}^2$ and thus compact, we know that u achieves its maximum in some point $p \in \bar{D}$. The above theorem now tells us that this maximum value will indeed always be achieved on the boundary, though does not exclude that the maximum is also achieved at further points which might be in the interior.

In fact however the so called *strong maximum principle* (which is off syllabus) asserts that a function u with $\Delta u \geq 0$ cannot have an interior maximum unless it is constant.

Proof:

If we denote the boundary of D by ∂D , then as $D \cup \partial D$ is a closed bounded set and thus compact, u must attain its maximum somewhere in D or on its boundary. The proof now proceeds in two parts.

Suppose first that $u_{xx} + u_{yy} > 0$ in D.

If u has an interior maximum at some point (x_0, y_0) inside D, then the following conditions must be satisfied at (x_0, y_0):

$$u_x = u_y = 0, \quad u_{xx} \leq 0, \quad u_{yy} \leq 0.$$

But, as we assumed that $u_{xx} + u_{yy} > 0$ in all of D it is impossible for both u_{xx} and u_{yy} to be non positive . Hence u cannot have an interior maximum within D. so it must attain its maximum value on the boundary ∂D.

Suppose now that we only have $u_{xx} + u_{yy} \geq 0$ in D. We perturb u to get a function v which satisfies $v_{xx} + v_{yy} \geq 0$, so we can apply the first part of the proof.

Consider the function

$$v(x, y) = u(x, y) + \frac{\epsilon}{4}(x^2 + y^2),$$

where ϵ is a positive constant.

Then

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \epsilon > 0$$

in D. So using the result just proved, v attains its maximum value on ∂D.

Now, suppose that the maximum value of u on ∂D is M and the maximum value of $(x^2 + y^2)$ on ∂D is R^2 , then the maximum value of v on ∂D (and thus throughout D) is $M + (\epsilon/4)R^2$. In other words, the inequality

$$u + \frac{\epsilon}{4}(x^2 + y^2) = v \leq M + \frac{\epsilon}{4}R^2$$

holds for all $(x, y) \in D$. Letting $\epsilon \to 0$, we see that $u \leq M$ throughout D, i.e. that u attains its maximum value on ∂D. □

It obviously follows (by using the above result with u replaced by $-u$) that, if $\Delta u \leq 0$ in D, then u attains its minimum value on ∂D.

In the case $\Delta u = 0$, u therefore attains both its maximum and minimum values on ∂D. This is an important property of *Laplace's equation.*

Corollary 4.2. *(a) Consider the Dirichlet problem* (4.18), (4.19). *Then if the solution exists, it is unique.*

(b) The Dirichlet problem (4.18), (4.19) *has continuous dependence on the data.*

Proof: (a) Suppose that u_1, u_2 are two solutions, so $u = u_1 - u_2$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in } D && (4.21)\\ u &= 0 \quad \text{on } \partial D. && (4.22) \end{aligned}$$

By (4.21) the maximum and minimum of u occur on ∂D, so by (4.22) $u \leq 0$ and $u \geq 0$ in D. Thus $u = 0$ in D as required.

(b) We have to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that if u_i, $i = 1, 2$ are solutions with boundary data g_i, then

$$\sup_{(x,y)\in\partial D} |g_1(x, y) - g_2(x, y)| < \delta \quad \Rightarrow \quad \sup_{(x,y)\in D} |u_1(x, y) - u_2(x, y)| < \epsilon.$$

By linearity $u = u_1 - u_2$ satisfies

$$u_{xx} + u_{yy} = 0 \text{ in } D \tag{4.23}$$

$$u = g_1 - g_2, \text{ on } \partial D. \tag{4.24}$$

We now apply the maximum principle to see that $u \leq \max_{(x,y)\in\partial D}(g_1 - g_2)$, and applying the same result to $-u$, $-u \leq \max_{(x,y)\in\partial D} -(g_1 - g_2)$.

Hence in D

$$|u_1 - u_2| \leq \max_{(x,y)\in\partial D} |g_1 - g_2|, \tag{4.25}$$

so we may take $\delta = \epsilon$. □

4.4.2 The heat equation

In parabolic PDEs it is usually the case that one independent variable represents time, so we now use x and t as independent variables instead of x and y. The normal form for second-order parabolic equations is

$$u_{xx} = F(x, t, u, u_t, u_x)$$

and specific examples include the inhomogeneous heat equation, often called the diffusion equation:

$$u_t = u_{xx} + f(x, t).$$

and the reaction-diffusion equation

$$u_t = u_{xx} + f(x, t, u),$$

Well posed boundary data: Typical boundary data for a diffusion equation are to give an initial condition for u at $t = 0$ and one boundary condition on each of two curves C_1 and C_2 in the (x, t)-plane that do not meet and are nowhere parallel to the x-axis.

For example: The inhomogeneous heat equation

$$u_t = u_{xx} + f(x, t)$$

is a simple model for the temperature $u(x, t)$ in a uniform bar of conductive material, with heat source $f(x, t)$, where x is position and t is time. Suppose the bar is of length L, its initial temperature is given via $u_0(x)$, and its ends are kept at zero temperature. Then the initial and boundary conditions are

$$u = u_0(x) \text{ at } t = 0, \quad u = 0 \text{ at } x = 0; \; u = 0 \text{ at } x = L.$$

If, instead of being held at constant temperature, an end is insulated, then the Dirichlet boundary condition, $u = 0$, there is replaced by the Neumann boundary condition, $u_x = 0$. Alternatively , the boundary conditions at $x = 0$ and $x = L$ may, in general, be replaced by conditions at moving boundaries, say $x = x_1(t)$ and $x = x_2(t)$.

Theorem 4.3. *(The Maximum principle for the heat equation) Suppose that $u(x,t)$ satisfies*

$$u_t - u_{xx} \leq 0 \tag{4.26}$$

in a region D_τ bounded by the lines $t = 0$, $t = \tau > 0$, and two non-intersecting smooth curves C_1 and C_2 that are nowhere parallel to the x-axis. Suppose also that $f \leq 0$ in D_τ. Then u takes its maximum value either on $t = 0$ or on one of the curves C_1 or C_2 .

Proof:

The proof is similar to that for Poisson's equation.

We first observe that since u is a continuous function on a compact set $\bar{D}_\tau$ it will achieve its maximum on $\bar{D}_\tau$.

Suppose first that $u_t - u_{xx} < 0$ in D_τ. At an internal maximum inside D_τ, u must satisfy

$$u_x = u_t = 0 \ u_{xx} \leq 0, \ \ (u_{tt} \leq 0).$$

On the other hand, if u has a maximum at a point on $t = \tau$, then there it must satisfy

$$u_x = 0, \ \ u_t \geq 0, u_{xx} \leq 0.$$

With $u_t - u_{xx}$ assumed to be strictly negative, both of these lead to contradictions, and it follows that u must take its maximum value somewhere on ∂D_τ but not on $t = \tau$. We are done.

Suppose now that $u_t - u_{xx} \leq 0$, then define

$$v(x,t) = u(x,t) + \frac{\epsilon}{2}x^2,$$

where ϵ is a positive constant. Then v satisfies

$$v_t - v_{xx} = u_t - u_{xx} - \epsilon < 0$$

in D_τ. So by the earlier step v takes its maximum value on ∂D_τ but not on $t = \tau$.

Now if the maximum value of u over these three portions of ∂D_τ is M, and the maximum value of $|x|$ on C_1 and C_2 is L, then

$$u \leq v \leq \frac{L^2 \epsilon}{2} + M.$$

Now we let $\epsilon \to 0$ and conclude that $u \leq M$, i.e. u takes its maximum value on ∂D_τ but not on $t = \tau$ $\square$

If $u_t - u_{xx} \geq 0$ in D_τ, then a similar argument shows that u attains its minimum value on ∂D_τ but not on $t = \tau$. Thus, for the homogeneous equation (the heat equation) u attains both its maximum and its minimum values on ∂D_τ but not on $t = \tau$.

Remark: Physical interpretation: In a rod with no heat sources the hottest and the coldest spot will occur either initially or at an end. (Because heat flows from a hotter area to a colder area.)

Corollary 4.4. *Consider the IBVP consisting of* (4.26) *in* D_τ *with* u *given on* $\partial D_\tau \backslash \{t = \tau\}$. *Then if the solution exists, it is unique and depends continuously on the initial data.*

Proof: As for Poisson's equation (see problem sheet).

Sketch: Suppose that u_i $(i = 1, 2)$ are solutions of (4.26) in D_τ with data

$u_i = g_i$ on C_1, $u_i = h_i$ on C_2, $u_i(x, 0) = k_i(x)$ for x between C_1 and C_2 .

Let $u = u_1 - u_2$, $g = g_1 - g_2$, $h = h_1 - h_2$, $k = k_1 - k_2$. Then u satisfies (4.26) with $f = 0$, and $u = g$ on C_1, $u = h$ on C_2, $u(x, 0) = k(x)$ for x between C_1 and C_2 . Thus u and $-u$ take their maximum values on $\partial D_\tau \backslash \{t = \tau\}$. Hence

$$\sup_{D_\tau} u \leq \max\{\sup g, \sup h, \sup k\}$$

and

$$\sup_{D_\tau} -u \leq \max\{\sup -g, \sup -h, \sup -k\}.$$

So

$$\sup_{D_\tau} |u| \leq \max\{\sup |g|, \sup |h|, \sup |k|\},$$

and continuous dependence follows. In particular, if $g = h = k = 0$ then $u = 0$ and uniqueness follows.

5 Where does this course lead?

The course leads to DEs2, where, among other topics, boundary value problems for ODEs are discussed. Further discussion of differential equations comes in the Part B courses 'Non-linear Systems' and 'Applied Partial Differential Equations'. The use of abstract methods such as the Contraction Mapping Theorem to investigate the solutions of differential equations is taken further in various C4 courses, which require some knowledge of Banach and Hilbert spaces.

The techniques taught in DEs1 and DEs2 will be useful in various applied maths courses such as the Part A short course 'Modelling in Mathematical Biology' and the Part B courses 'Mathematical Ecology and Biology', 'Viscous Flow' and 'Waves and Compressible Flow'.

5.1 Section 1

Fixed point results such as the CMT provide very powerful methods for proving existence of solution for ODEs and PDEs. For PDEs we have to work in Banach spaces of functions rather than $\mathbb{R}^n$. For example for parabolic equations such as the reaction-diffusion equation:

$$u_t = u_{xx} + f(t, u),$$

with suitable boundary data, our proof of Picard's theorem can be extended to prove local existence (in t), provided f is continuous and is Lipschitz in u, where for each time t the solution u lives within a Banach space of x-dependent functions (which is infinite dimensional space) rather than the Euclidean n-dimensional space that we considered previously for ODEs. The technical details of this require methods from Functional Analysis as covered in the courses B4,1, B4.2 and C4.1 courses, but the basic ideas are just the same - there is just more to do, because more can go wrong!

An example of a reaction diffusion equations which occurs in applications is Fisher's equation:

$$u_t = u_{xx} + u(1 - u),$$

Another example of a second order parabolic equation is the Black-Scholes equation in mathematical finance.

Existence theorems play an important role in the theory of PDEs, which is a large and active field of current research, and you will be able to learn more about this in the Part C courses on Functional Analytic methods of PDEs and on Fixed Point Methods for Nonlinear PDEs.

5.2 Section 2

Phase plane analysis is a very important tool in mathematical modelling. It will be used, for example, in the Part A short course 'Modelling in Mathematical Biology' and the Part B course, 'Mathematical Ecology and Biology'.

The theory will be taken further in the Part B course, 'Nonlinear Systems'.

5.3 Section 3

We have considered only semi-linear first order PDEs. Similar methods can be extended to quasi-linear and fully non-linear equations. In these cases the characteristic equations are generally more difficult to solve. Such equations allow for the formation of shocks - see Part B 'Applied PDEs'.

These first order PDEs model many physical processes, including particularly conservation laws, and will appear in many of the modelling courses in Part B and beyond.

There are other methods for producing explicit solutions of PDEs:

Transform methods are very useful for linear PDEs- see the Part A short course in HT.

Similarity solutions can be used for both linear or non-linear PDEs - see Part B course 'Applied PDEs' - and involve reducing the PDE to an ODE in a 'similarity variable' involving both independent variables in the PDE.

5.4 Section 4

As we have already observed, proving the existence of solution, even of semi-linear second order PDEs is challenging. The different types of equation demand different approaches. For example:

Semi-linear hyperbolic equations with the solution u and its normal derivative prescribed on a given initial curve: One method to prove existence of solutions proceeds by showing that on initial curves other than characteristic curves we can find all the derivatives of u and that if all coefficients etc in the equation are very smooth (analytic) the solution is given by a power series near the initial curve. This is a version of the Cauchy-Kowalevski theorem.

Elliptic equations are treated in the Part C course 'Functional analytic methods for PDEs'.

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