BO1 History of Mathematics
Lecture VIII
Establishing rigorous thinking in analysis

MT 2022 Week 4

Summary

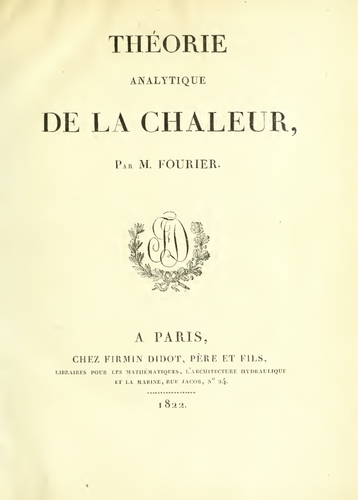
- French institutions
- Fourier series
- Early-19th-century rigour
- Limits, continuity, differentiability
- Mathematics of small quantities
- The baton passes from France to Germany

France at the turn of the 19th century

French revolution (1789) led to

- the establishment and prestige of the Grandes écoles
- a new concentration of mathematical talent in Paris: Lagrange, Laplace, Ampère, Fourier, Legendre, Poisson, Cauchy, ...
- a wealth of personal interactions
- a major new journal: *Journal de l'École polytechnique*
- and a new focus on rigour

Fourier series


THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

Joseph Fourier, *Analytic theory of heat*, 1822

See: Bernard Maurey, 'Fourier, one man, several lives', *European Mathematical Society Newsletter*, no. 113 (Sept 2019), 8–20

Fourier series

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{2}x^2\phi''(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi}\int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous and shaky (see: *Mathematics emerging*, §8.4.1).

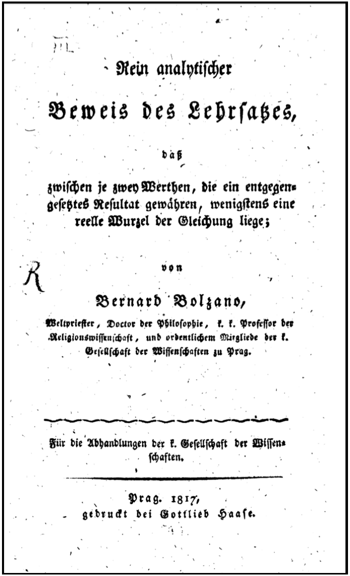
BUT it led to profound results

Establishing rigour

The development of ‘rigour’:

- Cauchy sequences
- continuity
- limits
- differentiability
- ϵ, δ notation

Cauchy sequences: Bolzano (1817)


Rein analytischer
Beweis des Lehrsatzes,
daß
zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;
von
Bernard Bolzano,
Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

Prag. 1817,
gedruckt bei Gottlieb Haase.

Bernard Bolzano, *Purely analytic proof of the theorem that between any two values which give opposite values lies at least one real root of the equation*, 1817

Cauchy sequences: Bolzano (1817)

Began with a discussion of previous proofs of the Intermediate Value Theorem (by “mathematicians of great repute”)

> *The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical propositon. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*

Cauchy sequences: Bolzano (1817)

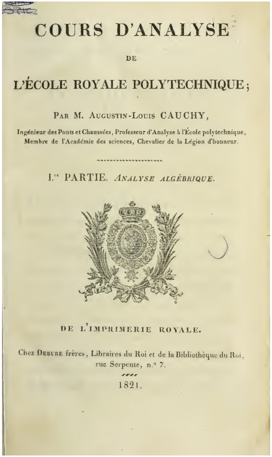
If a series of quantities has the property that the difference between its n-th term and every later one remains smaller than any given quantity ... then there is always a certain constant quantity ... which the terms of this series approach.

Proof: The hypothesis that there exists a quantity X which the terms of this series approach ... contains nothing impossible ...

(See: *Mathematics emerging*, §16.1.1; for a full translation, see: S. B. Russ, A translation of Bolzano's paper on the intermediate value theorem, *Historia Mathematica* **7**(2) (1980), 156–185)

Cauchy's *Cours d'analyse*

Augustin-Louis Cauchy, *Cours d'analyse de l'École royale polytechnique* (1821)



COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

(Annotated translation by Robert E. Bradley and C. Edward Sandifer, Springer, 2009)

Cauchy sequences: Cauchy (1821)

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit s: in other words, it is necessary and sufficient that for infinitely large values of the number n, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit s, and consequently from each other, by infinitely small quantities.*

(See: *Mathematics emerging*, §16.1.2.)

More from Cauchy (1821)

Further results from Cauchy's *Cours d'analyse*:

- ratio test;
- root test;
- alternating series test (proof uses Cauchy sequences);
- and many more.

(See *Mathematics emerging*, §16.1.2)

Cauchy sequences concluded

Early uses of Cauchy sequences:

- in da Cunha's *Principios mathematicos* (1782)
- in Bolzano's proof of the existence of a least upper bound (1817)
- in Cauchy's further results on sequences and series (1821)
- in Abel's proof of the general binomial theorem (1826)

BUT the convergence of Cauchy sequences themselves remained unproved

Continuity

Early definitions of continuity:

Wallis (1656): a curve that doesn't 'jump about'

Euler (1748): a curve described by a single expression

Later definitions of continuity:

Bolzano (1817): *$f(x+\omega) - f(x)$ can be made smaller than any given quantity, provided ω can be taken as small as we please*

Cauchy (1821): $f(x+a) - f(x)$ decreases with a

[Question: dependence? plagiarism? or a common source?]

Limits: early definitions

Wallis (1656):	a quantity ‘less than any assignable’ quantity is zero
Newton (1687):	adopted and ‘proved’ Wallis’s definition; also used ‘limit’ in the sense of a ‘bound’ or ‘ultimate value’; developed theory of ‘first and last ratios’
D’Alembert (1751):	‘one may approach a limit as closely as one wishes ... but never surpass it’; example: polygons and circle; he assumed that $\lim AB = \lim A \times \lim B$; a dictionary definition only — no theory

Limits: a later definition

Cauchy, *Cours d'analyse* (1821), p. 4:

> *When the values successively given to a variable approach indefinitely to a fixed value, so as to finish by differing from it by as little as one would wish, the latter is called the <u>limit</u> of all the others.*

Examples:

- an irrational number is a limit of rationals;
- in geometry a circle is a limit of polygons.

BUT still no formal definition of

- 'as small as one wishes',
- 'as closely as one wishes', ...

Differentiability: early ideas

For Leibniz and his immediate followers, any 'function' you could write down was automatically differentiable (by the usual rules).

For Lagrange, the 'Taylor' series

$$f(x+h) = f(x) + f'(x)h + \cdots$$

led naturally to consideration of

$$\frac{f(x+h) - f(x)}{h}$$

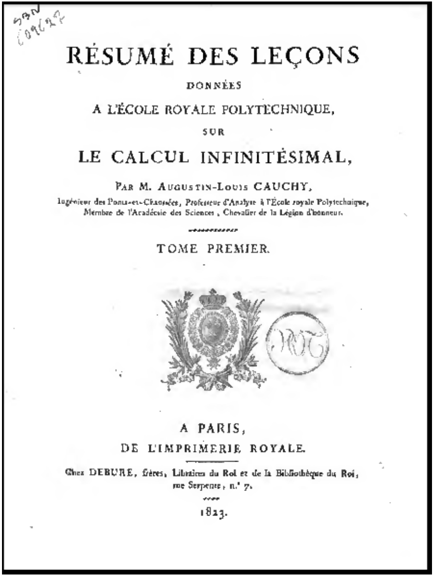
as an approximation to $f'(x)$, for small h

Ampère (1806) struggled with the meaning of

$$\frac{f(x+h) - f(x)}{h}$$

— why isn't it just zero or infinite?

Differentiability: Cauchy's *Résumé*



RÉSUMÉ DES LEÇONS
DONNÉES
A L'ÉCOLE ROYALE POLYTECHNIQUE,
SUR
LE CALCUL INFINITÉSIMAL,
PAR M. AUGUSTIN-LOUIS CAUCHY,
Ingénieur des Ponts-et-Chaussées, Professeur d'Analyse à l'École royale Polytechnique,
Membre de l'Académie des Sciences, Chevalier de la Légion d'honneur.

TOME PREMIER.

A PARIS,
DE L'IMPRIMERIE ROYALE.
Chez DEBURE, frères, Libraires du Roi et de la Bibliothèque du Roi,
rue Serpente, n.° 7.
1823.

Cauchy, *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, 1823

(Translation by Dennis M. Cates, Fairview Academic Press, 2012)

Differentiability: Cauchy's *Résumé*

> *... those who read my book will I hope be convinced that the principles of the differential calculus and its most important applications can easily be set out without the use of series.*

Defined the derivative as the limit of

$$\frac{f(x+h)-f(x)}{h}$$

with many particular examples: ax, a/x, $\sin x$, $\log x$, ...

but no concerns about existence in general

(See: *Mathematics emerging*, §14.1.4.)

Arbitrarily small intervals

A theorem of Lagrange (1797):

> *If the first derived function of a function f is strictly positive on an interval $[a, b]$, then $f(b) > f(a)$.*

Proof: Divide the interval $[a, b]$ into n subintervals, taking n as large as necessary ...

Unconvincing to modern eyes, but a useful technique.

(See: *Mathematics emerging*, §11.2.3.)

IVT revisited

Cauchy, *Cours d'analyse* (1821), Note III, p. 460 (On the numerical solution of equations):

Theorem: Let f be a real function of the variable x, which remains continuous with respect to this variable between the limits $x = x_0$, $x = X$. If the two quantities $f(x_0)$, $f(X)$ are of opposite signs, the equation $f(x) = 0$ will be satisfied by one or more real values of x contained between x_0 and X.

(See: *Mathematics emerging*, §11.2.6.)

IVT revisited

Cauchy's proof:

Choose $m > 1$. Divide the interval $[x_0, X]$ into m equal parts; find neighbouring division points x_1, X' such that $f(x_1)$, $f(X')$ are of opposite signs. Subdivide the interval $[x_1, X']$ into m equal parts; find neighbouring division points x_2, X'' such that $f(x_2)$, $f(X'')$ are of opposite signs. Continue in this way to obtain an increasing sequence $x_0,\ x_1, \ldots$ and a decreasing sequence $X,\ X', \ldots$. The difference $X^{(n)} - x_n$ is $(X - x_0)/m^n$, which may be made as small as one wishes. The sequences $x_0,\ x_1, \ldots$ and $X,\ X', \ldots$ therefore converge to a common limit a, at which $f(a) = 0$.

Note: Cauchy offered this as a fast method of approximation to roots of equations.

But it also provides a much more convincing proof of the Intermediate Value Theorem than that appearing earlier in Cauchy's text (*Cours d'analyse*, Ch. II, Theorem 4: p. 44).

ε and δ appear

A theorem of Cauchy, *Résumé* (1823):

Suppose that in the interval $[x_0, X]$ we have $A < f'(x) < B$. Then we also have

$$A < \frac{f(X) - f(x_0)}{X - x_0} < B$$

Proof: Choose two quantities $\epsilon, \ \delta, \ldots$ such that for $i < \delta$

$$f'(x) - \epsilon < \frac{f(x+i) - f(x)}{i} < f'(x) + \epsilon$$

etc.

(See: *Mathematics emerging*, §14.1.5.)

Hints of a broader class of functions

If a Taylor series exists for a given function, and all the coefficients vanish, then surely the function itself must vanish . . .

However, Cauchy gave the example $f(x) = e^{-x^2} + e^{-x^{-2}}$, which is clearly never zero, but all of its derivatives vanish

So not every function can be expanded into a Taylor series, and it appears to be possible to conceive of functions to which the calculus is not immediately or naturally applicable . . .

Modern rigour in analysis

Karl Weierstrass (1815–1897):

- taught at University of Berlin from 1856 onwards
- completed the rigorisation of calculus via systematic use of ε/δ methods

BUT we have no direct sources, only lecture notes or books by his pupils and followers

From France to Germany

By the later 19th century the mathematical centre of gravity in Europe had moved from the Parisian Écoles to the German universities:

Göttingen (est. 1734):	Gauss, Dirichlet, [Dedekind], Riemann, Klein, Hilbert, ...
Berlin (est. 1810):	Crelle (editor), Dirichlet, Eisenstein, Kummer, [Jacobi], Kronecker, Weierstrass, ...

with a focus on both research and teaching.