Space filling curves

Paul Balister

This is a non-examinable, but fun, application of Theorem 7.2 (uniform limit of continuous functions is continuous) from the lecture notes. It should be read after one has completed the section of the course on **uniform convergence** (Section 7 of the notes).

We define a **curve** in $\mathbb{R}^2$ as a continuous function $\gamma\colon [a,b] \to \mathbb{R}^2$. If we write $\gamma(t) = (x(t), y(t))$ then continuity of γ is in fact equivalent to each coordinate $x(t)$ and $y(t)$ being a continuous function $[a,b] \to \mathbb{R}$.

Lemma. $\gamma(t) = (x(t), y(t))\colon [a,b] \to \mathbb{R}^2$ *is continuous iff both coordinate functions* $x(t), y(t)\colon [a,b] \to \mathbb{R}$ *are continuous.*

Proof. $\Longrightarrow$: Suppose $\gamma(t)$ is continuous. Then for any $t_0 \in [a,b]$ and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow |\gamma(t) - \gamma(t_0)| := \sqrt{(x(t) - x(t_0))^2 + (y(t) - y(t_0))^2} < \varepsilon$$

But then $|x(t) - x(t_0)|, |y(t) - y(t_0)| \le |\gamma(t) - \gamma(t_0)| < \varepsilon$ and so $x(t)$ and $y(t)$ are continuous.

$\Longleftarrow$: Suppose that both $x(t)$ and $y(t)$ are continuous. Then for any $t_0 \in [a,b]$ and $\varepsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$|t - t_0| < \delta_1 \Longrightarrow |x(t) - x(t_0)| < \tfrac{\varepsilon}{2} \qquad \text{and} \qquad |t - t_0| < \delta_2 \Longrightarrow |y(t) - y(t_0)| < \tfrac{\varepsilon}{2}.$$

But then for $\delta := \min\{\delta_1, \delta_2\} > 0$ we have

$$|t - t_0| < \delta \Longrightarrow |\gamma(t) - \gamma(t_0)| \le |x(t) - x(t_0)| + |y(t) - y(t_0)| < \varepsilon$$

by the triangle inequality for distances in the plane. Hence $\gamma(t)$ is continuous. $\square$

We naturally think of the curve in terms of its image $\{\gamma(t) : t \in [a,b]\} \subseteq \mathbb{R}^2$ and expect a curve to be some sort of '1-dimensional' shape sitting in the plane. But things can get weird!

Theorem. *There exists a space filling curve, i.e., a curve whose image has positive area in the plane.*

More specifically, we will define a continuous function $\gamma\colon [0,1] \to \mathbb{R}^2$ whose image fills the square $[0,1]^2$.

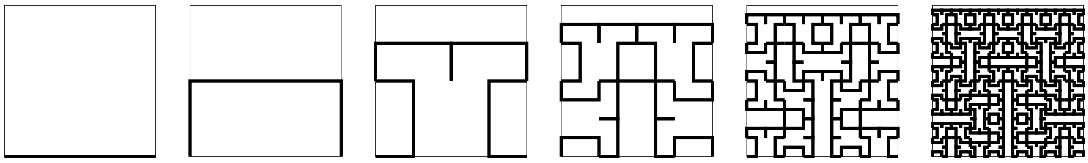
Define the function $\gamma_0\colon [0,1] \to \mathbb{R}^2$ by

$$\gamma_0(t) = (x_0(t), y_0(t)) := (t, 0).$$

Then for $n \geq 0$ inductively define

$$\gamma_{n+1}(t) = (x_{n+1}(t), y_{n+1}(t)) := \begin{cases} (\frac{1}{2}y_n(4t), \frac{1}{2}x_n(4t)), & t \in [0, \frac{1}{4}]; \\ (\frac{1}{2}x_n(4t-1), \frac{1}{2} + \frac{1}{2}y_n(4t-1)), & t \in [\frac{1}{4}, \frac{1}{2}]; \\ (\frac{1}{2} + \frac{1}{2}x_n(4t-2), \frac{1}{2} + \frac{1}{2}y_n(4t-2)), & t \in [\frac{1}{2}, \frac{3}{4}]; \\ (1 - \frac{1}{2}y_n(4t-3), \frac{1}{2} - \frac{1}{2}x_n(4t-3)), & t \in [\frac{3}{4}, 1]. \end{cases}$$

The images of the first few cases are drawn below.



γ_0 γ_1 γ_2 γ_3 γ_4 γ_5

Lemma. *γ_n is a well-defined continuous function from $[0,1]$ to $[0,1]^2$.*

Proof. It is easy to see by induction that $\gamma_n(0) = (0,0)$ and $\gamma_n(1) = (1,0)$. Thus the above definitions are consistent at the points $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and so give a well defined function $[0,1] \to \mathbb{R}^2$.

Clearly γ_0 is continuous, and assuming γ_n is continuous on $[0,1]$, it is easy to see from the definitions that both coordinate functions $x_{n+1}(t)$ and $y_{n+1}(t)$ (and hence $\gamma_{n+1}(t)$) are continuous on each of the intervals $[0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{3}{4}, 1]$ separately. But at each of the points $t_0 \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, $\lim_{t \to t_0^-} x_{n+1}(t) = x_{n+1}(t_0) = \lim_{t \to t_0^+} x_{n+1}(t)$ and $\lim_{t \to t_0^-} y_{n+1}(t) = y_{n+1}(t_0) = \lim_{t \to t_0^+} y_{n+1}(t)$, so $x_{n+1}(t)$, $y_{n+1}(t)$ are also continuous at these points (see Example 3.14 from the notes). Thus γ_{n+1} is continuous on $[0,1]$ and so by induction all γ_n are continuous

The image of γ_0 clearly lies in $[0,1]^2$, so inductively assume the image of γ_n lies in $[0,1]^2$. Then $\gamma_{n+1}(t)$ lies in $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ for $t \in [0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$ respectively. In all cases the image is in $[0,1]^2$, so by induction the image of γ_n lies in $[0,1]^2$ for all n. $\square$

Lemma. *For all $n \geq 1$ and $t \in [0,1]$, $|\gamma_n(t) - \gamma_{n-1}(t)| \leq 2^{2-n}$.*

Proof. As $\gamma_0(t)$ and $\gamma_1(t)$ both lie in $[0,1]^2$, $|\gamma_1(t) - \gamma_0(t)| \leq \sqrt{2} < 2$ for all $t \in [0,1]$, so the result holds for $n = 1$. Now assume $|\gamma_n(t) - \gamma_{n-1}(t)| \leq 2^{2-n}$ for all $t \in [0,1]$. Then applying the above definition for γ_{n+1} and γ_n in terms of γ_n and γ_{n-1} respectively, we have $|\gamma_{n+1}(t) - \gamma_n(t)| = \frac{1}{2}|\gamma_n(t') - \gamma_{n-1}(t')| \leq 2^{2-(n+1)}$ where $t' = 4t - \lfloor 4t \rfloor$. Thus by induction $|\gamma_n(t) - \gamma_{n-1}(t)| \leq 2^{2-n}$ for all $n \geq 1$ and all $t \in [0,1]$. $\square$

Corollary. *The sequence of functionc $\gamma_n(t)$ converges uniformly to a continuous function $\gamma\colon [0,1] \to [0,1]^2$ which satisfies*

$$\gamma(t) = (x(t), y(t)) := \begin{cases} (\frac{1}{2}y(4t), \frac{1}{2}x(4t)), & t \in [0, \frac{1}{4}]; \\ (\frac{1}{2}x(4t-1), \frac{1}{2} + \frac{1}{2}y(4t-1)), & t \in [\frac{1}{4}, \frac{1}{2}]; \\ (\frac{1}{2} + \frac{1}{2}x(4t-2), \frac{1}{2} + \frac{1}{2}y(4t-2)), & t \in [\frac{1}{2}, \frac{3}{4}]; \\ (1 - \frac{1}{2}y(4t-3), \frac{1}{2} - \frac{1}{2}x(4t-3)), & t \in [\frac{3}{4}, 1]. \end{cases}$$

Proof. From the previous lemma we see that for any $n > m$ and any $t \in [0,1]$,

$$|\gamma_n(t) - \gamma_m(t)| \leq \sum_{k=m+1}^{n} |\gamma_k(t) - \gamma_{k-1}(t)| \leq \sum_{k=m+1}^{n} 2^{2-k} < 2^{2-m}.$$

So $(\gamma_n(t))$, and thus both coordinate sequences $(x_n(t))$ and $(y_n(t))$, are uniformly Cauchy. Hence, by the Cauchy Criterion for uniformly convergence sequences (Theorem 7.9 of the notes), $x_n(t)$ and $y_n(t)$ converge uniformly to functions $x(t)$ and $y(t)$, which must be continuous by Theorem 7.2 of the notes (uniform limit of continuous functions is continuous). Hence $\gamma(t) := (x(t), y(t))$ is continuous. As $x_n(t), y_n(t) \in [0,1]$ for all n and $[0,1]$ is closed, we have $\gamma(t) = (x(t), y(t)) \in [0,1]^2$. The given equation follows by taking limits as $n \to \infty$ in the definition of γ_{n+1} and applying AOL. □

We now claim that the image is in fact the whole of $[0,1]^2$, finishing the proof of the Theorem.

Lemma. *The image of γ is $[0,1]^2$.*

Proof. Note that for any point $\mathbf{x} \in [0,1]^2$ there is a t_0, say $t_0 = 0$, such that $|\gamma(t_0) - \mathbf{x}| < 2$, say. Now assume that for every $\mathbf{x} \in [0,1]^2$ we can find a t_n with $|\gamma(t_n) - \mathbf{x}| < 2^{1-n}$. Fix $\mathbf{x} \in [0,1]^2$. Then $\mathbf{x}$ lies in one of the squares $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ or $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$. Thus we can write $\mathbf{x} = (x, y)$ in one of the forms $(\frac{y'}{2}, \frac{x'}{2})$, $(\frac{x'}{2}, \frac{1}{2} + \frac{y'}{2})$, $(\frac{1}{2} + \frac{x'}{2}, \frac{1}{2} + \frac{y'}{2})$ or $(1 - \frac{y'}{2}, \frac{1}{2} - \frac{x'}{2})$ for some $\mathbf{x}' = (x', y') \in [0,1]^2$. By induction we can find a $t'_n \in [0,1]$ with $|\gamma(t'_n) - \mathbf{x}'| < 2^{1-n}$. Thus we can find a $t_{n+1} = \frac{1}{4}(t'_n + i)$ for $i = 0$, 1, 2 or 3, respectively, for which $|\gamma(t_{n+1}) - \mathbf{x}| = \frac{1}{2}|\gamma(t'_n) - \mathbf{x}'| < 2^{1-(n+1)}$ as required.

Now by the Bolzano–Weierstrass Theorem, there is a subsequence (t_{s_n}) that converges to $t \in [0,1]$ say. But then by continuity of γ,

$$\gamma(t) = \lim_{n \to \infty} \gamma(t_{s_n}) = \mathbf{x}.$$

As $\mathbf{x}$ was arbitrary, the image of γ is the whole of $[0,1]^2$. □

Note that the t obtained above is not in general unique. In fact it can be proved that no space filling curve can be injective (even if we adjust each γ_n slightly to be injective). To see this however needs results from the Part A *Metric Spaces and Complex Analysis* course.