B1.2 Set Theory MT22

Problem Set 4

1. (a) Let X be a well-ordered set, and $x \in X$. Show that either x is the greatest element in X or x has an immediate successor (that is an element $x^* \in X$ with $x < x^*$ such that there is no $y \in X$ with $x < y < x^*$).

(b) Let $X \subset \mathbb{R}$ such that the inherited order $<$ from $\mathbb{R}$ is a well-order on X. Prove that X must be countable. [Hint: consider the intervals (x, x^*).]

2. Let $<_A, <_B$ be strict total orders on sets A, B respectively. We define the *sum* $(A, <_A) + (B, <_B)$ and the *product* $(A, <_A) \times (B, <_B)$ of the orders as follows.

For the sum, we assume A, B are disjoint (which can always be arranged by replacing them by $A' = \{0\} \times A, B' = \{1\} \times B$ with the obvious orders on them). Then $(A, <_A) + (B, <_B)$ is the set $A \cup B$ with the order $<_+$ in which elements of A or B are ordered by $<_A, <_B$ respectively and all elements of A precede all elements of B.

The product $(A, <_A) \times (B, <_B)$ is $A \times B$ with the *reverse lexicographic order*, that is $(a, b) <_\times (a', b')$ iff $b < b'$, or $b = b'$ and $a < a'$.

(i) Draw illustrative pictures (coloured pens may be helpful) of the orders

$$\omega + 4, \quad 4 + \omega, \quad \omega + \omega, \quad \omega \cdot \omega$$

(ii) Prove that $<_+, <_\times$ are well-orders if $<_A, <_B$ are well-orders. [*You may omit the (tedious) verification that they are strict orders and that they are total.*]

3. Let α, β, γ be ordinals. Show that

(i) if $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$ (Hint: induction on γ, or use Theorem 14.7).
(ii) if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$, i.e. left cancellation holds.
(iii) right cancellation "$\alpha + \gamma = \beta + \gamma$ implies $\alpha = \beta$" fails, by giving a counterexample.
(iv) if γ is a limit ordinal then $\alpha + \gamma$ is a limit ordinal.

4. For any two ordinals α, β, exactly one of $\alpha \in \beta, \alpha = \beta, \beta \in \alpha$ hold. Determine which of these holds when

(i) $\alpha = (\omega + 1).2, \quad \beta = 2.(\omega + 1)$
(ii) $\alpha = (\omega + 1).\omega, \quad \beta = \omega.(\omega + 1)$

5. Ordinal exponentiation $\beta \mapsto \alpha^\beta$ for any ordinal $\alpha > 0$ was defined in lectures. Prove that if α, β are countable, with $\alpha > 0$, then α^β is countable. [Observe the difference with cardinal exponentiation on this point]. Assume AC. It can be done without with more work.

6. Let P be a non-empty partially strictly ordered set and assume no element of P is maximal (i.e. for every $x \in P$ there exists $y \in P$ with $x < y$). Use AC to show there exists a function $f : \omega \to P$ with $f(n) < f(n^+)$ for all $n \in \omega$.

7. Let R be a commutative ring with identity $1 \neq 0$.

(i) Prove that the union of a chain of proper ideals is a proper ideal.

(ii) Use Zorn's Lemma to prove that R has a maximal ideal.

8. Let $\mathcal{A}$ be a non-empty set of non-empty sets and $X = \bigcup \mathcal{A}$. Assume Zorn's Lemma and prove the existence of a function $F : \mathcal{A} \to X$ with $F(A) \in A$ for each $A \in \mathcal{A}$. [Hint: apply ZL to the partially ordered set P of partial maps from $\mathcal{A}$ to X, regarded as a subset of $\mathcal{P}(\mathcal{A} \times X)$, ordered by inclusion.]

Deduce that Zorn's Lemma implies the Axiom of Choice.