BO1 History of Mathematics
Lecture VII
Infinite series

MT 2022 Week 4

Summary

- A non-Western prelude
- Newton and the Binomial Theorem
- Other 17th century discoveries
- Ideas of convergence
- Much 18th century progress: power series
- Doubts — and more on convergence

The Kerala School

Flourished in Southern India from the 14th to the 16th centuries, working on mathematical and astronomical problems

Names associated with the school: Narayana Pandita, Madhava of Sangamagrama, Vatasseri Parameshvara Nambudiri, Kelallur Nilakantha Somayaji, Jyeṣṭhadeva, Achyuta Pisharati, Melpathur Narayana Bhattathiri, Achyutha Pisharodi, Narayana Bhattathiri

Treatises on arithmetic, algebra, geometry, inc. methods for approximation of roots of equations, discussion of magic squares, infinite series, . . .

The Kerala School

Flourished in Southern India from the 14th to the 16th centuries, working on mathematical and astronomical problems

Names associated with the school: Narayana Pandita, Madhava of Sangamagrama, Vatasseri Parameshvara Nambudiri, Kelallur Nilakantha Somayaji, Jyeṣṭhadeva, Achyuta Pisharati, Melpathur Narayana Bhattathiri, Achyutha Pisharodi, Narayana Bhattathiri

Treatises on arithmetic, algebra, geometry, inc. methods for approximation of roots of equations, discussion of magic squares, infinite series, . . .

Tantrasamgraha (1501)

Completed by Kelallur Nilakantha Somayaji (1444–1544) in 1501; concerns astronomical computations

Keralan series

Infinite series for trigonometric functions appear in Sanskrit verse in an anonymous commentary on the *Tantrasamgraha*, entitled the *Tantrasamgraha-vyakhya*, of c. 1530:

इष्टज्यात्रिज्ययोर्घातात् कोट्याप्तं प्रथमं फलम् ।
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम् ॥
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः ।
एकत्र्याद्योज संख्याभिर्भक्तेष्वेतेष्वनुक्रमात् ॥
ओजानां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत् ।
दोःकोट्योरल्पमेवेह कल्पनीयमिह स्मृतम् ।
लब्धीनामवसानं स्यान्नान्यथापि मुहुः कृते ॥

Proof supplied by Jyeṣṭhadeva in his *Yuktibhāṣā* (1530)

Keralan series

From the *Tantrasamgraha-vyakhya*:

The product of the given Sine and the radius divided by the Cosine is the first result. From the first, [and then, second, third] etc., results obtain [successively] a sequence of results by taking repeatedly the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide [the above results] in order by the odd numbers one, three, etc. [to get the full sequence of terms]. From the sum of the odd terms, subtract the sum of the even terms. [The results] become the arc. In this connection, it is laid down that the [Sine of the] arc or [that of] its complement, which ever is smaller, should be taken here [as the given Sine]; otherwise, the terms obtained by the [above] repeated process will not tend to the vanishing magnitude.

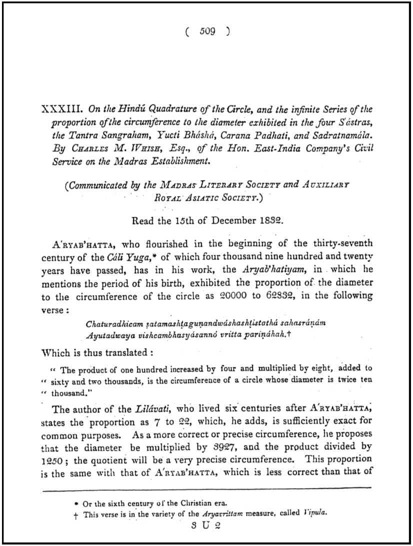
Modern interpretation:

$$R\theta = \frac{R(R\sin\theta)^1}{1(R\cos\theta)^1} - \frac{R(R\sin\theta)^3}{3(R\cos\theta)^3} + \frac{R(R\sin\theta)^5}{5(R\cos\theta)^5} - \cdots \quad (R\sin\theta < R\cos\theta)$$

Keralan series

But these results were unknown in the West until the 1830s

As we will see, the series for arctan was reproduced independently in Scotland in the 1670s



(509)

XXXIII. *On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four S'ástras, the Tantra Sangraham, Yucti Bháshá, Carana Padhati, and Sadratnamála. By* Charles M. Whish, *Esq., of the Hon. East-India Company's Civil Service on the Madras Establishment.*

(*Communicated by the* Madras Literary Society *and* Auxiliary Royal Asiatic Society.)

Read the 15th of December 1832.

A'ryab'hatta, who flourished in the beginning of the thirty-seventh century of the *Cáli Yuga*,* of which four thousand nine hundred and twenty years have passed, has in his work, the *Aryab'hatiyam*, in which he mentions the period of his birth, exhibited the proportion of the diameter to the circumference of the circle as 20000 to 62832, in the following verse :

Chaturadhicam ṣatamashṭaguṇandwáshashṭistathá sahasráṇám
Ayutadwaya vishcambhasyásannó vritta pariṇáhah.†

Which is thus translated :

"The product of one hundred increased by four and multiplied by eight, added to "sixty and two thousands, is the circumference of a circle whose diameter is twice ten "thousand."

The author of the *Lilávati*, who lived six centuries after A'ryab'hatta, states the proportion as 7 to 22, which, he adds, is sufficiently exact for common purposes. As a more correct or precise circumference, he proposes that the diameter be multiplied by 3927, and the product divided by 1250 ; the quotient will be a very precise circumference. This proportion is the same with that of A'ryab'hatta, which is less correct than that of

* Or the sixth century of the Christian era.
† This verse is in the variety of the *Aryavrittam* measure, called *Vipula*.

3 U 2

Keralan series

Warning!

It has sometimes been claimed that there must be a link between European and Keralan ideas about infinite series, because the same results occur in both places.

However, there is no documentary evidence of such a link.

In general,

$$\text{conceptual similarities} \neq \text{evidence of transmission}$$

Question: 'what are or should be the criteria for accepting a hypothesis of cross-cultural transmission as plausible, and are those criteria culturally dependent?' (Kim Plofker, *Mathematics in India*, Princeton University Press, 2009, p. 252)

Infinite series 1600–1900: an overview

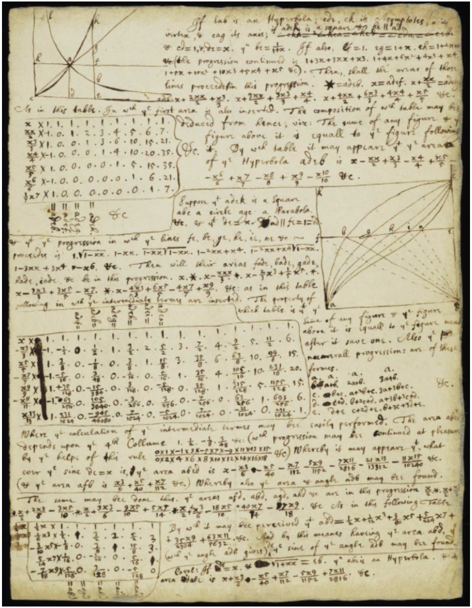
Lecture VII:

- mid–late 17th century: many discoveries
- early 18th century: much progress
- later 18th century: doubts and questions

Lecture VIII:

- early 19th century: Fourier series
- early 19th century: convergence better understood

Newton and the general binomial theorem



CUL Add. MS 3958.3, f. 72

(See lecture IV)

Recall: Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int(1+x)^n dx$. (NB. Newton did not use the notation $\int(1+x)^n dx$.)

Newton's method of extrapolation

In fact, this method extends easily to any integer n

Newton's explanation:

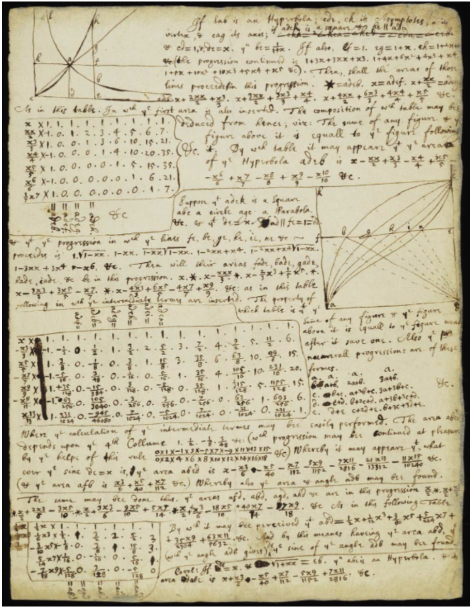
> *The property of which table is y^{t} y^{e} sum of any figure and y^{e} figure above it is equal to y^{e} figure next after it save one. Also y^{e} numerall progressions are of these forms.*

a	a	a	a	
b	$a+b$	$2a+b$	$3a+b$	
c	$b+c$	$a+2b+c$	$3a+3b+c$	$\&c.$
d	$c+d$	$b+2c+d$	$a+3b+3c+d$	
e	$d+e$	$c+2d+e$	$b+3c+3d+e$	

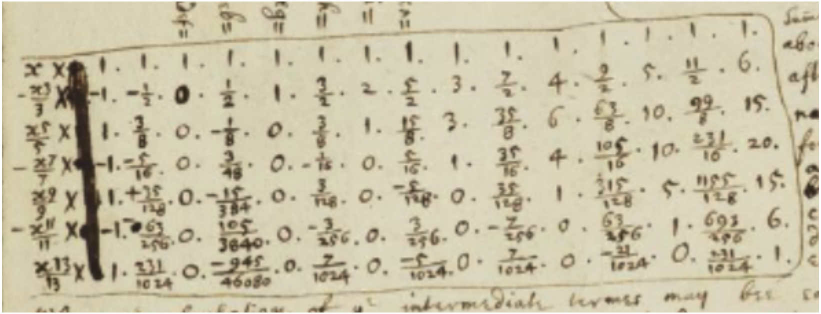
(See: *Mathematics emerging*, §8.1.1.)

Newton and the general binomial theorem

CUL Add. MS 3958.3, f. 72



Newton's method of interpolation



$x \times$	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
$-\frac{x^3}{3} \times$	-1.	$-\frac{1}{2}$.	0.	$\frac{1}{2}$.	1.	$\frac{3}{2}$.	2.	$\frac{5}{2}$.	3.	$\frac{7}{2}$.	4.	$\frac{9}{2}$.	5.	$\frac{11}{2}$.	6.
$\frac{x^5}{5} \times$	1.	$\frac{3}{8}$.	0.	$-\frac{1}{8}$.	0.	$\frac{3}{8}$.	1.	$\frac{15}{8}$.	3.	$\frac{35}{8}$.	6.	$\frac{63}{8}$.	10.	$\frac{99}{8}$.	15.
$-\frac{x^7}{7} \times$	-1.	$-\frac{5}{16}$.	0.	$\frac{3}{48}$.	0.	$-\frac{1}{16}$.	0.	$\frac{5}{16}$.	1.	$\frac{35}{16}$.	4.	$\frac{105}{16}$.	10.	$\frac{231}{16}$.	20.
$\frac{x^9}{9} \times$	1.	$+\frac{35}{128}$.	0.	$-\frac{15}{384}$.	0.	$\frac{3}{128}$.	0.	$-\frac{5}{128}$.	0.	$\frac{35}{128}$.	1.	$\frac{315}{128}$.	5.	$\frac{1155}{128}$.	15.
$-\frac{x^{11}}{11} \times$	-1.	$-\frac{63}{256}$.	0.	$\frac{105}{3840}$.	0.	$-\frac{3}{256}$.	0.	$\frac{3}{256}$.	0.	$-\frac{7}{256}$.	0.	$\frac{63}{256}$.	1.	$\frac{693}{256}$.	6.
$\frac{x^{13}}{13} \times$	1.	$\frac{231}{1024}$.	0.	$\frac{-945}{46080}$.	0.	$\frac{7}{1024}$.	0.	$\frac{-5}{1024}$.	0.	$\frac{7}{1024}$.	0.	$-\frac{21}{1024}$.	0.	$\frac{231}{1024}$.	1.

Newton's method of interpolation

	$(1-x^2)^{-1}$	$(1-x^2)^{-\frac{1}{2}}$	$(1-x^2)^0$	$(1-x^2)^{\frac{1}{2}}$	$(1-x^2)^1$	$(1-x^2)^{\frac{3}{2}}$	$(1-x^2)^2$	$\cdots$
x	1	1	1	1	1	1	1	$\cdots$
$-\frac{x^3}{3}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\cdots$
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\cdots$
$-\frac{x^7}{7}$	-1	$-\frac{5}{16}$	0	$\frac{3}{48}$	0	$-\frac{1}{16}$	0	$\cdots$
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\pm\frac{x^m}{m}$ and the column labelled $(1-x^2)^n$ is the coefficient of $\pm\frac{x^m}{m}$ in $\int(1-x^2)^n dx$.

(NB: possible slips in the last two rows of the original table)

Newton's method of interpolation

Can fill in some initial values by other methods

Newton applied the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

to fractional n, so that

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{1/2(1/2-1)}{2!} = -\frac{1}{8}$$

and so on

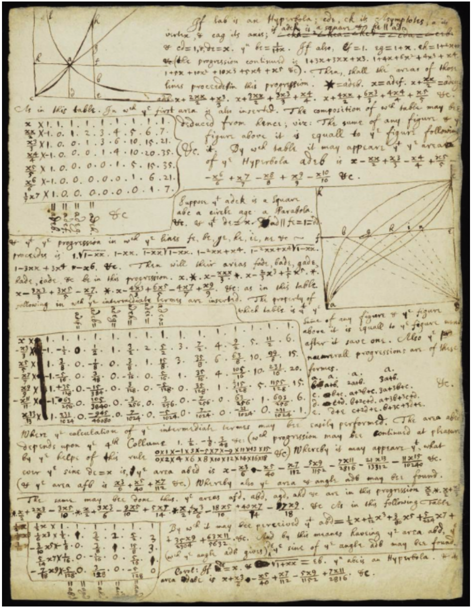
Newton's and the general binomial theorem

Newton went on to extend this method to other fractional powers, and also to $(a + bx)^n$, thereby convincing himself of the truth of the general binomial theorem — but this was not proved until the 19th century

On Newton and the binomial theorem, see https://www.youtube.com/watch?v=xv_PWwdDWDk

One more table

The table at the bottom of the page gives the interpolations for $(1+x)^n$ for half-integer n



Further discoveries by Newton

By further interpolations and integrations (based on strong geometric intuition) Newton found further series for:

- $(1 + x)^{p/q}$
- log, antilog
- sin, tan, ...   (NB: cosine was not yet much in use)
- arcsin, arctan, ...

(See: *Mathematics emerging*, §§8.1.2–8.1.3.)

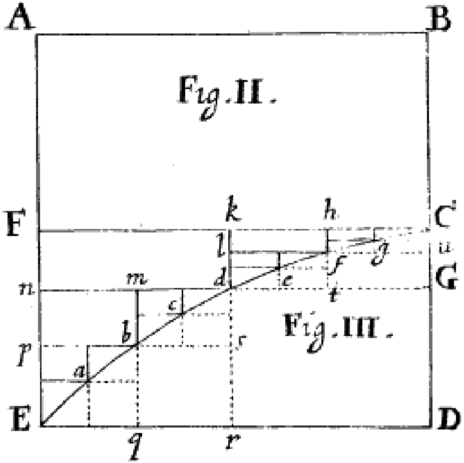
Newton on the move from finite to infinite series

> *And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.*

(*De analysi*, 1669; Derek T. Whiteside, *The mathematical papers of Isaac Newton*, CUP, 1967–1981, vol. II, p. 241)

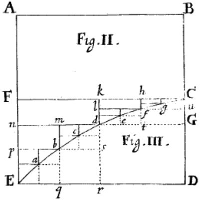
Other 17th-century discoveries (1a)

Brouncker, c. 1655, published 1668: area under the hyperbola given by $\dfrac{1}{1\times 2}+\dfrac{1}{3\times 4}+\dfrac{1}{5\times 6}+\cdots$



Other 17th-century discoveries (1b)

$$\left.\begin{aligned} \text{I say ABCdEA} &= \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \frac{1}{7\times 8} + \frac{1}{9\times 10}\ \&c. \\ \text{EdCDE} &= \frac{1}{2\times 3} + \frac{1}{4\times 5} + \frac{1}{6\times 7} + \frac{1}{8\times 9} + \frac{1}{10\times 11}\ \&c. \\ \text{EdCyE} &= \frac{1}{2\times 3\times 4} + \frac{1}{4\times 5\times 6} + \frac{1}{6\times 7\times 8} + \frac{1}{8\times 9\times 10}\ \&c. \end{aligned}\right\}\ \textit{in infinitum.}$$



And that therefore in the firſt ſeries half the firſt term is greater than the ſum of the two next, and half this ſum of the ſecond and third greater than the ſum of the four next, and half the ſum of thoſe four greater than the ſum of the next eight, *&c. in infinitum.* For $\frac{1}{2}$ *dD* = *br* + *bn*; but *bn* > *fG*, therefore $\frac{1}{2}$ *dD* > *br* + *fG*, *&c.* And in the *ſecond* ſeries half the firſt term is leſs then the ſum of the two next, and half this ſum leſs then the ſum of the four next, *&c. in infinitum.*

That the firſt *ſeries* are the *even* terms, *viz.* the 2^d^, 4^th^, 6^th^, 8^th^, 10^th^, *&c.* and the *ſecond*, the *odd*, *viz.* the 1^ſt^, 3^d^, 5^th^, 7^th^, 9^th^, *&c.* of the following ſeries, *viz.* $\frac{1}{1\times 2}$. $\frac{1}{2\times 3}$. $\frac{1}{3\times 4}$. $\frac{1}{4\times 5}$. $\frac{1}{5\times 6}$. $\frac{1}{6\times 7}$. *&c. in infinitum* = 1. Whereof *a* being put for the number of terms taken at pleaſure, $\frac{1}{a+a}$ is the laſt, $\frac{a}{a+1}$ is the ſum of all thoſe terms from the beginning, and $\frac{1}{a+1}$ the ſum of the reſt to the end.

That $\frac{1}{4}$ of the firſt terme in the *third* ſeries is leſs than the ſum of the two next, and a quarter of this ſum, leſs than the ſum of the four next, and one fourth of this laſt ſum leſs than the next eight, I thus demonſtrate.

Let a = the 3^d^ or laſt number of any term of the firſt Column, *viz.* of Diviſors,

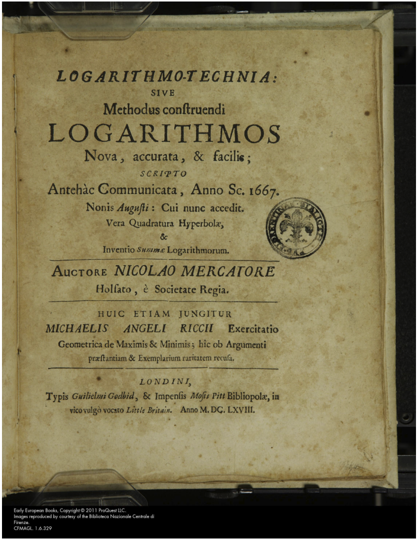
⋮

Other 17th-century discoveries (2)

Mercator's series (1668), found by long division:

$$\frac{1}{1+a} = 1 - a + aa - a^3 + a^4 \text{ (\&c.)}$$

Gives rise to series for log


LOGARITHMO-TECHNIA:
SIVE
Methodus conſtruendi
LOGARITHMOS
Nova, accurata, & facilis;
SCRIPTO
Antehàc Communicata, Anno Sc. 1667.
Nonis *Auguſti*: Cui nunc accedit.
Vera Quadratura Hyperbolæ,
&
Inventio *Summæ* Logarithmorum.

AUCTORE *NICOLAO MERCATORE*
Holſato, è Societate Regia.

HUIC ETIAM JUNGITUR
MICHAELIS ANGELI RICCII Exercitatio
Geometrica de Maximis & Minimis; hîc ob Argumenti
præſtantiam & Exemplarium raritatem recuſa.

LONDINI,
Typis *Guilielmi Godbid*, & Impenſis *Moſis Pitt* Bibliopolæ, in
vico vulgò vocato *Little Britain*. Anno M.DC.LXVIII.

Other 17th-century discoveries (3)

James Gregory (1671):

- general binomial expansion
- series for tan, sec, and others, including

$$\theta = \tan\theta - \frac{1}{2}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

Gregory to Collins, 23rd November 1670:

> *I suppose these series I send here enclosed, may have some affinity with those inventions you advertise me that Mr. Newton had discovered.*

(On Gregory's work, see: *Mathematics emerging*, §8.1.4.)

Other 17th-century discoveries (4)

Gottfried Wilhelm Leibniz (1675):

The area of a circle with unit diameter is given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$$

The error in the sum is successively less than $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, etc.

> *Therefore the series as a whole contains all approximations at once, or values greater than correct and less than correct: for according to how far it is understood to be continued, the error will be smaller than a given fraction, and therefore also less than any given quantity. Therefore the series as a whole expresses the exact value.*

(See: *Mathematics emerging*, §8.3.)

Series in the 17th century: 'convergence'

John Wallis (1656), *Arithmetica infinitorum*:

$$\square = \frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \cdots}$$

(Determined that

$$\square > \sqrt{\frac{3}{2}}, \quad \square < \frac{3}{2}\sqrt{\frac{3}{4}}, \quad \square > \left(\frac{3 \times 3}{2 \times 4}\right)\sqrt{\frac{5}{4}},$$

and so on)

Brouncker (1668): grouping of terms

Leibniz (1675): 'alternating' series

Power series in the 17th century

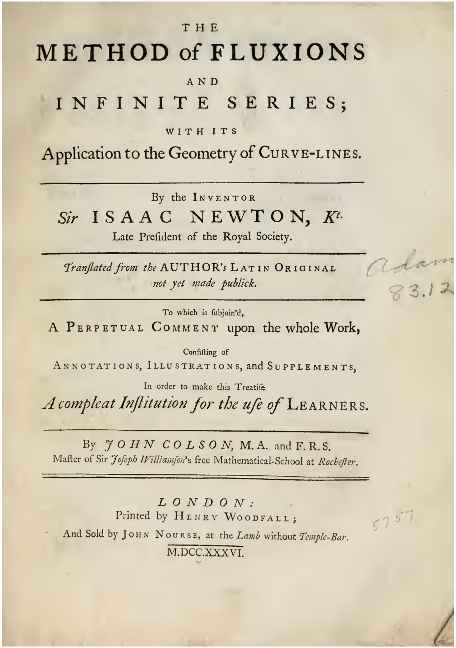
Power series (infinite polynomials):

- enabled term-by-term integration for difficult quadratures;
- helped establish sine, log, ... as ‘functions’ (transcendental);
- encouraged a move from geometric to algebraic descriptions;
- for Newton (and others) inextricably linked with calculus.

Power series rank with calculus as a major advance of the 17th century

Calculus and series combined

Newton's treatise of 1671, published 1736



THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, *Kt.*

Late Prefident of the Royal Society.

Tranflated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is fubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Confifting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatife

A compleat Inftitution for the ufe of LEARNERS.

By *JOHN COLSON*, M. A. and F. R. S.

Mafter of Sir *Jofeph Williamfon*'s free Mathematical-School at *Rochefter*.

LONDON:

Printed by HENRY WOODFALL;

And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.

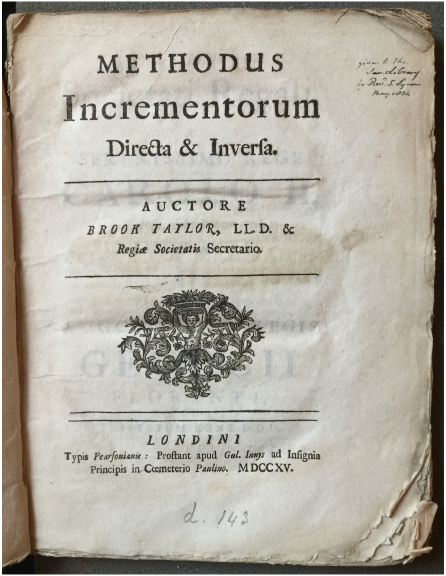
M.DCC.XXXVI.

Move on to the 18th century

Eighteenth century:

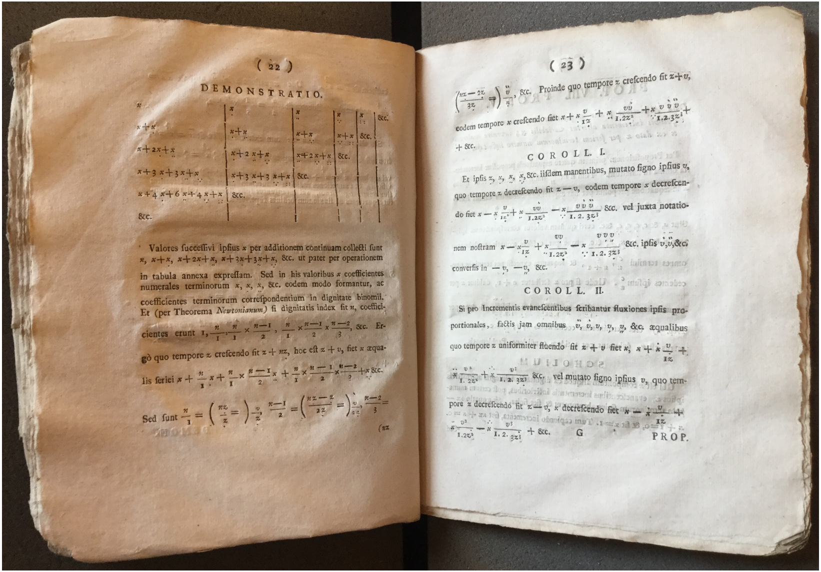
- as in 17th century, much progress;
- also many questions and doubts

Taylor series



Brook Taylor,
The method of direct and inverse increments (1715)

Taylor series



DEMONSTRATIO.

x	$\underset{.}{x}$	$\underset{..}{x}$	$\underset{\because}{x}$	$\underset{::}{x}$	&c.
$x+\underset{.}{x}$	$\underset{.}{x}+\underset{..}{x}$	$\underset{..}{x}+\underset{\because}{x}$	$\underset{\because}{x}+\underset{::}{x}$	&c.	
$x+2\underset{.}{x}+\underset{..}{x}$	$\underset{.}{x}+2\underset{..}{x}+\underset{\because}{x}$	$\underset{..}{x}+2\underset{\because}{x}+\underset{::}{x}$	&c.		
$x+3\underset{.}{x}+3\underset{..}{x}+\underset{\because}{x}$	$\underset{.}{x}+3\underset{..}{x}+3\underset{\because}{x}+\underset{::}{x}$	&c.			
$x+4\underset{.}{x}+6\underset{..}{x}+4\underset{\because}{x}+\underset{::}{x}$	&c.				
&c.					

Valores succeſſivi ipſius x per additionem continuam collecti ſunt x, $x+\underset{.}{x}$, $x+2\underset{.}{x}+\underset{..}{x}$, $x+3\underset{.}{x}+3\underset{..}{x}+\underset{\because}{x}$, &c. ut patet per operationem in tabula annexa expreſſam. Sed in his valoribus x coefficientes numerales terminorum $\underset{.}{x}$, $\underset{..}{x}$, $\underset{\because}{x}$, &c. eodem modo formantur, ac coefficientes terminorum correſpondentium in dignitate binomii. Et (per Theorema *Newtonianum*) ſi dignitatis index ſit n, coefficientes erunt $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. Ergò quo tempore z creſcendo fit $z+n\underset{.}{z}$, hoc eſt $z+v$, fiet x æqualis feriei $x+\frac{n}{1}\underset{.}{x}+\frac{n}{1}\times\frac{n-1}{2}\underset{..}{x}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\underset{\because}{x}$ &c.

Sed ſunt $\frac{n}{1}=\left(\frac{n\underset{.}{z}}{\underset{.}{z}}=\right)\frac{v}{\underset{.}{z}}$, $\frac{n-1}{2}=\left(\frac{n\underset{.}{z}-\underset{.}{z}}{2\underset{.}{z}}=\right)\frac{\overset{.}{v}}{2\underset{.}{z}}$, $\frac{n-2}{3}=$ (nz

$\left(\frac{n\underset{.}{z}-2\underset{.}{z}}{3\underset{.}{z}}=\right)\frac{\overset{..}{v}}{\underset{.}{z}}$, &c. Proinde quo tempore z creſcendo fit $z+v$, eodem tempore x creſcendo fiet $x+\underset{.}{x}\frac{v}{1\underset{.}{z}}+\underset{..}{x}\frac{v\overset{.}{v}}{1.2\underset{.}{z}^2}+\underset{\because}{x}\frac{v\overset{.}{v}\overset{..}{v}}{1.2.3\underset{.}{z}^3}+$ &c.

COROLL. I.

Et ipſis z, $\underset{.}{x}$, $\underset{..}{x}$, $\underset{\because}{x}$, &c. iiſdem manentibus, mutato ſigno ipſius v, quo tempore z decreſcendo fit $z-v$, eodem tempore x decreſcendo fiet $x-\underset{.}{x}\frac{v}{1\underset{.}{z}}+\underset{..}{x}\frac{v\overset{.}{v}}{1.2\underset{.}{z}^2}-\underset{\because}{x}\frac{v\overset{.}{v}\overset{..}{v}}{1.2.3\underset{.}{z}^3}$ &c. vel juxta notationem noſtram $x-\underset{.}{x}\frac{v}{1\underset{.}{z}}+\underset{..}{x}\frac{v\underset{,}{v}}{1.2\underset{.}{z}^2}-\underset{\because}{x}\frac{v\underset{,}{v}\underset{,,}{v}}{1.2.3\underset{.}{z}^3}$ &c. ipſis $\overset{.}{v}$, $\overset{..}{v}$, &c. converſis in $-\underset{,}{v}$, $-\underset{,,}{v}$, &c.

COROLL. II.

Si pro Incrementis evaneſcentibus ſcribantur fluxiones ipſis proportionales, factis jam omnibus $\overset{..}{v}$, $\overset{.}{v}$, v, $\underset{,}{v}$, $\underset{,,}{v}$, &c. æqualibus quo tempore z uniformiter fluendo fit $z+v$ fiet x, $x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}+\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}$ &c. vel mutato ſigno ipſius v, quo tempore z decreſcendo fit $z-v$, x decreſcendo fiet $x-\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}-\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}+$ &c.

G PROP.

(See: *Mathematics emerging*, §8.2.1.)

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt; after a further interval of δt, x has become $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$; continuing:

$$x + \frac{n}{1}\delta x + \frac{n(n-1)}{1 \cdot 2}\delta(\delta x) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\delta(\delta(\delta x)) + \cdots$$

$$= x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant
- $\delta x \propto \dot{x}$ and $\delta z \propto \dot{z}$, so in each case the former can be replaced by the latter

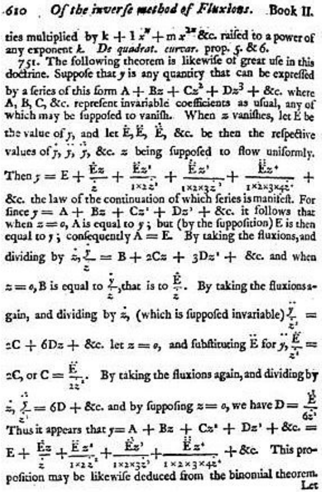
In essence (in modern terms): $\dfrac{\delta x}{\delta z} \to \dfrac{dx}{dz}$, $\dfrac{\delta(\delta x)}{(\delta z)^2} \to \dfrac{d^2x}{dz^2}$, and so on

Again in modern terms, we arrive at:

$$x + \frac{dx}{dz}v + \frac{d^2x}{dz^2}\frac{v^2}{1 \cdot 2} + \frac{d^3x}{dz^3}\frac{v^3}{1 \cdot 2 \cdot 3} + \cdots$$

Cf. Taylor's notation in *Mathematics Emerging*, §8.1.2

Maclaurin's *Treatise of fluxions*, vol. II, p. 610



ties multiplied by $k + l x^n + m x^{2n}$ &c. raiſed to a power of any exponent *k*. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewiſe of great uſe in this doctrine. Suppoſe that *y* is any quantity that can be expreſſed by a ſeries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. repreſent invariable coefficients as uſual, any of which may be ſuppoſed to vaniſh. When *z* vaniſhes, let E be the value of *y*, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the reſpective values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. *z* being ſuppoſed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times2\times3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times2\times3\times4\dot{z}^4} +$ &c. the law of the continuation of which ſeries is manifeſt. For ſince $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = o$, A is equal to *y*; but (by the ſuppoſition) E is then equal to *y*; conſequently A = E. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = o$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is ſuppoſed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = o$, and ſubſtituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by ſuppoſing $z = o$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times2\times3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times2\times3\times4\dot{z}^4} +$ &c. This propoſition may be likewiſe deduced from the binomial theorem.

Let

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

z is assumed to flow uniformly, so that $\dot{z} = \text{const}$

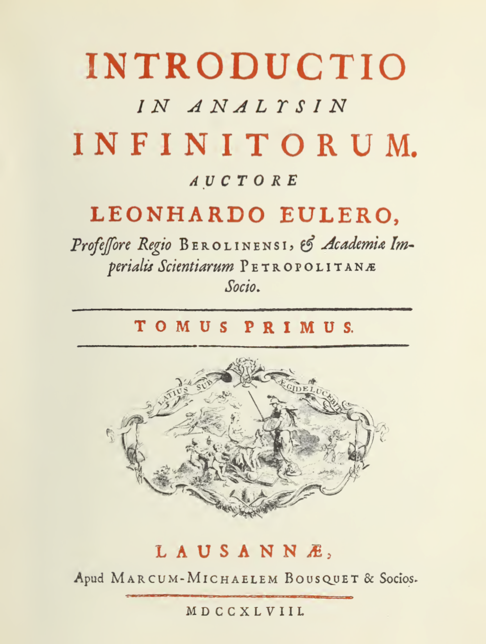
By repeatedly taking fluxions, we may calculate in turn: $A = E$, $B = \dot{E}\dot{z}$, $C = \frac{\ddot{E}}{2\dot{z}^2}$, $D = \frac{\dddot{E}}{6\dot{z}^3}$, etc.

"the law of the continuation of [the] series is manifest"

(*Mathematics emerging*, §8.2.2.)

Euler's *Introductio*

Leonhard Euler, *Introduction to analysis of the infinite* (1748)



INTRODUCTIO

IN ANALYSIN

INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

Euler's *Introductio*

Incorporated power series into the definition of a function:

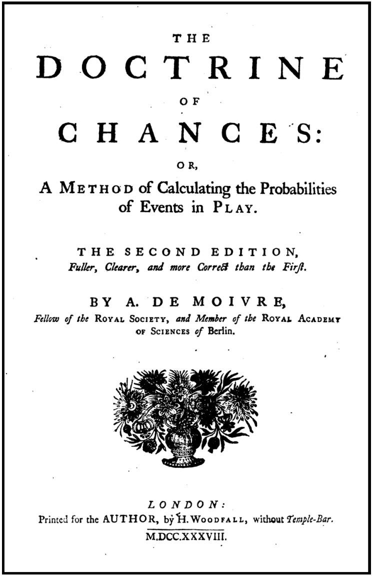
> *A* function *of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Euler derived series for sine, cosine, exp, log, etc.;

he also discovered relationships between them, for example:

$$\cos v = \frac{1}{2}(e^{iv} + e^{-iv})$$

An application of series



THE

DOCTRINE

OF

CHANCES:

OR,

A METHOD of Calculating the Probabilities of Events in PLAY.

THE SECOND EDITION,

Fuller, Clearer, and more Correct than the First.

BY A. DE MOIVRE,

Fellow of the ROYAL SOCIETY, *and Member of the* ROYAL ACADEMY OF SCIENCES *of* Berlin.

LONDON:

Printed for the AUTHOR, by H. WOODFALL, without *Temple-Bar.*

M.DCC.XXXVIII.

Abraham de Moivre posed this problem about confidence intervals:

What are the Odds that after a certain number of Experiments have been made concerning the happening or failing of Events, the Accidents of Contingency will not afterwards vary from those of Observation beyond certain Limits?

His answer involved clever (but non-rigorous) summation and manipulation of infinite series.

(*Mathematics emerging*, §7.1.3.)

Doubts

XXXV^ME MÉMOIRE.

Réflexions ſur les Suites & ſur les Racines imaginaires.

§. I.

Réflexions ſur les ſuites divergentes ou convergentes.

1. SI on éleve $1+\mu$ à la puiſſance m, le terme n^e de la ſerie ſera $\mu^{n-1} \times \frac{m(m-1)\dots(m-n+2)}{2.3.4\dots n-1}$, & le ſuivant, c'eſt-à-dire le $(n+1)^e$, ſera $\mu^n \times \frac{m(m-1)\dots(m-n+2)(m-n+1)}{2.3.4\dots n-1.n}$; donc le rapport du $(n+1)^e$ terme au n^e ſera $\frac{\mu(m-n+1)}{n}$; or pour que la ſerie ſoit convergente, il faut que ce rapport (abſtraction faite du ſigne qu'il doit avoir) ſoit $<$ que l'unité.

2. Remarquons d'abord que la formule précédente donnera le moyen de former très-promptement les termes d'une ſuite : par exemple, ſi $m = \frac{1}{2}$, il faudra multiplier le premier terme par $\mu \times \frac{1}{2}$ pour avoir le ſecond ;

D'Alembert, 1761:

... all reasoning and calculation based on series that do not converge, or that one may suppose not to, always seems to me extremely suspect, even when the results of this reasoning agree with truths known in other ways.

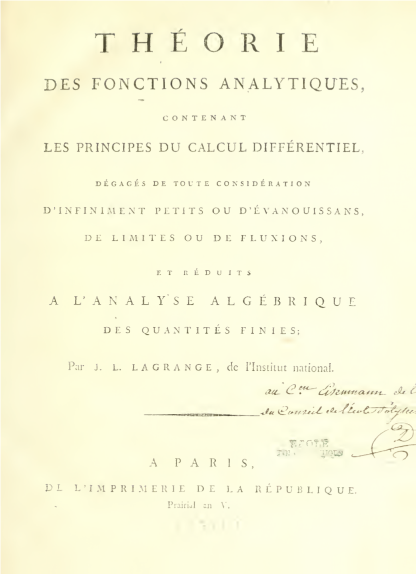
Introduced, without proof, what came to be known (in a more general setting) as d'Alembert's ratio test.

(See: *Mathematics emerging*, §8.3.1.)

Lagrange's use of series

J.-L. Lagrange, *Théorie des fonctions analytiques* (1797)

Lagrange's use of series: an attempt to liberate calculus from infinitely small quantities (essentially by treating only those functions that may be described by power series)


THÉORIE

DES FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION

D'INFINIMENT PETITS OU D'ÉVANOUISSANS,

DE LIMITES OU DE FLUXIONS,

ET RÉDUITS

A L'ANALYSE ALGÉBRIQUE

DES QUANTITÉS FINIES;

Par J. L. LAGRANGE, de l'Institut national.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

Prairial an V.

Lagrange and convergence

> *... [one needs] a way of stopping the expansion of the series at any term one wants and of estimating the value of the remainder of the series.*
>
> *This problem, one of the most important in the theory of series, has not yet been resolved in a general way*

Lagrange found bounds for the 'remainder' ...

and applied his findings to the binomial series ...

thus proving what Newton had taken for granted

(See: *Mathematics emerging*, §8.3.2.)