Exercise sheet 2. Week 6. Chapters 1-8.

Q1. Let $i \in \{0, \ldots, n\}$ and let $u_i : k^n \to \mathbb{P}^n(k)$ be the standard map (with image the coordinate chart U_i). Let $C \subseteq k^n$ be a closed subvariety of k^n (ie an algebraic set in k^n). For any $P \in k[x_0, \ldots, x_{i-1}, \check{x}_i, x_{i+1}, \ldots, x_n]$ let

$$\beta_i(P) := x_i^{\deg(P)} P(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{\breve{x_i}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}) \in k[x_0, \ldots, x_n].$$

(1) Let $\bar{C}$ be the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. Show that $(\beta_i(\mathcal{I}(C))) = \mathcal{I}(\bar{C})$ (where $(\beta_i(\mathcal{I}(C)))$ is the ideal of $k[x_0, \ldots, x_n]$ generated by all the elements of $\beta_i(\mathcal{I}(Z))$).

(2) Suppose that $\mathcal{I}(C) = (J)$ (ie $\mathcal{I}(C)$ is a principal ideal with generator J). Show that $(\beta_i(J)) = \mathcal{I}(\bar{C})$.

(3) Suppose that $n = 3$ and that C is the variety considered in Q3 of Sheet 1. Describe the closure of $u_0(C)$ in $\mathbb{P}^3(k)$. Find homogenous polynomials $(H_1, \ldots, H_h)$ such that $Z(H_1, \ldots, H_h)$ is the closure of $u_0(C)$ in $\mathbb{P}^3(k)$.

Q2. Let V (resp. W) be a closed subvariety of $\mathbb{P}^n(k)$ (resp. $\mathbb{P}^t(k)$). Let $V_0 \subseteq V$ (resp. $W_0 \subseteq W$) be an open subset of V (resp. and open subset of W). View V_0 (resp. W_0) as an open subvariety of V (resp. W). Let $Q_0, \ldots, Q_t \in k[x_0, \ldots, x_n]$ be homogenous polynomials. Suppose that $V_0 \cap Z((Q_0, \ldots, Q_t)) = \emptyset$. Let $f : V_0 \to \mathbb{P}^t(k)$ be the map given by the formula $f(\bar{v}) := [Q_0(\bar{v}), \ldots, Q_t(\bar{v})]$. Suppose finally that $f(V_0) \subseteq W_0$. Show that the induced map $V_0 \to W_0$ is a morphism of varieties.

Q3. Prove Lemma 7.1.

Q4. Let T be a topological space.

(1) Let $S \subseteq T$ be a subset. Suppose that S is irreducible. Show that the closure of S in T is also irreducible.

(2) Suppose that T is noetherian. Show that T is Hausdorff iff T is finite and discrete.

(3) Let V be a variety. Show that V is irreducible iff the ring $\mathcal{O}_V(U)$ is an integral domain for all open subsets $U \subseteq V$.

(4) Suppose T is noetherian. Show that T is quasi-compact.

Q5. Prove Lemma 8.1.

Q6. Let T be a topological space. Let $\{V_i\}$ be an open covering of T. Let $C \subseteq T$ be an irreducible closed subset (hence non empty).

(1) Show that $C \cap V_i$ is irreducible if $C \cap V_i \neq \emptyset$ and that $\sup_{i, C \cap V_i \neq \emptyset} \operatorname{cod}(C \cap V_i, V_i) = \operatorname{cod}(C, T)$ and $\sup_i \dim(V_i) = \dim(T)$.

(2) Prove Proposition 8.6. Give an example of a noetherian topological space of infinite dimension.

Q7. (1) Show that any element of $\mathrm{GL}_{n+1}(k)$ (= group of $(n+1) \times (n+1)$-matrices with entries in k and with non zero determinant) defines an automorphism of $\mathbb{P}^n(k)$.

(2) Show that if V is a projective variety, then for any two points $v_1, v_2 \in V$, there is an open affine subvariety $V_0 \subseteq V$ such that $v_1, v_2 \in V_0$.

Q8. (optional) (1) Let $P(x_0, \ldots, x_n)$ be a homogenous polynomial. Show that all the irreducible factors of P are also homogenous.

(2) Let $D \subseteq \mathbb{P}^n(k)$ be a closed subvariety. Suppose that D is irreducible and that $\operatorname{cod}(D, \mathbb{P}^n(k)) = 1$. Show that there is a homogenous irreducible polynomial $P \in k[x_0, \ldots, x_n]$ such that $D = Z(P)$.