Exercise sheet 1. Week 4. Chapters 1-4.

Q1. (1) Describe the Zariski topology of k.

(2) Show that the Zariski topology of k^2 is not the product topology of $k \times k = k^2$.

Solution. (1) The non empty closed sets of k are the vanishing sets of polynomials in one variable, so the non empty closed sets are precisely the finite subsets of k.

(2) Let $W \subseteq k^2$ the vanishing set of the polynomial $P(x, y) := xy - 1$. We will show that if U and V are open sets in k, then $U \times V \cap W \neq \emptyset$. This shows that the open subset $k^2 \backslash W$ is not a union of open subsets of the type $U \times V$, and so the Zariski topology on k^2 is not the product topology. By (1), we have $U = k\backslash A$, where A is finite (resp. $V = k\backslash B$, where B is finite). We need to show that $xy-1$ vanishes at some point of $U \times V$. For any $a \neq 0$, we have $P(a, 1/a) = 0$. As a runs through $U\backslash\{0\}$, the function $1/a$ takes infinitely many values (since U is infinite), so for some $a_0 \in U$ we have $1/a_0 \in V$. We then have $(a_0, 1/a_0) \in W$ and $(a_0, 1/a_0) \in U \times V$. Hence $W \cap U \times V \neq \emptyset$.

Q2. Let $V \subseteq k^n$ be an algebraic set. Show that V is the disjoint union of two non empty algebraic sets in k^n iff there are two non-zero finitely generated reduced k-algebras T_1 and T_2 and an isomorphism of k-algebras $T_1 \oplus T_2 \simeq \mathcal{C}(V)$.

Solution. Suppose that V is the disjoint union of two non empty algebraic sets V_1 and V_2 in k^n. Let $I_1 := \mathcal{I}(V_1) \subseteq \mathcal{C}(V)$ and let $I_2 := \mathcal{I}(V_1) \subseteq \mathcal{C}(V)$ be the radical ideals in $\mathcal{C}(V)$ corresponding to V_1 and V_2. The intersection $I_1 \cap I_2$ consists of the regular functions on V which vanish on both V_1 and V_2 and thus on all of V. Thus $I_1 \cap I_2 = 0$. On the other hand, if $f : V \to k$ is a regular function, then the function f_1 which is 0 on V_2 and equal to f on V_1 is a regular function by Proposition 4.5. Similarly, the function f_2 which is 0 on V_1 and equal to f on V_2 is a regular function. By construction, we have $f_1 \in I_2$, $f_2 \in I_1$ and $f_1 + f_2 = f$. We conclude that $I_1 + I_2 = \mathcal{C}(V)$. We might now apply the Chinese remainder theorem to conclude that $\mathcal{C}(V) \simeq \mathcal{C}(V)/I_1 \oplus \mathcal{C}(V)/I_2$.

Conversely, suppose that $\mathcal{C}(V) \simeq R_1 \oplus R_2$, where R_1 and R_2 are finitely generated reduced k-algebras. Let $I_1 := \{(a, 0) \,|\, a \in R_1\}$ and $I_2 := \{(0, a) \,|\, a \in R_2\}$. We clearly have $I_1 \cap I_2 = 0$ and $I_1 + I_2 = \mathcal{C}(V)$. Also, I_1 and I_2 are easily seen to be radical. Hence V is the disjoint union of $Z(I_1)$ and $Z(I_2)$ by Lemma 2.8 and the following discussion.

Q3. Let $V \subseteq k^3$ be the set

$$V := \{(t, t^2, t^3) \,|\, t \in k\}.$$

Show that V is an algebraic set and that it is isomorphic to k as an algebraic set. Provide generators for $\mathcal{I}(V)$.

Solution. We have $V = Z((x_2 - x_1^2, x_3 - x_1^3)$ so V is an algebraic set. If we let $A : k \to k^3$ be the polynomial map such that $A(t) := (t, t^2, t^3)$ and $B : k^3 \to k$ be the polynomial map such that $B(x_1, x_2, x_3) = x_1$, then $A(k) \subseteq V$, $B|_V \circ A = \mathrm{Id}_k$, $A \circ B|_V = \mathrm{Id}_V$ so A and $B|_V$ are regular maps from k to V and from V to k respectively, which are inverse to each other. So they gives an isomorphism between V and k. We still have to provide generators for $\mathcal{I}(V)$. For this, consider the map of k-algebras $\phi : k[x] \to k[x_1, x_2, x_3]/(x_2 - x_1^2, x_3 - x_1^3)$ sending $x \to x_1$ (resp. the map of k-algebras $\psi : k[x_1, x_2, x_3]/(x_2 - x_1^2, x_3 - x_1^3) \to k[x]$ sending x_1 to x, x_2 to x^2, x_3 to x^3). By construction, these maps are inverse to each other and thus the ideal $(x_2 - x_1^2, x_3 - x_1^3)$ is prime and in particular radical. So $\mathcal{I}(V) = (x_2 - x_1^2, x_3 - x_1^3)$.

Q4. (1) Let $V \subseteq k^2$ be the set of solutions of the equation $y = x^2$. Show that V is isomorphic to k as an algebraic set.

(2) Let $V \subseteq k^2$ be the set of solutions of the equation $xy = 1$. Show that V is not isomorphic to k as an algebraic set.

(3) [difficult] (optional) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $V \subseteq k^2$ be the set of zeroes of $P(x, y)$. Show that V is isomorphic to one of the algebraic sets defined in (1) and (2).

Solution. (1) This is similar to Q3, with $\{(t, t^2) \,|\, t \in k\}$ instead of V.

(2) If V is isomorphic to k then $k[x, y]/(xy - 1) \simeq k[t]$ by Theorem 3.7. Now note that x is a unit in $k[x, y]/(xy-1)$ by construction and that x is not in the image of k in $k[x, y]/(xy-1)$. Since the only units of $k[t]$ lie in the image of k, there is thus no isomorphism $k[x, y]/(xy - 1) \simeq k[t]$.

(3) See

math.stackexchange.com/questions/3577406/affine-conics-over-an-algebraically-closed-field-of-char-2

Q5. Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let $\psi : V \to W$ be a regular map.

(1) Show that $\psi(V)$ is dense in W iff the map of rings $\psi^* : \mathcal{C}(W) \to \mathcal{C}(V)$ is injective.

(2) Show that ψ^* is surjective iff $\psi(V)$ is closed and the induced map $V \to \psi(V)$ is an isomorphism of algebraic sets.

Solution. (1) By definition of the closure of a set, $\psi(V)$ is dense in W iff any closed subset of W containing $\psi(V)$ is W. In view of Lemma 2.8 (and the following discussion) and Lemma 3.4, we thus see that $\psi(V)$ is dense in W iff I is any radical ideal of $\mathcal{C}(W)$ contained in $\psi^*(\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Spm}(\mathcal{C}(V))$ then I is the 0 ideal. Since $\cap_{\mathfrak{m}\in\mathrm{Spm}(\mathcal{C}(V))}\psi^*(\mathfrak{m})$ is a radical ideal by construction, we thus see that $\psi(V)$ is dense in W iff $\cap_{\mathfrak{m}\in\mathrm{Spm}(\mathcal{C}(V))}\psi^*(\mathfrak{m}) = 0$. Now we have

$$\cap_{\mathfrak{m}\in\mathrm{Spm}(\mathcal{C}(V))}\psi^*(\mathfrak{m}) = \psi^*(\cap_{\mathfrak{m}\in\mathrm{Spm}(\mathcal{C}(V))}\mathfrak{m}) = \psi^*(0) = \ker(\psi^*)$$

where the equality before last holds because $\mathcal{C}(V)$ is a reduced Jacobson ring. The equivalence follows.

(2) Suppose that ψ^* is surjective. Then ψ^* induces an isomorphism $\mathcal{C}(W)/\ker(\psi^*) \simeq \mathcal{C}(V)$ and thus the map $\mathrm{Spm}(\psi^*)$ is injective and its image is the set of maximal ideals of $\mathcal{C}(W)$ which contain $\ker(\psi^*)$. This proves that $\psi(V)$ is closed and that the induced map $V \to \psi(V)$ is bijective. Furthermore, its shows that $\mathcal{I}(\psi(V)) = \ker(\psi^*)$ (note that $\ker(\psi^*)$ is radical since $\mathcal{C}(V)$ is reduced). To summarise, the maps of algebraic sets $\psi(V) \to V$ and $V \to W$ give a diagram of surjective maps of k-algebras

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\psi^*} & \mathcal{C}(V) \\ \downarrow & & \\ \mathcal{C}(\psi(V)) & & \end{array}$$

where the kernels of the two maps are equal. Hence $\mathcal{C}(\psi(V))$ is isomorphic to $\mathcal{C}(V)$ as a $\mathcal{C}(W)$-algebra. In particular there is an isomorphism of k-algebras $\mathcal{C}(\psi(V)) \to \mathcal{C}(V)$ making the diagram commutative. Using Theorem 3.7, this gives an isomorphism $V \xrightarrow{\sim} \psi(V)$ which is compatible with the maps $V \to W$ and $\psi(V) \to W$.

Now suppose that $\psi(V)$ is closed and that the induced map $V \to \psi(V)$ is an isomorphism of algebraic sets. Let $I := \mathcal{I}(\psi(V)) \subseteq \mathcal{C}(W)$. By Theorem 3.7 the map ψ^* factors through $\mathcal{C}(W)/I$ and the induced map $\mathcal{C}(W)/I \to \mathcal{C}(V)$ is an isomorphism. In particular ψ^* is surjective.

Q6. Let $V \subseteq k^3$ be the algebraic set described by the ideal $(x^2-yz, xz-x)$. Show that V has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Solution. Treat x, y, z as variable elements of k. If $x \neq 0$, then $z = 1$ and $y = x^2$. Also we have $\langle 0, 0, 1\rangle \in V$ and hence $Z((x^2 - yz, xz - x)) \supseteq \{\langle x, x^2, 1\rangle \,|\, x \in k\}$. We have $\{\langle x, x^2, 1\rangle \,|\, x \in k\} = Z((y - x^2, z - 1))$ and the first projection gives an isomorphism between this algebraic set and k. Hence $\{\langle x, x^2, 1\rangle \,|\, x \in k\}$ is an irreducible algebraic set in k^3, which is contained in V. If $x = 0$, then the simultaneous solutions of the equation $x^2 = yz$ and $xz = x$ are precisely the solutions of the equation $yz = 0$. The simultaneous solutions of $yz = 0$ and $x = 0$ consist of the union of the z-axis and the y-axis. So V is the union of the z-axis, the y-axis and $\{\langle x, x^2, 1\rangle \,|\, x \in k\}$, which are all three irreducible. One easily checks that none of these three sets are contained in the union of the two others, so they are the irreducible components of V.

Q7. Let $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic subsets. Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be open subsets. View V_0 and W_0 as open subvarieties of V and W respectively. For $i \in \{1, \dots, t\}$ let $\pi_i : k^t \to k$ be the projection on the i-coordinate. Let $\psi : V_0 \to W_0$ be a map. Show that ψ is a morphism of varieties iff $\pi_i \circ \psi$ is a regular function on V_0 for all $i \in \{1, \dots, t\}$.

Solution. The direction $\Rightarrow$ of the stated equivalence is clear, since compositions of regular maps are regular and the projections π_i are polynomial maps (and regular functions are regular maps with target k). For the direction $\Leftarrow$ of the stated equivalence, recall that by Proposition 4.5 a function is regular on V_0 iff it is regular when restricted to any member of an open covering of V_0. Now choose an open covering of V_0 by open subsets of the form $V\backslash Z(f)$, where $f \in \mathcal{C}(V)$ (this exists by Lemma 4.1). The set $V\backslash Z(f)$ is the image of a regular injective map of algebraic sets $T \to V$, such that a function on $V\backslash Z(f)$ is regular iff its composition with the map $T \to V$ is regular (by Lemma 4.3). Hence we may suppose wrog that $V_0 = V$ (effectively replacing V_0 by T). If $V_0 = V$ and $\pi_i \circ \psi$ is a regular function on V for all $i \in \{1, \dots, t\}$, then the induced map $V \to W$ is by definition the restriction of a polynomial map $k^n \to k^t$ and is thus a regular map. Since $W_0 \subseteq W$ is open, the map ψ is thus automatically regular.

Q8. (optional) Show that the open subvariety $k^2\backslash\{0\}$ of k^2 is not affine.

Solution. Let $f : k^2\backslash\{0\} \to k$ be a regular function. Then the restriction of f to the complement of the x-axis is of the form $P(x, y)/x^n$ by Corollary 4.4. Similarly, the restriction to the complement of the y-axis is of the form $Q(x, y)/y^m$. Dividing by powers of x (resp. y), we may assume that P is not divisible by x (resp. Q is not divisible by y). Now we have $P(x, y)/x^n = Q(x, y)/y^m$ for all $x, y \neq 0$ and thus $Q(x, y)x^n = P(x, y)y^m$ (as polynomials) since k is infinite. Since $k[x, y]$ is a UFD, we deduce that $m = 0$ and that $n = 0$ and hence that $Q = P$. Thus $f(a, b) = P(a, b)$ for all a, b. In other words, the regular maps $k^2\backslash\{0\} \to k$ are all restrictions of polynomial maps $k^2 \to k$. Now suppose for contradiction that $k^2\backslash 0$ is affine. We have just seen that the natural map $\mathcal{C}(k^2) \to \mathcal{C}(k^2\backslash\{0\})$ is surjective and so $k^2\backslash\{0\}$ is closed in k^2 by Q5 (2). But this is a contradiction, so $k^2\backslash\{0\}$ is not affine.