B1.2 Set Theory MT22

Problem Set 3

1. Prove that there is no descending sequence $X_0 \ni X_1 \ni \ldots$ of sets, that is, there is no function f with domain ω such that $f(n^+) \in f(n)$ for all $n \in \omega$. Hint: Apply the Axiom of Foundation to a suitably chosen set.

2. Use the Axiom of Foundation to show that, if A is a non-empty set, then $A \neq A \times A$. Hint: Apply the Axiom of Foundation to a suitably chosen set.

3. Prove that a subset of a finite set is finite. Hint: First show, by induction, that, for $n \in \omega$, every subset of n is equinumerous with some natural number. (A set is defined to be finite if it is equinumerous with an element of ω.)

4. Prove that the following properties of a set X are equivalent:

   (1) $\omega \preceq X$ (i.e. there is an injective function $f : \omega \to X$)

   (2) there exists a function $g : X \to X$ which is injective but not surjective.

Hint: For (2)$\Rightarrow$(1) use the Recursion Theorem, and induction to verify that the function you define is indeed injective.

5. Suppose κ, λ, μ are cardinals. Prove (no need to check obvious bijections)

(i) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$
(ii) $(\kappa.\lambda).\mu = \kappa.(\lambda.\mu)$
(iii) $\kappa.(\lambda + \mu) = \kappa.\lambda + \kappa.\mu$
(iv) $\kappa^{\lambda+\mu} = \kappa^\lambda.\kappa^\mu$
(v) $\kappa^{\lambda.\mu} = \left(\kappa^\lambda\right)^\mu$
(vi) $\left(\kappa.\lambda\right)^\mu = \kappa^\mu.\lambda^\mu$

6. (a) Let A, X, Y be sets such that $X \preceq A$. Prove that $X^Y \preceq A^Y$. Deduce that, for cardinals κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.

(b) Now let A, B, X, Y be sets with $X \preceq A$ and $Y \preceq B$. Prove that, apart from exceptional case(s), $X^Y \preceq A^B$. [You need to show that the map you give from X^Y to A^B is really injective.] What are the exceptional cases?

7. Calculate the cardinalities of the following sets, simplifying your answers as far as possible: your answer in each case should be a cardinal from the list $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$.

(i) the set of all finite sequences of natural numbers [*Note that the axioms given so far do not prove that a countable union of countable sets is countable. Use unique factorization of non-zero natural numbers into powers of primes.*]

(ii) the set of functions $f : \mathbb{R} \to \mathbb{R}$

(iii) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ Hint: a continuous function is determined by its values on $\mathbb{Q}$.

(iv) The set of equivalence relations on ω. Hint: To get a lower bound think about partitions of ω.

8. Let $f : X \to Y$ be surjective. Prove that $\mathcal{P}(Y) \preceq \mathcal{P}(X)$. [*You should not assume there exists an injective map $g : Y \to X$ as the axioms we have so far do not suffice to prove this.*]

9. (a) Let κ be any cardinal number and $n \in \omega$. Prove that (for cardinal addition)

(i) $\kappa + 0 = \kappa$ and $\kappa.0 = 0$

(ii) $\kappa.n^+ = \kappa.n + \kappa$

(b) We now have two definitions of addition and multiplication for elements of ω. Prove that they agree.