

B1.2 Set Theory MT22**Problem Set 2**

1. (a) Let R be a relation (that is R is a set of ordered pairs). Prove that $\text{dom}(R)$, which we define to be $\{x : \exists y((x, y) \in R)\}$, is a set.

(b) Let X, Y be sets. Prove there exists a set whose elements are the functions from X onto Y . I.e. surjections.

(c) Let X be a set. Prove that there is a set consisting precisely of all strict total orders on X .

2. Prove, using induction and the fact that each $n \in \omega$ is transitive (by sheet 1) that $n \in n$ is false for every $n \in \omega$. (Do not use the Axiom of Foundation.)

3. Prove that the function sending n to $n!$ exists (as a set). Hint: Use Recursion in the form of Theorem 5.1 with $X = \omega \times \omega$.

4. Prove that multiplication on ω is commutative by proving the following statements for $n, m \in \omega$ by induction. You may use the other arithmetic properties established in lectures; or (hint) prove that $m * n = n.m$ satisfies the same recursion as $m.n$.

(i) $0.n = 0$

(ii) $m^+.n = m.n + n$

(iii) $m.n = n.m$

5. Write $1 = 0^+, 2 = 1^+$. Define $n \in \omega$ to be *even* if it is of the form $2.k$ for some $k \in \omega$ and *odd* if it is of the form $2.h + 1$ for some $h \in \omega$. Prove that

(i) every element of ω is either even or odd

(ii) no element of ω is both even and odd

6. A *Peano system* is a triple (A, s, a_0) in which A is a set, $a_0 \in A$, and $s : A \rightarrow A$ is a function with is (a) one-to-one, (b) does not include a_0 in its range, and (c) satisfies the Principle of Induction: that is, if $S \subseteq A$, $a_0 \in S$ and $\forall a(a \in S \rightarrow s(a) \in S)$, then $S = A$.

(i) Prove that $(\omega, x \mapsto x^+, 0)$ is a Peano system.

(ii) Suppose (A, s, a_0) is a Peano system. Prove that there exists an isomorphism from $(\omega, ^+, 0)$ to (A, s, a_0) , that is, there is a bijection $f : \omega \rightarrow A$ such that $f(0) = a_0$ and, for all $n \in \omega$, $f(n^+) = s(f(n))$.

[Hence, up to isomorphism, $(\omega, ^+, 0)$ is the unique Peano system.]

Hint: Define f by recursion and verify the required properties.

PTO

7. Let $X = X_0$ be a set. By the axiom of Unions, the sets $X_1 = \bigcup X, X_2 = \bigcup X_1, \dots$ are sets. The *transitive closure* of X is defined to be $T(X) = \bigcup_{n=0}^{\infty} X_n = \bigcup \{X_0, X_1, \dots\}$. Prove that

- (i) $T(X)$ is a set
- (ii) $T(X)$ is transitive
- (iii) $X \subseteq T(X)$
- (iv) If $X \subseteq Y$ and Y is transitive then $T(X) \subseteq Y$
- (v) If X is transitive then $T(X) = X$.

8. A set X is called *hereditarily finite* if its transitive closure $T(X)$ is a finite set.

- (i) Prove that the following sets are hereditarily finite

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

- (ii) Prove that a subset of a hereditarily finite set is hereditarily finite, and an element of a hereditarily finite set is hereditarily finite. [You may assume: a subset of a finite set is finite]
- (iii) Let \mathbf{H} be the class of hereditarily finite sets. It turns out that \mathbf{H} is in fact a set. Prove that the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme are all true in \mathbf{H} . [For example, the Axiom of Pairs is true in \mathbf{H} provided that, if a, b are hereditarily finite sets, there is a hereditarily finite set c whose only hereditarily finite elements are a and b . This will be true if indeed $\{a, b\}$ is hereditarily finite.]
- (iv) Is $\omega \in \mathbf{H}$?
- (v) (For those with B1.1 Logic or equivalent) Show that the Axiom of Infinity is not a consequence (in first order predicate logic) of the Empty Set Axiom, Extensionality, Pairs, Union and Comprehension.
- (vi) A set is called *hereditarily countable* if $T(X)$ is a countable set (i.e. is finite or is in bijection with ω). Let \mathbf{K} be the class of hereditarily countable sets. In fact \mathbf{K} is a set. Now $\omega \in \mathbf{K}$. Which of the axioms Extensionality, Empty Set, Pairs, Unions, Comprehension Scheme, Infinity, Powerset hold in \mathbf{K} ? [You may use that a countable union of countable sets is countable, though this does not follow from the axioms so far given.]
- (vii) (For those with B1.1 Logic or equivalent) Is it possible to prove the Powerset Axiom from the above axioms (now including Axiom of Infinity)?