Topics in Fluid Mechanics

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These lecture notes are largely based on the notes by A. C. Fowler.
Some sections have been removed and a few have been added.

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Chapter 1

Thin film flows

1.1 Lubrication theory

Lubrication theory refers to a class of approximations of the Navier–Stokes equations which are based on a large *aspect ratio* of the flow. The aspect ratio is the ratio of two different directional length scales of the flow, as for example the depth and the width. Typical examples of flows where the aspect ratio is large (or small, depending on which length is in the numerator) are lakes, rivers, atmospheric winds, waterfalls, lava flows, and in an industrial setting, oil flows in bearings (whence the term lubrication theory). Lubrication theory forms a basic constituent of a viscous flow course and will not be dwelt on here.

In brief the Navier–Stokes equations for an incompressible take the form

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \rho[\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla} p + \mu\nabla^2\mathbf{u}, \end{aligned} \tag{1.1}$$

at least in Cartesian coordinates. It should be recalled that the actual definition of $\nabla^2 \equiv \boldsymbol{\nabla}\,\boldsymbol{\nabla}. - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$, and the components of $\nabla^2\mathbf{u} = \nabla^2 u_i \mathbf{e}_i$ (we use the summation convention) is only applicable in Cartesian coordinates. For other systems, one can for example consult the appendix in Batchelor (1967).

We begin by non-dimensionalising the equations by choosing scales

$$\mathbf{x} \sim l, \quad t \sim \frac{l}{U}, \quad \mathbf{u} \sim U, \quad p - p_a \sim \frac{\mu U}{l}; \tag{1.2}$$

this is the usual way to scale the equations, except that we have chosen to balance the pressure with the viscous terms. The pressure p_a is an ambient pressure, commonly atmospheric pressure. The resulting dimensionless equations are

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ Re\,\dot{\mathbf{u}} \equiv Re\,[\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla} p + \nabla^2\mathbf{u}, \end{aligned} \tag{1.3}$$

where

$$Re = \frac{\rho U l}{\mu} \tag{1.4}$$

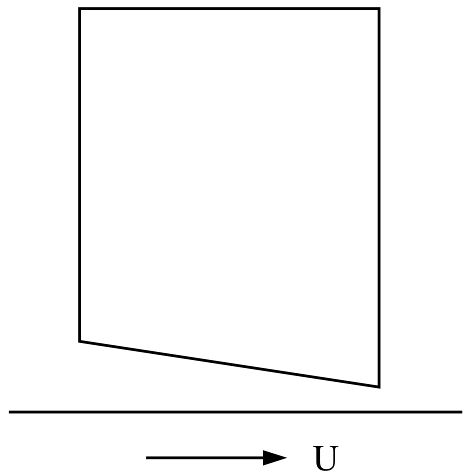

Figure 1.1: A slider bearing.

is the Reynolds number; the overdot denotes the material derivative. For $Re \ll 1$ we have Stokes flow, where the inertial terms can be neglected, and for $Re \gg 1$, boundary layers generally occur (and the pressure would be rescaled to balance the inertia terms, thus $p \sim Re$).

Lubrication theory describes a situation where the geometry of the flow allows the neglect of the inertial terms, even if the Reynolds number is not small. Suppose for example that l measures the extent of the flow in the x direction, but the fluid thickness in the (say) z direction is small. A simple example is the slider bearing, shown in figure 1.1, in which the fluid is confined between two surfaces, which we might take to be $z = 0$ and $z = h(x)$, and one of the surfaces moves at speed U relative to the other. To be specific, we assume a two-dimensional flow in which the coordinates are (x, z), the velocity components are (u, w), the bearing $(z = h)$ is of finite length l and lies above a flat surface $z = 0$ which moves at speed U; the bearing is open to the atmosphere at each end, and the gap width $h \sim d \ll l$. We define the small parameter

$$\varepsilon = \frac{d}{l}, \tag{1.5}$$

so that in non-dimensional terms, the bearing is at $z = \varepsilon h(x)$ (where we scaled the dimensional h with d, so that the dimensionless h is $O(1)$). It is then appropriate to rescale the variables as follows:

$$z \sim \varepsilon, \quad w \sim \varepsilon, \quad p \sim \frac{1}{\varepsilon^2}, \tag{1.6}$$

and the equations then take the form

$$\begin{aligned} u_x + w_z &= 0, \\ \varepsilon^2 Re\, \dot{u} &= -p_x + u_{zz} + \varepsilon^2 u_{xx}, \\ \varepsilon^4 Re\, \dot{w} &= -p_z + \varepsilon^2 (w_{zz} + \varepsilon^2 w_{xx}), \end{aligned} \tag{1.7}$$

with boundary conditions

$$\begin{aligned} u = 1,\ w = 0 \quad &\text{at} \quad z = 0, \\ u = w = 0 \quad &\text{at} \quad z = h, \\ p = 0 \quad &\text{at} \quad x = 0, 1. \end{aligned} \tag{1.8}$$

At leading order we then have $p = p(x, t)$, and thus, integrating, we obtain

$$u = \frac{z}{h} - \tfrac{1}{2} p_x (hz - z^2). \tag{1.9}$$

The final part of the solution comes from integrating the mass conservation equation from $z = 0$ to $z = h$. This gives

$$0 = -[w]_0^h = -\int_0^h w_z\, dz = \int_0^h u_x\, dz = \frac{\partial}{\partial x} \int_0^h u\, dz, \tag{1.10}$$

where we can take the differentiation outside the integral because u is zero at $z = h$. In fact we can write down (1.10) directly since it is an expression of conservation of mass across the layer; and this applies more generally, even if the base is not flat, and indeed even if both surfaces depend on time, and the result can be extended to three dimensions; see question 1.2. Calculating the flux from (1.9), we obtain

$$\int_b^s u\, dz = \tfrac{1}{2} h - \tfrac{1}{12} h^3 p_x = K \tag{1.11}$$

is constant. Given h, the solution for p can be found as a quadrature, and is

$$p = 6 \left[f_2(x) - \frac{f_2(1) f_3(x)}{f_3(1)} \right], \quad f_n(x) = \int_0^x \frac{dx}{h^n}. \tag{1.12}$$

In three dimensions, exactly the same procedure leads to the equation

$$\tfrac{1}{12} \boldsymbol{\nabla}_H . (h^3 \boldsymbol{\nabla}_H p) = \tfrac{1}{2} h_x, \tag{1.13}$$

where the plate flow direction is taken along the x axis; derivation of this is left as an exercise.

1.2 Droplet dynamics

When one of the surfaces is a free surface (meaning it is free to deform), such as a droplet of liquid resting on a surface, or a rivulet flowing down a window pane, there are two differences which must be accounted for in formulating the problem. One is that the free surface is usually a material surface, so that a kinematic condition is appropriate. In three dimensions, this takes the form

$$w = s_t + u s_x + v s_y - a. \tag{1.14}$$

Here, $z = s$ is the free surface, and (u, v, w) is the velocity; the term a is normally absent, but a non-zero value describes surface accumulation (which might for example be due to condensation); if $a < 0$ it describes ablation due for example to evaporation.

The other difference is that the boundary conditions at the free surface are generally not ones of prescribed velocity but of prescribed stress. In the common case of a droplet of liquid with air above, these conditions take the form

$$\sigma_{nn} = -p_a, \quad \sigma_{nt} = 0, \tag{1.15}$$

representing the fact that the atmosphere exerts a constant pressure on the surface, and no shear stress. Commonly the pressure is taken as *gauge* pressure, i. e., measured relative to atmospheric pressure, which is equivalent to taking $p_a = 0$ in (1.15). To unravel these conditions, we will consider the case of a two-dimensional incompressible flow. In this case, the components of the stress tensor are

$$\sigma_{11} = -p + \tau_1, \quad \sigma_{13} = \sigma_{31} = \tau_3, \quad \sigma_{33} = -p - \tau_1, \tag{1.16}$$

where

$$\tau_1 = 2\mu u_x, \quad \tau_3 = \mu(u_z + w_x), \tag{1.17}$$

and then with

$$\mathbf{n} = \frac{(-s_x, 1)}{(1 + s_x^2)^{1/2}}, \quad \mathbf{t} = \frac{(1, s_x)}{(1 + s_x^2)^{1/2}}, \tag{1.18}$$

we have

$$\begin{aligned} \sigma_{nn} = \sigma_{ij} n_i n_j &= -p - \frac{[\tau_1(1 - s_x^2) + 2\tau_3 s_x]}{1 + s_x^2}, \\ \sigma_{nt} = \sigma_{ij} n_i t_j &= \frac{[\tau_3(1 - s_x^2) - 2\tau_1 s_x]}{1 + s_x^2}. \end{aligned} \tag{1.19}$$

The dimensionless equations are virtually the same, as we initially scale $p - p_a, \tau_1$ and τ_3 with $\mu U/l$, and then when the rescaling in (1.6) is done (note that consequently we rescale $\tau_3 \sim 1/\varepsilon$), the surface boundary conditions become

$$\begin{aligned} p + \frac{\varepsilon^2[\tau_1(1 - \varepsilon^2 s_x^2) + 2\tau_3 s_x]}{1 + \varepsilon^2 s_x^2} &= 0, \\ \tau_3(1 - \varepsilon^2 s_x^2) - 2\varepsilon^2 \tau_1 s_x &= 0, \end{aligned} \tag{1.20}$$

where

$$\tau_1 = 2u_x, \quad \tau_3 = u_z + \varepsilon^2 w_x. \tag{1.21}$$

Putting $\varepsilon = 0$, we thus obtain the leading order conditions

$$p = \tau_3 = 0 \quad \text{on} \quad z = s. \tag{1.22}$$

We can then integrate $u_{zz} = p_x$, assuming also a no slip base at $z = b$, to obtain an expression for the flux

$$\int_b^s u\, dz = -\tfrac{1}{3} h^3 p_x, \tag{1.23}$$

and the conservation of mass equation then integrates (see question 1.2) to give the evolution equation for $h = s - b$ in the form

$$h_t = \tfrac{1}{3}\frac{\partial}{\partial x}[h^3 p_x]. \tag{1.24}$$

1.2.1 Gravity

The astute reader will notice that something is missing. Unlike the slider bearing, nothing is driving the flow! Indeed, since $p = p(x,t)$ and $p = 0$ at $z = s$, $p = 0$ everywhere. Related to this is the fact that there is nothing to determine the velocity scale U. Commonly such droplet flows are driven by gravity. If we include gravity in the z momentum equation, then it takes the dimensional form $\ldots = -p_z - \rho g \ldots$, and since in the rescaled model all the other terms are negligible, the pressure will be hydrostatic, $p \approx p_a + \rho g(s - z)$, and this gives a natural scale for $p - p_a \sim \rho g d$, and equating this with the eventual pressure scale $\mu U l/d^2$ determines the velocity scale as

$$U = \frac{\rho g d^3}{\mu l}. \tag{1.25}$$

The dimensionless pressure then becomes $p = s - z$, so that $p_x = s_x$, and (1.24) now takes the form of a nonlinear diffusion equation,

$$h_t = \tfrac{1}{3}\frac{\partial}{\partial x}[h^3 s_x]. \tag{1.26}$$

One might wonder how the length scales l and d should be chosen; the answer to this, at least if the base is flat, is that it can be taken from the initial condition for s. The reason for this is that, since (1.26) is a diffusion equation, the drop will simply continue to spread out: there is no natural length scale in the model. Associated with this is the consequent fact that for an initial concentration of liquid at the origin (again on a flat base), the solution takes the form of a similarity solution (see question 1.6). On the other hand, if b is variable, then it provides a natural length scale. Indeed, for a basin shaped b (for example x^2, dimensionlessly), the initial volume (or cross-sectional area) determines the eventual steady state as a lake with s constant, and both d and l prescribed.

1.2.2 Surface tension

Another way in which a natural length scale can occur in the model is through the introduction of surface tension at the interface. Let us digress for a moment to consider how surface tension arises. Surface tension is a property of interfaces, whereby they have an apparent strength. This is most simply manifested by the ability of small objects which are themselves heavier than water to float on the interface. The experiment is relatively easily done using a paper clip, and certain insects (water striders) have the ability to stay on the surface of a pond.

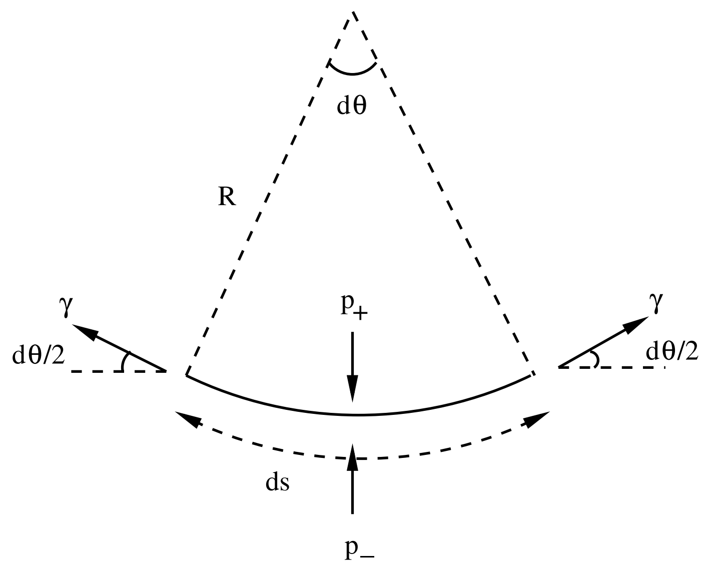

Figure 1.2: The simple mechanical interpretation of surface tension.

The simplest way to think about surface tension is mechanically. The interface between two fluids has an associated tension, such that if one draws a line in the interface of length l, then there is a force of magnitude γl which acts along this line: γ is the surface tension, and is a force per unit length. The presence of a surface tension causes an imbalance in the normal stress across the interface, as is indicated in figure 1.2, which also provides a means of calculating it. Taking ds as a short line segment in an interface subtending an angle $d\theta$ at its centre of curvature, a force balance normal to the interface leads to the condition

$$p_+ - p_- = \frac{\gamma}{R}, \tag{1.27}$$

where

$$R = \frac{ds}{d\theta} \tag{1.28}$$

is the *radius of curvature*, and its inverse $1/R$ is the curvature.

For a two-dimensional surface, the curvature is described by two *principal radii of curvature* R_1 and R_2, the mean curvature is defined by

$$\kappa = \tfrac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{1.29}$$

and the pressure jump condition is

$$p_+ - p_- = 2\gamma\kappa = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{1.30}$$

although this is not much use to us unless we have a way of calculating the curvature of a surface. This leads us off into the subject of differential geometry, and we do not want to go there. A better way lies along the following path.

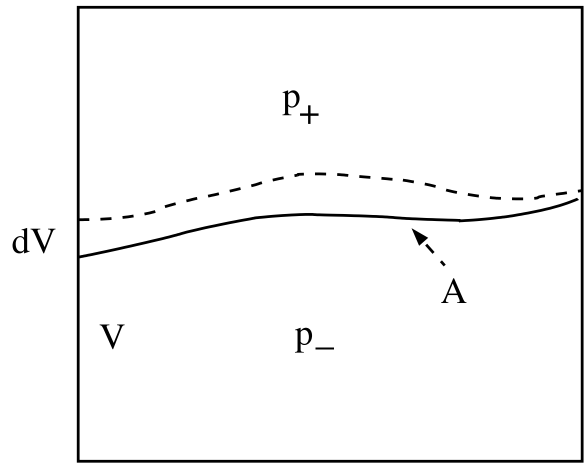

Figure 1.3: The energetic basis of surface tension.

The sceptical reader will in any case wonder what this surface tension actually is. It manifests itself as a force, but along a *line*? And what is its physical origin? The answer to this question veers towards the philosophical. We think we understand force, after all it pops up in Newton's second law, but how do we measure it? Pressure, for example, we conceive of as being due to the collision of molecules with a surface, and the measure of the force they exert is due to the momentum exchange at the surface. We pull on a rope, exerting a force, but the measure of the force is in the extension of the rope via Hooke's law. Force is apparently something we measure via its effect on momentum exchange, or on mechanical displacement; we can actually define force through these laws.

The more basic quantity is energy, which has a direct interpretation, whether as kinetic energy or internal energy (the vibration of molecules). And in fact Newton's second law for a particle is equivalent to the statement that the rate of change of energy is equal to the rate of doing work, and this might be taken as the fundamental law.

The meaning of surface tension actually arises through the property of an interface, which has a surface energy γ with units of energy per unit area. The interfacial condition then arises through the (thermodynamic) statement that in equilibrium the energy of the system is minimised.

To be specific, consider the situation in figure 1.3, where two fluids at pressures p_- and p_+ are separated by an interface with area A. Consider a displacement of the interface causing a change of volume dV as shown. Evidently the work done on the upper fluid is $p_+\,dV$, which is thus its change of energy, and correspondingly the change for the lower fluid is $-p_-\,dV$. If the change of interfacial surface area is dA, then the total change of energy[1] is

$$dF = (p_+ - p_-)\,dV + \gamma\,dA, \tag{1.31}$$

[1] This energy is the *Helmholtz free energy*.

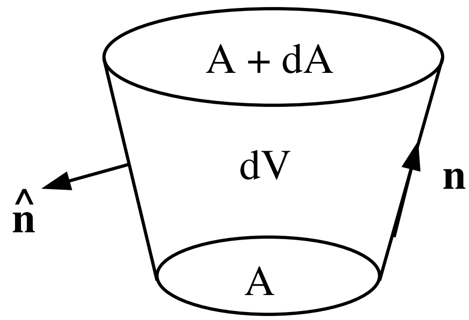

Figure 1.4: Calculation of $\dfrac{\partial A}{\partial V}$.

and at equilibrium this must be zero (since F is minimised). The equilibrium interfacial boundary condition is therefore

$$p_+ - p_- = -\gamma \frac{\partial A}{\partial V}, \tag{1.32}$$

which, it turns out, is equivalent to (1.30).

Computation of $\dfrac{\partial A}{\partial V}$ can be done as follows. We consider a displacement of the interface as shown in figure 1.4. An element of surface A is displaced to $A + dA$, and we can form a connecting volume dV such that the normal $\mathbf{n}$ to the interface is always parallel to the connecting surface between the end faces A and $A + dA$. We need to distinguish between the normal $\hat{\mathbf{n}}$ to the surface of the connecting volume and the normal to the interfacial surface. Evidently we have $\mathbf{n} = \hat{\mathbf{n}}$ at the end faces, but $\mathbf{n}.\hat{\mathbf{n}} = 0$ on the connecting cylindrical surface.

Applying the divergence theorem, we see that the change in area is

$$dA = \int_{\partial(dV)} \mathbf{n}.\hat{\mathbf{n}}\, dS = \int_{dV} \boldsymbol{\nabla}.\,\mathbf{n}\, dV, \tag{1.33}$$

and thus

$$\frac{\partial A}{\partial V} = \boldsymbol{\nabla}.\,\mathbf{n}. \tag{1.34}$$

For example, if the interface is represented as $z = s(x, y, t)$, then

$$\boldsymbol{\nabla}.\,\mathbf{n} = -\boldsymbol{\nabla}.\left[\frac{\boldsymbol{\nabla} s}{(1 + |\boldsymbol{\nabla} s|^2)^{1/2}}\right], \tag{1.35}$$

where on the right hand side $\boldsymbol{\nabla} = \boldsymbol{\nabla}_H = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$, and for small interfacial displacement, this may be linearised to obtain

$$2\kappa = -\frac{\partial A}{\partial V} = -\boldsymbol{\nabla}.\,\mathbf{n} = \boldsymbol{\nabla}.\left[\frac{\boldsymbol{\nabla} s}{(1 + |\boldsymbol{\nabla} s|^2)^{1/2}}\right] \approx \nabla^2 s. \tag{1.36}$$

1.2.3 The capillary droplet

Now we use this in the droplet equation. Again we restrict attention to two-dimensional droplets. For three-dimensional droplets, see question 1.7. The surface boundary condition is now approximately $p - p_a = -\gamma s_{xx}$, and non-dimensionally

$$p = -\frac{1}{B} s_{xx} \quad \text{on} \quad z = s, \tag{1.37}$$

where B (commonly also written Bo) is the Bond number, given by

$$B = \frac{\rho g l^2}{\gamma}. \tag{1.38}$$

This gives a natural length scale for the droplet, by choosing $B = 1$, thus

$$l = \left(\frac{\gamma}{\rho g} \right)^{1/2}; \tag{1.39}$$

in this case the dimensionless pressure is $p = s - z - s_{xx}$, and thus mass conservation leads to

$$h_t = \tfrac{1}{3} \frac{\partial}{\partial x} \left[h^3 (s_x - s_{xxx}) \right], \tag{1.40}$$

and the surface tension term acts as a further stabilising term.[2]

Surface tension acts to limit the spread of a droplet. Indeed there is a steady state of (1.40) which is easily found. Suppose the base is flat, so $s = h$. We prescribe the cross-sectional area of the drop, A. In dimensionless terms, we thus require

$$\int h\, dx = 2\alpha = \left(\frac{\rho g}{\gamma} \right)^{1/2} \frac{A}{d}. \tag{1.41}$$

Let us choose d so that the maximum depth is one (note that the value of d remains to be determined). We can suppose that the drop is symmetric about the origin, and that its dimensionless half-width is λ, also to be determined. Thus

$$h(\pm\lambda) = 0, \quad h(0) = 1, \tag{1.42}$$

as well as (1.41), and both α and λ are to be determined.

A further condition is necessary at the margins. This is the prescription of a contact angle, which can be construed as arising through a balance of the surface tension forces at the three interfaces at the contact line: gas/liquid, liquid/solid, and solid/gas. All three interfaces have a surface energy, and minimisation of this corresponds to prescription of a contact angle. Specifically, if θ is the angle between the

[2] This can be seen by considering small perturbations about a uniform solution $h = s = 1$ (with a flat base), for which the linearised equation has normal mode solutions $\propto \exp(\sigma t + ikx)$, with $\sigma = -\frac{1}{3}(k^2 + k^4)$.

gas/liquid and liquid/solid interfaces, then resolution of the surface tension tangential to the wall leads to

$$\gamma_{\rm SL} + \gamma \cos\theta = \gamma_{\rm SG}, \tag{1.43}$$

where $\gamma_{\rm SL}$ is the solid/liquid surface energy, and $\gamma_{\rm SG}$ is the solid/gas surface energy. Defining $S = l \tan\theta/d$, this implies that

$$h_x = \mp S \quad \text{at} \quad x = \pm\lambda. \tag{1.44}$$

The steady state of (1.40) is easily found. The flux is zero, so $h_x - h_{xxx}$ is zero, and integration of this leads to

$$h = 1 - \left(\frac{\cosh x - 1}{\cosh\lambda - 1}\right), \tag{1.45}$$

and then (1.41) and (1.44) yield

$$\alpha = \frac{\lambda\cosh\lambda - \sinh\lambda}{\cosh\lambda - 1}, \qquad \frac{\sinh\lambda}{\cosh\lambda - 1} = S. \tag{1.46}$$

$S(\lambda)$ is a monotonically decreasing function of λ (why?), and tends to one as $\lambda \to \infty$, and therefore the second relation determines λ providing $S > 1$. It seems there is a problem if $S < 1$, but this is illusory since both α and S depend on the unknown d, so it is best to solve

$$\frac{\alpha}{S} = \frac{A}{2l^2\tan\theta} = \frac{\lambda\cosh\lambda - \sinh\lambda}{\sinh\lambda}; \tag{1.47}$$

the right hand side increases monotonically from 0 to ∞ as λ increases, and therefore provides a unique solution for λ for any values of A and θ; d is then determined by either expression in (1.46).

It is of interest to see when the assumption $d \ll l$ is then valid. From (1.46),

$$\varepsilon = \tan\theta\left(\frac{\cosh\lambda - 1}{\sinh\lambda}\right). \tag{1.48}$$

The expression in λ increases monotonically from 0 to 1 as λ increases. Thus $\varepsilon \ll 1$ if either $\theta \ll 1$, or (if $\tan\theta \sim O(1)$) $\lambda \ll 1$. From (1.47), this is the case provided $A \ll l^2$, i.e., $\dfrac{\rho g A}{\gamma} \ll 1$. For air and water, this implies $A \ll 7\ \text{mm}^2$.

1.2.4 Stability

We now consider the stability of steady solutions of (1.40), which we take in the form

$$h_t = \left[\tfrac{1}{3}h^3(h_x - h_{xxx})\right]_x. \tag{1.49}$$

Before doing so, we comment on the meaning of the fourth derivative term, which is present due to surface tension. The gravity term is clearly diffusive (with a nonlinear diffusion coefficient $\frac{1}{3}h^3$), but what does the surface tension term represent? In other contexts it is referred to as a long-range or non-local diffusion (or dispersion) term. To understand such a reference, suppose that the flux of a quantity having density ρ is given not by Fick's law $\mathbf{J} = -D\boldsymbol{\nabla}\rho$, but by

$$\mathbf{J} = -D\boldsymbol{\nabla}W, \quad W = \int_{\mathbf{R}^3} \rho(\mathbf{x}+\boldsymbol{\xi}, t)K(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.50}$$

where the kernel function $K = K(\xi)$ (here $\xi = |\boldsymbol{\xi}|$) is spherically symmetric in an isotropic medium, and can be taken (by choice of D) to have integral over all space equal to one. If K is a delta function, $K = \delta(\mathbf{x} - \boldsymbol{\xi})$, then we regain Fick's law, but more generally we might suppose it is a Gaussian, for example. (1.50) allows a diffusive motion due to non-local concentrations. An example of such dependence might be in traffic flow, where the motion of individual 'molecules' (cars) is affected by the observation of conditions further ahead. Another example might be in herd migration.

If we suppose that K is delta function-like, in the sense that it varies rapidly with $\boldsymbol{\xi}$, then it is appropriate to approximate (1.50) by Taylor expansion of ρ, and this leads to

$$\mathbf{J} = -D\boldsymbol{\nabla}\rho - D_2\boldsymbol{\nabla}\nabla^2\rho + \ldots, \tag{1.51}$$

where

$$D_2 = \tfrac{1}{6}D\int_{\mathbf{R}^3} \xi^2 K(\xi)\, d\boldsymbol{\xi} = \tfrac{2}{3}\pi D\int_0^\infty \xi^4 K(\xi)\, d\xi. \tag{1.52}$$

Solutions of the conservation law $\rho_t = -\boldsymbol{\nabla}.\mathbf{J}$, using the truncated expression in (1.51), have the normal mode form

$$\rho = e^{i\mathbf{k}.\mathbf{x}+\sigma t}, \quad \sigma = -Dk^2 + D_2k^4, \tag{1.53}$$

and we see that the well-posedness ($\sigma < 0$ as $k \to \infty$) in this truncated form requires $D_2 < 0$, which seems unlikely, unless K becomes negative at large ξ.

If we use the full expression in (1.50), then we find that (1.53) is replaced by

$$\sigma = -4\pi kDI(k), \quad I(k) = \int_0^\infty rK(r)\sin kr\, dr \tag{1.54}$$

(use spherical polar coordinates and take the z axis in the direction of $\mathbf{k}$). For example, the (normalised) Gaussian

$$K(\xi) = \frac{1}{(\pi\nu)^{3/2}}e^{-\xi^2/\nu} \tag{1.55}$$

leads to

$$\sigma = -k^2De^{-\frac{1}{4}\nu k^2}, \tag{1.56}$$

and expansion of this for small ν (or k) leads to the truncated version above. Note that for the full expression, the limits $\nu \to 0$ and $k \to \infty$ do not commute.

Returning to the matter at hand (equation (1.49)), we first consider the case of an infinite uniform layer of fluid, with constant solution $h = 1$. In this case we write $h = 1 + h_1$ and linearise on the basis that $h_1 \ll 1$. This simply gives

$$h_{1t} = \tfrac{1}{3}(h_{1xx} - h_{1xxxx}), \tag{1.57}$$

which has the normal mode solutions $h_1 = e^{ikx+\sigma t}$, and

$$\sigma = -\tfrac{1}{3}(k^2 + k^4), \tag{1.58}$$

and the steady solution is stable.

For the case of a finite droplet with solution $h_0(x)$ given by (1.45), we write $h = h_0 + h_1$, and again supposing $h_1 \ll h_0$, we linearise as before, which leads (since $h_0''' = h_0'$) to

$$h_{1t} = \left[\tfrac{1}{3} h_0^3 (h_{1x} - h_{1xxx})\right]_x, \tag{1.59}$$

and normal mode solutions are of the form $h_1 = H(x)e^{\sigma t}$, and then

$$\sigma H = \left[\tfrac{1}{3} h_0^3 (H_x - H_{xxx})\right]_x. \tag{1.60}$$

This equation requires boundary conditions, but there are issues. If the margins move, then the linearisation must become invalid, since it requires the assumption that $h_1 \ll h_0$, which cannot in general be true if the margins move. Consideration of this case requires a more subtle approach, which uses the method of strained coordinates, but will be foregone here.

Let us suppose, then, that the margins do not move. In this case we should prescribe

$$H = H' = 0 \quad \text{at} \quad x = \pm\lambda. \tag{1.61}$$

This provides four conditions, the gradient condition occurring because of the prescribed contact angle. However, we note that the equation is degenerate since $h_0(\pm\lambda) = 0$, so that the full complement of boundary conditions may not be able to be satisfied. Often in such singular problems (think of Bessel's equation), one only needs to suppress singular solutions. If (1.61) can be satisfied, then automatically $H \ll h_0$ as $x \to \pm\lambda$, which is required for the validity of the analysis.

Perhaps an ingenious exact solution of (1.60) can be found, but failing that, we resort to an energy-type argument. If we multiply both sides of the equation by $H - H_{xx}$ and integrate, then we find

$$\sigma = \frac{-\displaystyle\int_{-\lambda}^{\lambda} \tfrac{1}{3} h_0^3 (H_x - H_{xxx})^2 \, dx}{\displaystyle\int_{-\lambda}^{\lambda} (H^2 + H_x^2) \, dx}, \tag{1.62}$$

and thus $\sigma < 0$: the droplet is stable. (1.62) actually provides a variational principle for σ: see question 1.3.

Coming back to the issue of the behaviour of H at the end points, we put, for example, $X = x + \lambda$, so that

$$-\alpha H \approx [X^3 (H_X - H_{XXX})]_X, \quad \alpha = \frac{3|\sigma|}{S^3}, \tag{1.63}$$

and we find possible solution behaviours as $X \to 0$ of the form

$$
\begin{aligned}
H &\sim X^2 + cX^3 + \dots, \\
H &\sim 1 - bX \ln X,
\end{aligned}
\tag{1.64}
$$

where b and c are specific constants (see exercise 1.3). Therefore it seems in fact that only one condition can be applied at each end, in keeping with the degenerate nature of the equation, but that in fact the extra gradient condition in (1.61) is satisfied automatically.

It should be mentioned that when droplets move, there are issues both with the viability of prescribing a constant contact angle, because of experimentally observed *contact angle hysteresis*, and also with the application of the no-slip condition, which causes a contact line singularity. So the above discussion of stability is slightly inaccurate.

1.2.5 Advance and retreat

When a droplet is of finite extent, it is possible to describe the behaviour near the margins by a local expansion. Typically the surface approaches the base with local power law behaviour, and this depends on whether the droplet is advancing or retreating. Consider, for example, the gravity-driven droplet with an accumulation or ablation term:

$$
h_t = \tfrac{1}{3}\left(h^3 h_x\right)_x + a, \tag{1.65}
$$

where $a > 0$ for accumulation, and $a < 0$ for ablation. (1.65) represents a simple model for the motion of an ice sheet such as Antarctica, where $a > 0$ represents accumulation due to snowfall. If we suppose that near the margin $x = x_s$ in a two-dimensional motion, $h \sim C(x_s - x)^\nu$, then a local expansion shows that if the front is advancing, $\dot{x}_s > 0$, then $\nu = \frac{1}{3}$ and $\dot{x}_s \sim \frac{1}{9}C^3$; in advance the front is therefore steep. On the other hand, if the front is retreating, then this can only occur if $a < 0$ (as is in fact obvious), and in that case $\nu = 1$ and $\dot{x}_s \sim -|a|/C$. The fact that the front slope is infinite in advance and finite in retreat is associated with 'waiting time' behaviour, which occurs when the front has to 'fatten up' before it can advance.

We can try and carry out the same analysis for the droplet with gravity and surface tension. If the left hand margin is $x = x_s(t)$, we put $x = x_s + X$, so that in the (X, t) coordinates,

$$
h_t - \dot{x}_s h_X = \left[\tfrac{1}{3}h^3(h_X - h_{XXX})\right]_X; \tag{1.66}
$$

however, finding a local expansion is not so easy. Trying various choices, it seems that retreat ($\dot{x}_s > 0$) can be described by

$$
h \sim aX(-\ln X)^{1/3}, \quad \dot{x}_s \sim \tfrac{1}{9}a^3, \tag{1.67}
$$

but no such simple (!) behaviour describes advance. However, a balance is possible when there is a non-zero flux at the front q_s, and then

$$
h \sim aX^{3/4}, \quad q_s = \tfrac{5}{64}a^4. \tag{1.68}
$$

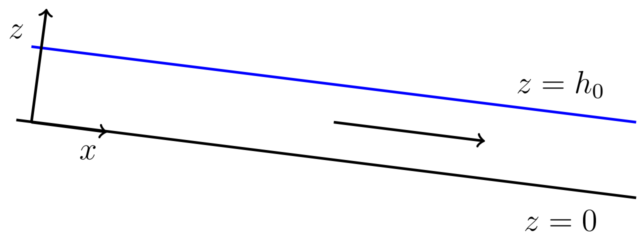

Figure 1.5: Schematic of a falling film e.g. rain flowing down a windshield.

But both these behaviours provide an infinite gradient at the margin, which is inconsistent with the prescription of a finite slope contact angle, and also with the lubrication theory linearisation of the curvature term, and for both these reasons, the model becomes suspect if the margins are allowed to move.

1.2.6 Falling films

In this section we consider a class of flows called falling films, for which there is a predominant background flow which plays an important role on the film dynamics. Examples of such flows include rain falling down a windshield, or industrial coating problems. As we will see, despite having a long-thin aspect ratio, inertia may still play an important role in such flows.

We consider a thin two-dimensional falling film on a tilted plane with angle α to the horizontal. We use rotated coordinates x, z, as illustrated in figure 1.5, such that the the impermeable base is located at $z = 0$. The dimensional Navier-Stokes equations in the tilted coordinates are given by

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \rho[\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla} p + \mu\nabla^2\mathbf{u} - \rho\mathbf{g}, \end{aligned} \tag{1.69}$$

where $\mathbf{g} = (-\sin\alpha, \cos\alpha)$. We impose no slip conditions $u = w = 0$ on $z = 0$. In the case where the upper surface is at constant level $z = h_0$, the stress conditions become $p = p_a$ and $u_z = 0$ at $z = h_0$. There is an exact solution for this scenario, which is given by

$$\begin{aligned} \bar{u} &= \frac{\rho g \sin\alpha}{2\mu}(2h_0 z - z^2), \\ \bar{w} &= 0, \\ \bar{p} &= p_a - \rho g \cos\alpha(z - h_0), \end{aligned} \tag{1.70}$$

where we use bar notation to indicate that this is the base state.

We consider long-wave perturbations to this flow, with aspect ratio $\varepsilon = h_0/l \ll 1$. Dimensional scalings are chosen as

$$x \sim l, \quad z \sim \varepsilon l, \quad \bar{u} \sim U = \frac{\rho g \sin\alpha h_0^2}{2\mu}, \quad \bar{w} \sim \varepsilon U, \quad \bar{p} - p_a \sim \rho g h_0 \sin\alpha. \tag{1.71}$$

We consider perturbations to the base state of the form

$$\begin{aligned} u &= \bar{u} + \hat{u}, \\ w &= \hat{w}, \\ p &= \bar{p} + \hat{p}, \end{aligned} \tag{1.72}$$

and we consider a variable profile for the thin film $z = h(x, t)$. Hence, the governing equations become

$$\begin{aligned} \hat{u}_x + \hat{w}_z &= 0, \\ \mathrm{Re}\,\varepsilon\left[\hat{u}_t + (\bar{u} + \hat{u})\hat{u}_x + \hat{w}(\bar{u}_z + \hat{u}_z)\right] &= -2\varepsilon\hat{p}_x + \hat{u}_{zz} + \varepsilon^2\hat{u}_{xx}, \\ \mathrm{Re}\,\varepsilon^2\left[\hat{w}_t + (\bar{u} + \hat{u})\hat{w}_x + \hat{w}\hat{w}_z\right] &= -2\hat{p}_z + \varepsilon(\hat{w}_{zz} + \varepsilon^2\hat{w}_{xx}), \end{aligned} \tag{1.73}$$

where Re$= \rho U h_0/\mu$. The no-slip boundary conditions at $z = 0$ become

$$\hat{u} = \hat{w} = 0, \tag{1.74}$$

whereas the kinematic and stress conditions at $z = h(x, t)$ become

$$\begin{aligned} \hat{w} &= h_t + (\bar{u} + \hat{u})h_x, \\ \bar{p} + \hat{p} &= -\frac{\gamma}{\rho g l^2 \sin\alpha} h_{xx} + \mathcal{O}(\varepsilon^2), \\ \bar{u}_z + \hat{u}_z &= \mathcal{O}(\varepsilon^2). \end{aligned} \tag{1.75}$$

It is assumed that $S = \gamma/(\rho g l^2 \sin\alpha)$ is an order $\mathcal{O}(1)$ constant. Conservation of mass can be written as

$$h_t + \frac{\partial}{\partial x}\left[\int_0^h (\bar{u} + \hat{u})\,\mathrm{d}z\right] = 0, \tag{1.76}$$

which holds true at all orders. In the limit of $\varepsilon \to 0$, the above system has solution

$$\begin{aligned} \hat{u}_0 &= 2z(h - 1), \\ \hat{w}_0 &= -z^2 h_x, \\ \hat{p}_0 &= (h - 1)\cot\alpha - S h_{xx}. \end{aligned} \tag{1.77}$$

Likewise, (1.76) indicates that

$$h_t + 2h^2 h_x = 0, \tag{1.78}$$

which is a nonlinear advection equation that has stable solutions. Next, we consider an asymptotic expansion solution of the form

$$\begin{aligned} \hat{u} &= \hat{u}_0 + \varepsilon\hat{u}_1 + \ldots, \\ \hat{w} &= \hat{w}_0 + \varepsilon\hat{w}_1 + \ldots, \\ \hat{p} &= \hat{p}_0 + \varepsilon\hat{p}_1 + \ldots. \end{aligned} \tag{1.79}$$

Inserting this into the x momentum equation (1.73) gives us

$$\mathrm{Re}\left[-4\,zh^2h_x + 2\,z^2hh_x\right] = -2\frac{\partial}{\partial x}\left[(h-1)\cot\alpha - Sh_{xx}\right] + \hat{u}_{1_{zz}}, \tag{1.80}$$

at first order. Likewise, the boundary conditions at first order indicate that

$$\begin{aligned} \hat{u}_{1_z} &= 0: \quad z = h, \\ \hat{u}_1 &= 0: \quad z = 0. \end{aligned} \tag{1.81}$$

Hence, the first order velocity correction is given by

$$\hat{u}_1 = \left[h_x\cot\alpha - Sh_{xxx}\right](z^2 - 2zh) + \frac{1}{6}\mathrm{Re}\,hh_x(z^4 - 4z^3h + 8h^3z). \tag{1.82}$$

Inserting this into (1.76) gives the thin film equation

$$h_t + 2h^2h_x + \varepsilon\frac{\partial}{\partial x}\left[h^3(-\frac{2}{3}h_x\cot\alpha + Sh_{xxx}) + \frac{8}{15}\mathrm{Re}\,h^6h_x\right] = 0. \tag{1.83}$$

This is sometimes referred to as the 'Benney equation' after a paper published by D.J. Benney in 1966. The second term represents the base flow, the third term is gravity-driven diffusion, the fourth term is surface-tension-driven diffusion, and the fifth term is a non-linear inertial term.

This example of lubrication theory is different to the previous examples because inertia plays an important role despite the fact that the flow is long and thin. The inertial term can cause waves to bunch up and grow. This can be seen by considering a small perturbation

$$h = 1 + \eta, \tag{1.84}$$

where $\eta(x,t) \ll 1$. Inserting this into the Benney equation and linearising yields

$$\eta_t + 2\eta_x + \varepsilon\left[-\frac{2}{3}\eta_{xx}\cot\alpha + S\eta_{xxxx} + \frac{8}{15}\mathrm{Re}\,\eta_{xx}\right] = 0. \tag{1.85}$$

Switching to the moving frame $\xi = x + 2t$ and imposing a wave-like perturbation of the form $\eta = \exp(\sigma t + ik\xi)$ results in the dispersion relation

$$\sigma = \varepsilon k^2\left[-\frac{2}{3}\cot\alpha + \frac{8}{15}\mathrm{Re} - Sk^2\right]. \tag{1.86}$$

Hence, we can see that we require a base flow which is faster than $\mathrm{Re} > (5/4)\cot\alpha$ for an instability to form.

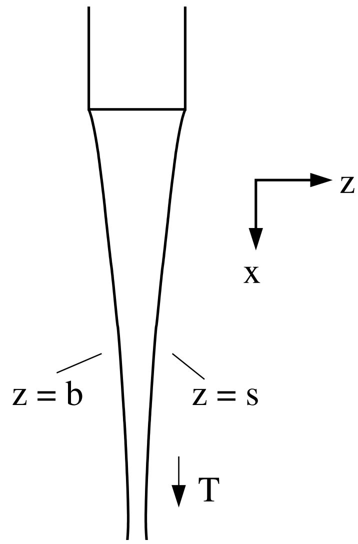

Figure 1.6: An elongational film flow.

1.3 Elongational flows

A different application of lubrication theory occurs in a falling sheet of fluid, such as occurs when a tap is switched on. At low velocities, the flow is continuous and laminar (though at very low flow rates it breaks up into droplets), and is also thin, but is distinguished from surface droplets or bearing flows by the fact that *both* surfaces of the fluid have zero stress acting on them.

To be specific, we consider the situation shown in figure 1.6. We consider flow from an orifice, and we take the flow to be two-dimensional, with the x direction in the direction of flow and z transverse to it. To begin with we ignore gravity and suppose that the flow is driven by an applied tension T (force per unit width in the y direction out of the page) at ∞; this is like drawing honey out of a jar with a spoon.

The basic equations are those as scaled in (1.3), and can be written in the form

$$\begin{aligned} u_x + w_z &= 0, \\ Re\,\dot{u} &= -p_x + \tau_{1x} + \tau_{3z}, \\ Re\,\dot{w} &= -p_z + \tau_{3x} - \tau_{1z}, \end{aligned} \tag{1.87}$$

where

$$\tau_1 = 2u_x, \quad \tau_3 = u_z + w_x. \tag{1.88}$$

If the two free surfaces are $z = s$ and $z = b$, then the boundary conditions on both surfaces are $\sigma_{nn} = \sigma_{nt} = 0$ (we subtract off the ambient pressure), or in other words $\sigma_{ni} = \sigma_{ij} n_j = 0$, and for $z = s$, this gives

$$\begin{aligned} (p - \tau_1)s_x + \tau_3 &= 0, \\ -\tau_3 s_x - p - \tau_1 &= 0. \end{aligned} \tag{1.89}$$

(These are actually equivalent to (1.19).)

Now we rescale the variables to account for the large aspect ratio. The difference with the earlier approach is that shear stresses are uniformly small, and so we also rescale τ_3 to be small. Thus we rescale the variables as

$$z \sim \varepsilon, \quad w \sim \varepsilon, \quad \tau_3 \sim \varepsilon, \tag{1.90}$$

and this leads to the rescaled equations

$$\begin{aligned} u_x + w_z &= 0, \\ Re\,\dot{u} &= -p_x + \tau_{1x} + \tau_{3z}, \\ \varepsilon^2 Re\,\dot{w} &= -p_z + \varepsilon^2 \tau_{3x} - \tau_{1z}, \end{aligned} \tag{1.91}$$

where

$$\tau_1 = 2u_x, \quad \varepsilon^2 \tau_3 = u_z + \varepsilon^2 w_x, \tag{1.92}$$

and on the free surfaces (e. g., $z = s$)

$$\begin{aligned} (p - \tau_1)s_x + \tau_3 &= 0, \\ -\varepsilon^2 \tau_3 s_x - p - \tau_1 &= 0. \end{aligned} \tag{1.93}$$

At leading order, we have $u = u(x,t)$, $p + \tau_1 = 0$, $p = -2u_x$, whence we find

$$\tau_{3z} = Re\,\dot{u} - 4u_{xx}, \tag{1.94}$$

with

$$\tau_3 = 4u_x s_x \quad \text{on} \quad z = s, \; \tau_3 = 4u_x b_x \quad \text{on} \quad z = b,$$

and from these we deduce

$$\begin{aligned} Re\,h(u_t + uu_x) &= 4(hu_x)_x, \\ h_t + (hu)_x &= 0, \end{aligned} \tag{1.95}$$

where the second equation is derived as usual to represent conservation of mass. Note in this derivation that the inertial terms are not necessarily small; nevertheless the asymptotic procedure works in the usual way.

1.3.1 Steady flow

For a long filament such as that shown in figure 1.6, it is appropriate to prescribe inlet conditions, and these can be taken to be

$$h = u = 1 \quad \text{at} \quad x = 0, \tag{1.96}$$

by appropriate choice of U and d. In addition, we prescribe the force (per unit width in the third dimension) to be T, and this leads to

$$hu_x \to 1 \quad \text{as} \quad x \to \infty, \tag{1.97}$$

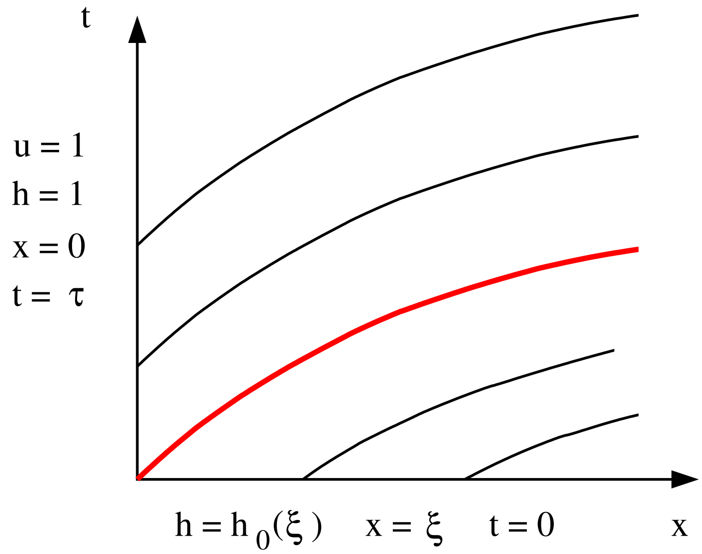

Figure 1.7: Characteristics for (1.95). The dividing characteristic from the origin is shown in red.

where the constant is set to one by choice of the length scale as

$$l = \frac{2\mu d U}{T}; \tag{1.98}$$

thus the aspect ratio is small ($d \ll l$) if $T \ll \mu U$.

If we consider a slow, steady flow in which the inertial terms can be ignored ($Re \to 0$), it is easy to solve the equations. We have $hu = 1$ and $hu_x = 1$, and thus

$$u = e^x, \quad h = e^{-x}. \tag{1.99}$$

As a matter of curiosity, one can actually solve the time-dependent problem (1.95), at least when $Re = 0$. We write the equations in the form

$$\begin{aligned} h_t + uh_x &= -1, \\ hu_x &= 1, \end{aligned} \tag{1.100}$$

with the boundary and initial conditions as shown in figure 1.7. The characteristic form of the first equation is

$$x_t = u[x(\xi, t), t], \quad h_t = -1, \tag{1.101}$$

where the partial derivatives are holding ξ fixed, i. e., we consider $x = x(\xi, t)$, $h = h(\xi, t)$. The dividing characteristic from the origin (which we define to be $t = t_d(x)$) divides the quadrant into two regions, in which the initial data is parameterised differently. For the lower region $t < t_d(x)$, we have

$$h = h_0(\xi) - t. \tag{1.102}$$

We take the first equation in (1.101), and differentiate with respect to ξ. Using the definition of u_x from (1.100), we find

$$x_{\xi t} = \frac{x_\xi}{h_0(\xi) - t}. \tag{1.103}$$

We can integrate this with respect to t, *holding* ξ *constant*, that is, the integral with respect to t is along a characteristic. It follows that

$$x_\xi = \frac{h_0(\xi)}{h_0(\xi) - t}, \tag{1.104}$$

in which we have applied the initial condition $x_\xi = 1$ at $t = 0$.

Next we integrate with respect to ξ *holding* t *constant*; since (1.104) only holds for $t < t_d(x)$, we integrate back to this, but note that this corresponds to the value $\xi = 0$; we then have

$$x = x_d(t) + \int_0^\xi \frac{h_0(s)\,ds}{h_0(s) - t}, \tag{1.105}$$

where x_d is the inverse of $t_d(x)$: to calculate this we need to solve for the upper region $t > t_d$.

To do this, we can proceed as above, but it is quicker to note that since the boundary conditions on $x = 0$ are constant, the solution is just the steady state solution (1.99). In particular, the characteristics are $e^{-x} = 1 - (t - \tau)$, and the dividing characteristic is that with $\tau = 0$, thus

$$t_d = 1 - e^{-x}, \quad x_d = -\ln(1 - t). \tag{1.106}$$

The solution in $t < t_d$ is thus

$$x = -\ln(1 - t) + \int_0^\xi \frac{h_0(s)\,ds}{h_0(s) - t}, \tag{1.107}$$

but the transient is of little interest since it disappears after finite time, $t = 1$. As a check, notice that if $h_0 = e^{-\xi}$, the steady state solution is regained everywhere.

The steady solution can be extended to positive Reynolds number. In steady flow we then find

$$u_x = Ku + \tfrac{1}{4} Re\, u^2 \tag{1.108}$$

for some constant K, and we see that there is no solution in which the filament can be drawn to ∞, as pinch-off always occurs. This is in keeping with experience.

1.3.2 Capillary effects

As for the shear-driven droplet flows, one can add gravity to the model, and this is done in question 1.4. In this section we consider the modification to the equations which occurs when capillary effects are included. The normal stress conditions are modified to

$$
\begin{aligned}
-\sigma_{nn} &= -\frac{\gamma s_{xx}}{(1+s_x^2)^{3/2}} \quad \text{on} \quad z=s, \\
\sigma_{nn} &= -\frac{\gamma b_{xx}}{(1+b_x^2)^{3/2}} \quad \text{on} \quad z=b.
\end{aligned} \tag{1.109}
$$

The definition of σ_{nn} is in (1.19), and with the basic scaling (all lengths scaled with l, etc.) this leads to

$$
-p - \frac{2\tau_3 s_x}{1+s_x^2} - \frac{\tau_1(1-s_x^2)}{1+s_x^2} = \frac{1}{Ca}\frac{\gamma s_{xx}}{(1+s_x^2)^{3/2}} \quad \text{on} \quad z=s, \tag{1.110}
$$

where

$$
Ca = \frac{\mu U}{\gamma} \tag{1.111}
$$

is the capillary number; a similar expression applies on $z=b$, with the opposite sign on the right hand side. When the equations are re-scaled ($z \sim \varepsilon$, etc.), then these take the approximate form

$$
\begin{aligned}
p+\tau_1 &\approx -\frac{1}{C}s_{xx} \quad \text{on} \quad z=s, \\
p+\tau_1 &\approx \frac{1}{C}b_{xx} \quad \text{on} \quad z=b,
\end{aligned} \tag{1.112}
$$

where we write

$$
Ca = \varepsilon C. \tag{1.113}
$$

Now the normal stress is constant across the filament, thus

$$
p+\tau_1 \approx -\frac{1}{C}s_{xx} \tag{1.114}
$$

everywhere, and this forces symmetry of the filament, $s_{xx} = -b_{xx}$. The rest of the derivation proceeds as before, except that (1.94) gains an extra term $-s_{xxx}/C$ on the right hand side; integrating this and applying the boundary conditions leads to the modification of (1.95) as (bearing in mind that $h = s - b$ and thus $h_{xx} = 2s_{xx}$)

$$
\begin{aligned}
h_t + (hu)_x &= 0, \\
Re\, h(u_t + uu_x) &= \frac{1}{2C}hh_{xxx} + 4(hu_x)_x.
\end{aligned} \tag{1.115}
$$

Steady flow

The extra derivatives for h require, apparently, two extra boundary conditions. If we suppose the pressure becomes atmospheric at ∞, then we might apply

$$
h_{xx} \to 0 \quad \text{as} \quad x \to \infty. \tag{1.116}
$$

Since this also implies $h_x \to 0$, it may be sufficient. On the other hand, if $h \to 0$ at ∞, the multiplication of the third derivative term by h may render an extra boundary condition unnecessary.

Again we can consider the steady state. Then $hu = 1$, and (1.115) has a first integral

$$K + \frac{Re}{h} = \frac{1}{2C}\left[hh_{xx} - \tfrac{1}{2}h_x^2\right] - \frac{4h_x}{h}, \tag{1.117}$$

where K is constant. Evidently there is no solution if $Re > 0$, as pinch-off must again occur. For the case of slow flow, taking $Re = 0$, we have $K = 4$ due to the far field stress condition, and

$$h^2 h_{xx} - \tfrac{1}{2}hh_x^2 - 8C(h_x + h) = 0. \tag{1.118}$$

We seek a solution of this with $h(0) = 1$ and $h(\infty) = 0$. Phase plane analysis shows that there is a unique such solution: see question 1.8.

Gravity

While we chose to model a thin filament pulled downwards by a tension, equally we might consider a filament descending under its own weight. In this case, the model can be derived much as before, but now the tension at infinity can be taken to be zero, and the length scale is then chosen to normalise the gravity term to equal one. The modification of (1.95) is then

$$\begin{aligned} h_t + (hu)_x &= 0, \\ h[Re\,(u_t + uu_x) - 1] &= 4(hu_x)_x. \end{aligned} \tag{1.119}$$

In this case, steady solutions extending to infinity exist, even if $Re > 0$, but if any non-zero tension is applied at infinity, the solution breaks down as before and pinch-out occurs. See also question 1.4.

Exercises

1.1 A thin incompressible liquid film flows in two dimensions (x, z) between a solid base $z = 0$ where the horizontal (x) component of the velocity is $U(t)$, and may depend on time, and a stationary upper solid surface $z = h(x)$, where a no slip condition applies. The upper surface is of horizontal length l, and is open to the atmosphere at the ends. Write down the equations and boundary conditions describing the flow, and non-dimensionalise them assuming that $U(t) \sim U_0$. (You may neglect gravity.)

Assuming $\varepsilon = d/l$ is sufficiently small, where d is a measure of the gap width, rescale the variables suitably, and derive an approximate equation for the pressure p. Hence derive a formal solution if the block is of finite length l, and the pressure is atmospheric at each end, and obtain an expression involving integrals of powers of h for the horizontal fluid flux, $q(t) = \int_0^h u\,dz$.

1.2 A two-dimensional incompressible fluid flow is contained between two surfaces $z = b(x,t)$ and $z = s(x,t)$, on which kinematic conditions hold:

$$w = s_t + us_x \quad \text{at} \quad z = s,$$

$$w = b_t + ub_x \quad \text{at} \quad z = b.$$

By integrating the equation of conservation of mass, show that the fluid thickness $h = s - b$ satisfies the conservation law

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_b^s u\,dx = 0.$$

Extend the result to three dimensions to show that

$$h_t + \boldsymbol{\nabla}_H.\left[\int_b^s \mathbf{u}_H\,dz\right] = 0,$$

where $\mathbf{u}_H = (u,v)$ is the horizontal velocity, and $\boldsymbol{\nabla}_H = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ is the horizontal gradient operator.

1.3 A two-dimensional droplet has thickness $h(x,t)$ and satisfies the dimensionless equation

$$h_t = \left[\tfrac{1}{3}h^3(h_x - h_{xxx})\right]_x,$$

with conditions that $|h_x| = S$ when $h = 0$. Show that for a steady solution $h_0(x)$,

$$h_0 = \frac{S(\cosh\lambda - \cosh x)}{\sinh\lambda},$$

where λ is an arbitrary (positive) parameter. If the (dimensionless) 'volume' of the drop V is prescribed, show that λ is uniquely determined, and that it increases monotonically with V. Find approximate expressions for λ as $V \to 0$ and $V \to \infty$.

By writing $h = h_0 + h_1$, linearising, and then putting $h_1 = H(x)e^{\sigma t}$, derive a linear equation for H, and give the boundary conditions for H, assuming the margins of the drop do not move. By writing σ as a functional $[H]$ in terms of integrals of H and its derivatives, show that $\sigma < 0$ for any solution of this, and thus that the drop is stable.

Suppose that H is a solution of its governing differential equation with corresponding eigenvalue $\sigma[H]$. By considering variations δH to H such that $\displaystyle\int_{-\lambda}^{\lambda}(H^2 + H_x^2)\,dx$ remains constant, show that the first variation $\sigma[H+\delta H] - \sigma[H]$ is zero.

Now let $X = x + \lambda$ so that $h_0 \approx SX$. By considering limiting forms of the resulting approximate equation for H, show that either $H \propto X^2 + cX^3 + \ldots$ or $H \propto 1 + bX\ln X + \ldots$, and find the values of b and c.

1.4 An incompressible two-dimensional flow from a slit of width d falls vertically under gravity. Define *vertical* and *horizontal* coordinates x and z, with corresponding velocity components u and w. The stream is symmetric with free interfaces at $z = \pm s$, on which no stress conditions apply. Write down the equations and boundary conditions in terms of the deviatoric stress components $\tau_1 = \tau_{11} = -\tau_{33}$ and $\tau_3 = \tau_{13} = \tau_{31}$, and by scaling lengths with l, velocities with the inlet velocity U, and choosing suitable scales for time t and the pressure and stresses, show that the equations take the form

$$u_x + w_z = 0,$$

$$Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} + 1,$$

$$Re\,\dot{w} = -p_z + \tau_{3x} - \tau_{1z},$$

where you should define $\dot{u}$, the Reynolds number Re, and write down expressions for τ_1 and τ_3.

Now define $\varepsilon = \dfrac{d}{l}$, and assume it is small. Find a suitable rescaling of the equations, and show that the vertical momentum equation takes the approximate form

$$h[Re\,\dot{u} - 1] = 4(hu_x)_x,$$

where $u = u(x,t)$ and h is the stream width.

Show also that

$$h_t + (hu)_x = 0.$$

Explain why suitable boundary conditions are

$$h = u = 1 \quad \text{at} \quad x = 0, \quad hu_x \to 0 \quad \text{as} \quad x \to \infty.$$

Write down a single second order equation for u in steady flow. If $Re = 0$, find the solution.

If $Re > 0$, find a pair of first order equations for $v = \ln u$ and $w = v_x$. (*Note: w here is no longer the horizontal velocity.*) Show that $(\infty, 0)$ is a saddle point, and that a unique solution satisfying the boundary conditions exists. If $Re \gg 1$ (but still $\varepsilon^2 Re \ll 1$), show (by rescaling $w = W/Re$ and $x = Re\,X$) that the required trajectory hugs the W–nullcline, and thus show that in this case

$$u \approx \left(1 + \frac{2x}{Re}\right)^{1/2}.$$

1.5 A (two-dimensional) droplet rests on a rough surface $z = b$ and is subject to gravity g and surface tension γ. Write down the equations and boundary conditions which govern its motion, non-dimensionalise them, and assuming the depth at the summit d is much less than the half-width l, derive an approximate equation for the evolution in time of the depth h. Show that the horizontal velocity scale is

$$U = \frac{\rho g d^3}{\mu l},$$

and derive an approximate set of equations assuming

$$\varepsilon = \frac{d}{l} \ll 1, \quad F = \frac{U}{\sqrt{gd}} \ll 1.$$

Hence show that

$$h_t = \frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left(s_x - \frac{1}{B}s_{xxx}\right)\right],$$

where you should define the Bond number B.

Find a steady state solution of this equation for the case of a flat base, assuming that the droplet area A and a contact angle $\theta = \varepsilon\phi$ are prescribed, with $\phi \sim O(1)$, and show that it is unique. Explain how the solution chooses the unknowns d and l.

1.6 A droplet of thickness h satisfies the equation

$$h_t = \frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3 h_x\right].$$

Find a similarity solution of this equation which describes the spread of a drop of area one which is initially concentrated at the origin (i. e., $h(x,0) = \delta(x)$).

1.7 A three-dimensional droplet , subject to gravity and resting on a flat horizontal surface $z = 0$, has surface $z = h(x,y,t)$, on which the pressure is given by $p = \gamma \boldsymbol{\nabla} . \mathbf{n}$, where $\mathbf{n}$ is the unit upward normal to the surface. Show that this condition can be written in the form

$$p = -\gamma \boldsymbol{\nabla} . \left[\frac{\boldsymbol{\nabla} h}{\{1 + |\boldsymbol{\nabla} h|^2\}^{1/2}}\right],$$

where now (and below) $\boldsymbol{\nabla}$ is the horizontal gradient $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$.

Use the assumptions of lubrication theory to derive the dimensionless droplet equation

$$h_t = \tfrac{1}{3}\boldsymbol{\nabla} . \left[h^3 \boldsymbol{\nabla}\left\{h - \frac{1}{Bo}\nabla^2 h\right\}\right],$$

and define the Bond number Bo.

Suppose that $Bo = \infty$ (what does this mean in terms of the surface tension?), and that a concentrated dollop of fluid of dimensionless volume 2π is released at $r = 0$ at $t = 0$. By seeking a similarity solution of the form

$$h = \frac{1}{t^\alpha} f(\eta), \quad \eta = \frac{r}{t^\beta},$$

derive and solve an equation for f, and hence show that the droplet is bounded by a moving front at

$$r \approx 1.55\, t^{1/8}.$$

[*Hint:* $\left(\dfrac{8192}{343}\right)^{1/8} \approx 1.55$.]

Now suppose that $Bo < \infty$. Explain why we may take $Bo = 1$. Assuming this, and a boundary condition that $h_r = -S$ where $h = 0$, show that the steady solution satisfies

$$h_{rr} + \frac{1}{r}h_r - h = -K,$$

where K is constant, and deduce that

$$h = \frac{S[I_0(\lambda) - I_0(r)]}{I_0'(\lambda)},$$

where $I_0(r)$ is the modified Bessel function of the first kind, and $r = \lambda$ is the drop margin.

Suppose that the dimensionless volume V of the drop is prescribed, so that

$$\int_0^\lambda r h(r)\, dr = \frac{V}{2\pi}.$$

We want to show that this determines λ uniquely. By consideration of the equation for h, show that

$$L(\lambda) \equiv \lambda \left[\frac{\lambda I_0(\lambda)}{2I_0'(\lambda)} - 1\right] = \frac{V}{2\pi S};$$

λ will thus be unique if $L(\lambda)$ is monotonically increasing.

Define

$$\eta(\lambda) = \frac{I_0'(\lambda)}{I_0(\lambda)},$$

and show that

$$\eta' = 1 - \frac{\eta}{\lambda} - \eta^2.$$

Assuming that $I_0(\lambda) \sim 1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \ldots$ as $\lambda \to 0$, find the limiting behaviour of η as $\lambda \to 0$, and by consideration of trajectory directions in the semi-phase plane (λ, η), show that $\eta(\lambda)$ is a monotonically increasing function of λ, with $\eta(\infty) = 1$. Derive a differential equation for $g(\lambda) = 2\eta/\lambda$, and by the same device (but now using the (λ, g) semi-phase plane), show that g is a monotonically decreasing function of λ. Hence show that $L(\lambda)$ is a strictly increasing function, as required.

Denoting this steady state as $h_0(r)$, perturb h as $h = h_0 + h_1$, and linearise the equation. Now put $h_1 = H(x, y)e^{\sigma t}$ (do not assume that H is cylindrically symmetric) and write down the resulting eigenvalue problem for σ. Assuming that the drop margin is not perturbed, show that σ is real and negative for any solution of this eigenvalue problem, and hence that the drop is stable.

1.8 A film of fluid is drawn downwards under the action of a tensile force. A model for the dimensionless thickness h and dimensionless downwards velocity u of the film is

$$h_t + (hu)_x = 0,$$

$$Re\, h(u_t + uu_x) = \frac{1}{2C} hh_{xxx} + 4(hu_x)_x,$$

with

$$h = u = 1 \quad \text{on} \quad x = 0, \quad hu_x \to 1 \quad \text{as} \quad x \to \infty.$$

Show that a steady state solution in which $h \to 0$ as $x \to \infty$ can only occur if $Re = 0$. In that case, determine a second order differential equation satisfied by h, and by writing $h = \frac{1}{2}U^2$ and $V = U' = U_x$, write the equation as a pair of first order equations for U and V. Show that the origin is a (degenerate) saddle, and therefore show that a solution exists which satisfies the boundary conditions.