ANALYTIC TOPOLOGY

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These are my lecture notes for in-person lectures in Michaelmas Term 2022. Videos are also available. The videos are not meant to be a run-through of the lecture notes. They're intended to flesh out bits of material that might be more difficult; some of the easier material is omitted. I do intend to go through everything in the actual lectures.

However, in the lectures I may deviate from the notes from time to time and will certainly add ad-libs, and these notes obviously also do not include questions from the audience and attempted answers by the lecturer!

I may edit these notes from time to time.

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Everything in the lectures or on the problem sheets is on the syllabus and examinable, unless otherwise indicated.*

Prerequisites: an introductory course in topology is assumed; knowledge of set theory and logic would be helpful, but is not presupposed.

0. What is topology?

Definition 0.1. *$\langle X, \mathscr{T} \rangle$ is a* topological space *iff $\mathscr{T} \subseteq \wp X$, and*

1. $\varnothing, X \in \mathscr{T}$
2. $U, V \in \mathscr{T} \Rightarrow U \cap V \in \mathscr{T}$
3. $\mathscr{U} \subseteq \mathscr{T} \Rightarrow \bigcup_{U \in \mathscr{U}} U \in \mathscr{T}$

These axioms are *complex*—especially axiom 3.—we cannot get to grips with topological spaces by looking at points one by one, as we can with a group.†

Bad: topologies are hard to understand—so we often find ways to avoid having to look at a whole topology: bases, homotopy groups, etc.

Good: The topological property of *connectedness* is the part of the "essence" of $\mathbb{R}$ that first order logic can't describe. Topology is powerful.

The axioms are *simple*.

Bad: They don't in themselves say much. So topological spaces are hugely varied—topology is classification rather than theory.

Good: Topologies are everywhere, wherever you have a notion of closeness: in geometry, analysis, computer science, etc. With topology, the world is your oyster.

How do we approach the diversity of topology?

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* Anything in the footnotes is not on the syllabus, and may refer to notions from, for example, logic and set theory which are not assumed known in the main text.

† The notion of complexity referred to here is logical: that topology is not first order. For instance, any ordered field which is connected in the order topology is isomorphic to $\mathbb{R}$ and so has cardinality $2^{\aleph_0}$; so by the Löwenheim Skolem Theorem, the concept of connectedness cannot be described in first order.

Connectedness of a total order without endpoints could be expressed in second order logic as follows:

$$\forall U \Bigg(\bigg(\Big(\forall x \left(Ux \to \exists y \exists z \left(y < x \,\&\, x < z \,\&\, \forall w \left((y < w \,\&\, w < z) \to Uw \right) \right) \right) \\ \wedge \forall x \left(\neg Ux \to \exists y \exists z \left(y < x \,\&\, x < z \,\&\, \forall w \left((y < w \,\&\, w < z) \to \neg Uw \right) \right) \right) \Big) \bigg) \\ \to \left((\forall x \, Ux) \vee (\forall x \, \neg Ux) \right) \Bigg).$$

We will proceed cautiously, starting with familiar, "nice" spaces like $[0,1]$ and $\mathbb{R}$, and ask, given a criterion of niceness:

1. What properties do nice spaces have?
2. What will guarantee that a space is nice?

Nice may be compact, metrisable, etc.

1. Separation Axioms

Topologies often arise where we have intuitive ideas of closeness and separation. We put these ideas under a microscope.

Our basic idea is: separation by means of open sets.

1.1. Basic separation axioms.

Definition 1.1.1. *A topological space X is T_0 iff whenever x and y are distinct points of X, there is an open set U such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.*

In a T_0 space, the topology is able to distinguish between points.

Definition 1.1.2. *A topological space X is T_1 iff whenever x and y are distinct points of X, there is an open set U such that $x \in U$ and $y \notin U$.*

Lemma 1.1.3. *X is T_1 iff every point of X is closed.*

Proof: $\Rightarrow$) Let $x \in X$. Let $y \neq x$. Then there is open V such that $y \in V$ and $x \notin V$. So $X \setminus \{x\}$ is open, so $\{x\}$ is closed.

$\Leftarrow$) Let $x \neq y$. Then let $U = X \setminus \{y\}$. Then U is open, $x \in U$, and $y \notin U$, as required. $\square$

Remember: U is open iff $X \setminus U$ is closed; and U is open iff for all $x \in U$, there is open W such that $x \in W \subseteq U$.

☡ *Never* say "U is open iff not closed"—this is FALSE (think about $[0,1)$ in $\mathbb{R}$—neither open nor closed).

Definition 1.1.4. *X is T_2, or Hausdorff, iff whenever $x \neq y \in X$, there exists open U, V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

Definition 1.1.5. *X is regular iff, whenever $x \in X$ and $C \subseteq X$ is closed, and $x \notin C$, there exist open U, V such that $x \in U$, $C \subseteq V$ and $U \cap V = \varnothing$.*

X is T_3 iff T_1 and regular.

Definition 1.1.6. *X is normal iff, whenever C and D are disjoint and closed in X, there exist open U and V such that $C \subseteq U$, $D \subseteq V$, and $U \cap V = \varnothing$.*

X is T_4 iff T_1 and normal.

Theorem 1.1.7. *$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.*

Proof: Easy exercise. $\square$

1.2. More about Normality

THEOREM 1.2.1. *Every T_2 compact space is regular (hence T_3, by 1.1.7).*

PROOF: This is question 6 on problem sheet 0. □

THEOREM 1.2.2. *Any T_3 compact space is normal (hence T_4, by Theorem 1.1.7.)*

PROOF: On problem sheet. □

COROLLARY 1.2.3. *If X is compact, X is T_2 iff T_3 iff T_4.*

PROOF: Theorems 1.1.7, 1.2.1 and 1.2.2. □

THEOREM 1.2.4. *If X is T_i (for $i \leq 3$) and Y is a subspace of X, then Y is T_i also.*

PROOF: Exercise. □

DEFINITION 1.2.5. *Two subsets A and B of a space X are* functionally separated *iff there is continuous $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.*

LEMMA 1.2.6. *X is normal iff, for every closed $C \subseteq U$ open, there exists open V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

PROOF: Problem sheets. □

THEOREM 1.2.7. *(Urysohn's Lemma) Let X be a normal space, and let C, D be disjoint closed subsets of X. Then C and D are functionally separated.*

PROOF: We construct a separating function with our bare hands.

Noting that $\mathbb{Q} \cap (0,1)$ is countable, write

$$\mathbb{Q} \cap (0,1) = \{r_n : n \in \mathbb{N}\},$$

with $r_0 = 1$, $r_1 = 0$. Now construct open sets U_q, for $q \in \mathbb{Q} \cap [0,1]$, by recursion, so that if $q < q'$, then $\overline{U_q} \subseteq U_{q'}$.

(1) Let $W = X \setminus D$, so that W is open and $C \subseteq W$.

Let $C \subseteq U_1 \subseteq \overline{U_1} \subseteq W$.

(2) Let $C \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$.

(3) Suppose we have constructed U_1, U_0 and also $U_{r_2}, \ldots, U_{r_n}$. We now construct $U_{r_{n+1}}$, as follows.

Writing out 0, r_2, …, r_n and 1 in order of size, let a_n be the member of this set next before r_{n+1} and let b_n be the one after.

Now let $\overline{U_{a_n}} \subseteq U_{r_{n+1}} \subseteq \overline{U_{r_{n+1}}} \subseteq U_{b_n}$.

Observe that if $r = r_i < r_{n+1}$, then $r \leq a_n$, so $\overline{U_r} \subseteq \overline{U_{a_n}}$, so $\overline{U_r} \subseteq U_{r_{n+1}}$; and if $r = r_i > r_{n+1}$, then $r \geq b_n$, so $U_{b_n} \subseteq U_r$, so $\overline{U_{r_{n+1}}} \subseteq U_r$.

So the inductive hypothesis is preserved.

Having constructed all the U_r, we now build f.

Define

$$f = \inf\bigl(\{1\} \cup \{r : x \in U_r\}\bigr).$$

We note

(1) If $x \in C$, then $x \in U_0$; so $f(x) = 0$. Hence $f(C) \subseteq \{0\}$.

(2) If $x \in D$, then for all r, $x \notin U_r$. So $f(x) = 1$. So $f(D) \subseteq \{1\}$.

(3) For all x, $0 \le f(x) \le 1$, so $f : X \to [0,1]$.

(4) Is f continuous?

a) We show that for each q, $f^{-1}(-\infty, q)$ is open, that is, forall $x \in f^{-1}(-\infty, q)$, there exists open $W \ni x$ such that $W \subseteq f^{-1}(-\infty, q)$.

Well, $x \in f^{-1}(-\infty, q)$ iff $f(x) < q$ iff $\inf\big(\{1\} \cup \{r : x \in U_r\}\big) < q$ iff *either* $q > 1$—so $f^{-1}(-\infty, q) = X$, which is open—*or* there exists $r < q$ such that $x \in U_r$, which is enough.

b) We now show that for all q, $f^{-1}(q, \infty)$ is open. Well, $x \in f^{-1}(q, \infty)$ iff $f(x) > q$ iff $\inf\big(\{1\} \cup \{r : x \in U_r\}\big) > q$.

So, in particular, $q < 1$; and *either* $q < 0$, when $f^{-1}(q, \infty) = X$, which is open, *or* $q \ge 0$. So, find $r > q$ such that $r \in \mathbb{Q} \cap [0,1]$ and $x \notin U_r$.

Now find $s \in \mathbb{Q} \cap (q, r)$.

Then $U_s \subseteq \overline{U_s} \subseteq U_r$. Now $x \notin U_r$, so $x \notin \overline{U_s}$, so $x \in X \setminus \overline{U_s}$, which is open.

Also, if $y \in X \setminus \overline{U_s}$, then $y \notin U_s$, so for all $t \le s$, $y \notin U_t$. Hence $f(y) \ge s > q$. That is, $x \in X \setminus \overline{U_s} \subseteq f^{-1}(q, \infty)$.

I now claim that f is continuous. I will need a little machinery to show this.

Definition 1.2.8. *$\mathscr{C}$ is a* subbasis *for $\mathscr{T}$ iff the collection of all finite intersections of elements of $\mathscr{C}$ is a basis. If $\mathscr{C}$ is a subbasis for $\mathscr{T}$, say $\mathscr{T}$ is* generated by *$\mathscr{C}$.*

Lemma 1.2.9. *$f : X \to Y$ is continuous iff for every U in some subbasis $\mathscr{C}$ for Y, $f^{-1}(U)$ is open.*

Proof: Let U be a finite intersection of elements of $\mathscr{C}$; say $U = \bigcap_{i=1}^n W_i$, $W_i \in \mathscr{C}$. Then $f^{-1}(W_i)$ is open for all i; and $f^{-1}(U) = f^{-1}(\bigcap_{i=1}^n W_i) = \bigcap_{i=1}^n f^{-1}(W_i)$ which is open. □

Lemma 1.2.10. *$\{(-\infty, q) : q \in \mathbb{R}\} \cup \{(q, \infty) : q \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}$.*

Proof: The set of all open intervals is a basis; and

$$(a, b) = (-\infty, b) \cap (a, \infty).$$

□

This observation completes the proof of Urysohn's Lemma. □

1.3. Subspaces of normal spaces: the Tychonoff property

Definition 1.3.1. *A topological space is said to be* completely regular *if, whenever C is a closed set, and $x \notin C$, $\{x\}$ and C are functionally separated (see Definition 1.2.5).*

It is Tychonoff, *or $T_{3\frac{1}{2}}$, if it is completely regular and T_1.*

Theorem 1.3.2. *Every completely regular space is regular.*

Proof: Let C be closed, $x \notin C$. Let $f : X \to [0,1]$ be such that $f(x) = 0$ and $f(C) \subseteq \{1\}$.

Let $U = f^{-1}(-\infty, \frac{1}{2})$ and $V = (\frac{1}{2}, \infty)$. Then $x \in U$, $C \subseteq V$, U and V are open, and $U \cap V = \varnothing$. □

Theorem 1.3.3. *Suppose X is T_4, $Y \subseteq X$. Then (in the subspace topology) Y is $T_{3\frac{1}{2}}$.*

PROOF: By Theorem 1.2.4, Y is T_1. We show Y is completely regular.

Let C be closed in Y, $x \in Y \setminus C$.

Then there exists D closed in X such that $C = D \cap Y$; and $x \notin D$, for $x \in Y$ and $x \notin D \cap Y$.

Since X is T_1, $\{x\}$ is closed.

By Urysohn's Lemma (Theorem 1.2.7), $\{x\}$ and D are functionally separated. So, find $g : X \to [0,1]$ which is continuous such that $g\{x\} = \{0\}$ and $g(D) \subseteq \{1\}$.

Let $f = g \restriction Y$. Then f is continuous, and, clearly, $f\{x\} = \{0\}$ and $f(C) \subseteq \{1\}$, as required. □

Sadly, the analogue of Theorem 1.2.4 for T_4 is FALSE (see problem sheet 1).

1.4. Separation axioms: Examples

EXAMPLE 1.4.1. (The Sierpiński Space). Let $X = \{0,1\}$, and let

$$\mathscr{T} = \{\varnothing, \{1\}, \{0,1\}\}.$$

Then $\langle X, \mathscr{T} \rangle$ is T_0, but not T_1, since $\{1\}$ isn't closed.*

EXAMPLE 1.4.2. (Sequence with two limits) Let $X = \mathbb{N} \cup \{a,b\}$, and let $\mathscr{T}$ be the topology generated by the following sets:

(a) Any $Y \subseteq \mathbb{N}$,

(b) Any set of the form $(n,\infty) \cup \{a\}$, or $(n,\infty) \cup \{b\}$ (intervals in $\mathbb{N}$).

$\langle X, \mathscr{T} \rangle$ is T_1, but not T_2, since we cannot find disjoint open sets $U \ni a$ and $V \ni b$.

EXAMPLE 1.4.3. A space which is $T_{3\frac{1}{2}}$ and not T_4. (Modified Tychonoff Plank)

Let $W_0 = \mathbb{N} \cup \{\omega\}$, with a topology generated by

(a) All subsets of $\mathbb{N}$, and

(b) all sets of the form $\{\omega\} \cup (n, \omega)$.

Then W_0 is compact T_2: in fact, it is a convergent sequence.

Let $W_1 = Z \cup \{*\}$, where Z is some uncountable set, with a topology generated by

(a) All subsets of Z, and

(b) all sets of the form $\{*\} \cup (Z \setminus F)$, where F is finite.

Then W_1 is compact T_2.

So $W_0 \times W_1$ is compact T_2, so it is T_4 by Corollary 1.2.3.

Let $X = W_0 \times W_1 \setminus \{\langle \omega, * \rangle\}$. Then X, as a subspace of a T_4 space, is $T_{3\frac{1}{2}}$ (Theorem 1.3.3).

However X is not normal.

For, let $C = \{\omega\} \times Z$, $D = \mathbb{N} \times \{*\}$.

C and D are disjoint closed; we show that they cannot be separated by open sets.

For, suppose $C \subseteq U$ open, $D \subseteq V$ open.

Well, if $\langle n, * \rangle \in D \subseteq V$, then, by definition of the product topology, there exists finite $F_n \subseteq Z$ such that

$$\{\langle n, * \rangle\} \cup \{n\} \times (Z \setminus F_n) \subseteq V.$$

* Spaces that are T_0 but not T_1 are important in the theory of partial orders, and have applications in logic, computer science, etc.

Let $F = \bigcup_{n\in\mathbb{N}} F_n$; then F is countable.

Then, $Z \setminus F = \bigcap_{n\in\mathbb{N}} Z \setminus F_n \subseteq Z \setminus F_n$ for all n, so for each n,

$$\{\langle n, *\rangle\} \cup \{n\} \times (Z \setminus F) \subseteq V.$$

That is, $\mathbb{N} \times (Z \setminus F) \subseteq V$.

Now, if $z \in Z \setminus F$, so $\langle \omega, z\rangle \in C$, then there exists n such that $\{\langle \omega, z\rangle\} \cup (n, \infty) \times \{z\} \subseteq U$.

Then $\langle n+1, z\rangle \in U \cap V$, so $U \cap V \neq \varnothing$.

We will be meeting $T_{3\frac{1}{2}}$ spaces later!

2. Compactness, connectedness and convergence

We take some nice properties of $[0, 1]$, and see how they behave: in particular, we show that any product of compact spaces is compact (Tychonoff's Theorem). We also work out how to talk about convergence in general topological spaces.

2.1. Covering properties

We look at compactness a little more closely.

Definition 2.1.1. *A space $\langle X, \mathscr{T}\rangle$ is* Lindelöf *iff every open cover has a countable subcover (sc. if $\mathscr{U} \subseteq \mathscr{T}$ and $\bigcup \mathscr{U} = X$, then there exists a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcup \mathscr{V} = X$.)*

Definition 2.1.2. *A space $\langle X, \mathscr{T}\rangle$ is* countably compact *iff every countable open cover has a finite subcover.*

Theorem 2.1.3. *Every Lindelöf, countably compact space is compact.*

Proof: Trivial! □

Theorem 2.1.4. *A T_1 space is countably compact iff every infinite subset has a limit point.*

Proof: On the problem sheets. □

Definition 2.1.5. *$D \subseteq X$ is* dense *iff $\overline{D} \subseteq X$ (iff each non-empty open subset hits D).*

X is separable *iff it has a countable dense subset.*

Theorem 2.1.6. *Any Lindelöf metric space is separable.*

Proof: Problem sheets. □

Theorem 2.1.7. *Any separable metric space is Lindelöf.*

Proof: Problem sheets. □

Theorem 2.1.8. *A metric space is separable iff it has a countable basis.*

Corollary 2.1.9. *$\mathbb{R}$ is Lindelöf.*

Proof: $\mathbb{Q}$ is a countable dense set. □

2.2. Convergence, and filters

In a metric space, $x \in \overline{A}$ iff there is a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ on A converging to x. This isn't true in general. So we need a more general idea of convergence.

EXAMPLE 2.2.1. Let Y be an uncountable set, let $X = Y \cup \{*\}$, and let $\mathscr{T}$ be the topology generated by the following sets:

(a) Any subset of Y,

(b) Any set $\{*\} \cup (Y \setminus C)$, where C is countable.

Then (1) $* \in \overline{Y}$.

For, let $U \ni *$ be any open set. Then for some countable C,

$$\{*\} \cup (Y \setminus C) \subseteq U,$$

so $U \cap Y \neq \varnothing$.

But (2) there is no sequence on Y converging to $*$.

For, let $\langle x_n \mid n \in \mathbb{N} \rangle$ be a sequence on Y. Let $C = \{x_n \mid n \in \mathbb{N} \rangle$, and let $U = \{*\} \cup (Y \setminus C)$.

Then U is open round $*$, but $U \cap \{x_n \mid n \in \mathbb{N}\} = \varnothing$, so $x_n \not\to *$.

So what do we do?

EXAMPLE 2.2.2. (An attempt to make convergence look difficult!)

Let X be a space containing a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ converging to a point x. We define a set $\mathscr{F}$ of subsets of X, containing all tails of the sequence, as follows:

$$Y \in \mathscr{F} \text{ iff } x_n \in Y, \text{ for all but finitely many } n.$$

Now notice that if U is open and contains x, then U contains all but finitely many x_n, so $U \in \mathscr{F}$.

A "small" element of $\mathscr{F}$ is like a "far-right-hand end" of the sequence $\langle x_n \mid n \in \mathbb{N} \rangle$—as the x_n get closer and closer to x, so also the small elements of $\mathscr{F}$ are concentrated close to x. Note that $\mathscr{F}$ has the following properties:

1. $\varnothing \notin \mathscr{F}$, $X \in \mathscr{F}$.
2. $F, G \in \mathscr{F} \Rightarrow F \cap G \in \mathscr{F}$.
3. $F \in \mathscr{F}$, $F \subseteq G \Rightarrow G \in \mathscr{F}$.

DEFINITION 2.2.3. *$\mathscr{F}$ is a* filter on X *iff $\mathscr{F} \subseteq \wp X$, and*

1. $\varnothing \notin \mathscr{F}$, $X \in \mathscr{F}$.
2. $F, G \in \mathscr{F} \Rightarrow F \cap G \in \mathscr{F}$.
3. $F \in \mathscr{F}$, $F \subseteq G \Rightarrow G \in \mathscr{F}$.

DEFINITION 2.2.4. *Let X be a topological space, $x \in X$. Then the* neighbourhood filter *of x, written $\mathscr{N}_x$, is the set of all $F \subseteq X$ such that $x \in F^\circ$.*

PROPOSITION 2.2.5. *$\mathscr{N}_x$ is indeed a filter.*

Proof: Exercise. $\square$

Definition 2.2.6. *Let X be a topological space, $x \in X$, and $\mathscr{F}$ be a filter on X.*

Then we say $\mathscr{F}$ converges *to x, written $\mathscr{F} \to x$, iff $\mathscr{N}_x \subseteq \mathscr{F}$.*

We say x is a cluster point *of x iff for all $F \in \mathscr{F}$, $x \in \overline{F}$.*

There are a number of ways in which filter convergence is like the usual sort.

Proposition 2.2.7. *X is Hausdorff iff every filter on x converges to at most one point.*

Proof: Problem sheets. $\square$

Theorem 2.2.8. *If X is compact, then every filter has a cluster point.*

Proof: Let $\mathscr{F}$ be a filter.

Let

$$\overline{F} = \{\overline{F} \mid F \in \mathscr{F}\}.$$

Since $\mathscr{F}$ is closed under finite intersections, so does $\overline{\mathscr{F}}$. But recall that in a compact space, any family of closed sets which is closed under finite intersections has non-empty intersection.

Let $x \in \bigcap \overline{\mathscr{F}}$. Well, then $x \in \overline{F}$ for each $F \in \mathscr{F}$, so x is a cluster point of $\mathscr{F}$. $\square$

In doing ordinary convergence, we often want to take subsequences, because they are more likely to converge to a single point. The corresponding idea in filter convergence is to add more sets to a filter: to make it bigger.

The ultimate end-point in such a process is to extend the filter to an *ultrafilter:*

Definition 2.2.9. *Suppose $\mathscr{U}$ is a filter on X. Then $\mathscr{U}$ is an* ultrafilter *iff for all $A \subseteq X$, either $A \in \mathscr{U}$, or $X \setminus A \in \mathscr{U}$.*

Proposition 2.2.10. *Let $\mathscr{U}$ be a filter on X. Then $\mathscr{U}$ is an ultrafilter iff it is a maximal filter (that is, if $\mathscr{F} \supseteq \mathscr{U}$ is a filter, then $\mathscr{F} = \mathscr{U}$).*

Proof: $\Rightarrow$) Suppose $\mathscr{U}$ is an ultrafilter, and $\mathscr{F} \supset \mathscr{U}$ is a filter.

Then there exists $F \in \mathscr{F} \setminus \mathscr{U}$.

Now either $F \in \mathscr{U}$ or $X \setminus F \in \mathscr{U}$, by Definition 2.2.9. Since $F \notin \mathscr{U}$, $X \setminus F \in \mathscr{U}$.

Now by Definition 2.2.3, clause 2, $F \cap (X \setminus F) = \varnothing \in \mathscr{F}$.

But this contradicts Defn 2.2.3 clause 1.

$\Leftarrow$) Suppose $\mathscr{U}$ is not an ultrafilter. Then there exists A such that A, $X \setminus A \notin \mathscr{U}$.

Let

$$\mathscr{F} = \{F \subseteq X \mid \exists U \in \mathscr{U} U \cap A \subseteq F\}.$$

Then $\mathscr{F} \supset \mathscr{U}$, since $X \cap A \in \mathscr{F}$, and $\mathscr{F}$ is a filter, which we can confirm by checking the clauses of Definition 2.2.3:

(1) $X \supseteq X \cap A$; and if $F \in \mathscr{F}$, then $F \cap A \neq \varnothing$, otherwise $X \setminus A \in \mathscr{F}$ by clause (3), so $\varnothing \notin \mathscr{F}$.

(2) Suppose $F_1 \supseteq U_1 \cap A$, and $F_2 \supseteq U_2 \cap A$, U_1, $U_2 \in \mathscr{U}$. Then

$$F_1 \cap F_2 \supseteq (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A;$$

so $F_1 \cap F_2 \in \mathscr{F}$.

(3) is obvious.

So $\mathscr{U}$ is not a maximal filter. □

Any filter can be refined to a maximal filter:

THEOREM 2.2.11. *Let $\mathscr{F}$ be a filter on X. Then there is an ultrafilter $\mathscr{U}$ such that $\mathscr{U} \supseteq \mathscr{F}$.*

To prove this, we need from Set Theory:

FACT 2.2.12. *(Zorn's Lemma) Let $\mathscr{A}$ be a family of sets such that whenever $\mathscr{C} \subseteq \mathscr{A}$ is a non-empty* chain *(that is, C_1, $C_2 \in \mathscr{C}$ implies either $C_1 \subseteq C_2$, or $C_2 \subseteq C_1$), then $\bigcup \mathscr{C} \in \mathscr{A}$.*

Then $\mathscr{A}$ has a maximal element; that is, there exists $A \in \mathscr{A}$ such that if $B \in \mathscr{A}$ and $B \supseteq A$, then $B = A$.

PROOF OF THEOREM 2.2.11: Let

$$\mathscr{A} = \{\mathscr{G} \mid \mathscr{G} \text{ is a filter on } X \ \& \ \mathscr{G} \supseteq \mathscr{F}\}.$$

By Proposition 2.2.10, any maximal element of $\mathscr{A}$ is an ultrafilter extending $\mathscr{F}$.

We use Fact 2.2.12 (Zorn's Lemma) to find a maximal element of $\mathscr{A}$.

We check that ZL is applicable.

Suppose that $\mathscr{C} \subseteq \mathscr{A}$ is a chain. We check that $\bigcup \mathscr{C} \in \mathscr{A}$ also.

We confirm that $\bigcup \mathscr{C}$ is a filter by checking the conditions of Definition 2.2.3.

(1) $\varnothing \notin \bigcup \mathscr{C}$, since $\varnothing \notin \mathscr{G}$ for any $\mathscr{G} \in \mathscr{C}$.

$X \in \bigcup \mathscr{C}$, since $X \in \mathscr{G}$ for all $\mathscr{G} \in \mathscr{C}$.

(2) Suppose F, $G \in \bigcup \mathscr{C}$. Then there exist $\mathscr{G}_1$, $\mathscr{G}_2 \in \mathscr{C}$ such that $F \in \mathscr{G}_1$ and $G \in \mathscr{G}_2$. Since $\mathscr{C}$ is a chain, $\mathscr{G}_1 \subseteq \mathscr{G}_2$ or $\mathscr{G}_2 \subseteq \mathscr{G}_1$. Either way, there is $\mathscr{G}_i$ such that F, $G \in \mathscr{G}_i$. $\mathscr{G}_i$ is a filter, so $F \cap G \in \mathscr{G}_i$, so $F \cap G \in \bigcup \mathscr{C}$.

(3) Suppose $F \in \bigcup \mathscr{C}$ and $F \subseteq G$. Well, there is some $\mathscr{G} \in \mathscr{C}$ such that $F \in \mathscr{G}$. Then $G \in \mathscr{G}$ also, so $G \in \bigcup \mathscr{C}$.

Also of course, $\mathscr{F} \subseteq \bigcup \mathscr{C}$, since $\mathscr{F} \subseteq \mathscr{G}$ for any $\mathscr{G} \in \mathscr{C}$.

So indeed $\bigcup \mathscr{C} \in \mathscr{A}$ whenever $\mathscr{C}$ is a chain on $\mathscr{A}$.

So by ZL, $\mathscr{A}$ has a maximal element, which is an ultrafilter extending $\mathscr{F}$. □

THEOREM 2.2.13. *Let $\mathscr{U}$ be an ultrafilter on X, let $x \in X$. Then $\mathscr{U} \to x$ iff x is a cluster point of $\mathscr{U}$.*

PROOF: $\Rightarrow$) Suppose that $\mathscr{U} \to x$. Suppose that $F \in \mathscr{U}$, and that $U \in \mathscr{N}_x$. Then since $\mathscr{U} \to x$, $U \in \mathscr{U}$. Now $\mathscr{U}$ is a filter, so $U \cap F \in \mathscr{U}$. Thus we can see that if $F \in \mathscr{U}$, then $x \in \overline{F}$, as required.

$\Leftarrow$) Suppose $\mathscr{U} \not\to x$.

Then $\mathscr{N}_x \nsubseteq \mathscr{U}$. Let $U \in \mathscr{N}_x \setminus \mathscr{U}$. Then by Definition 2.2.9, $X \setminus U \in \mathscr{U}$.

But U is a neighbourhood of x and $U \cap (X \setminus U) = \varnothing$, so $x \notin \overline{X \setminus U}$.

Hence $\mathscr{U}$ does not cluster at x.

COROLLARY 2.2.14. *If X is compact, every ultrafilter on X converges.*

PROOF: Theorem 2.2.13 and Theorem 2.2.8. □

THEOREM 2.2.15. *If every ultrafilter on X converges, then X is compact.*

PROOF: Let $\mathscr{V}$ be an open cover with no finite subcover. Let $\mathscr{F} = \{F \subseteq X : \exists$ a finite set $U_1, \ldots, U_n \in \mathscr{V}$ such that $X \setminus \bigcup_{i=1}^n U_i \subseteq F\}$.

Then $\mathscr{F}$ is a filter, for, checking the conditions of Definition 2.2.3,

(1) $\varnothing \notin \mathscr{F}$, since $\mathscr{V}$ has no finite subcover. $X \in \mathscr{F}$ trivially.

(2) If $F_1 \supseteq X \setminus \bigcup_{i=1}^{n_1} U_i^1 \in \mathscr{F}$, and $F_2 \supseteq X \setminus \bigcup_{i=1}^{n_2} U_i^2 \in \mathscr{F}$, then

$$F_1 \cap F_2 \supseteq X \setminus \left(\bigcup_{i=1}^{n_1} U_i^1 \cup \bigcup_{i=1}^{n_2} U_i^2 \right) \in \mathscr{F}$$

also.

(3) is trivial.

Now extend $\mathscr{F}$ to an ultrafilter $\mathscr{U}$.

If $\mathscr{U} \to x$, then $\mathscr{N}_x \subseteq \mathscr{U}$. But $\mathscr{V}$ is an open cover, so there exists $V \in \mathscr{V}$ such that $x \in V$. Then $V \in \mathscr{N}_x$. But $X \setminus V \in \mathscr{F} \subseteq \mathscr{U}$. So by clause (2), $V \cap (X \setminus V) = \varnothing \in \mathscr{U}$. But this contradicts clause (1).

So $\mathscr{U}$ is an ultrafilter that does not converge, as required. □

Finally,

THEOREM 2.2.16. *If $f : X \to Y$ is an onto function and $\mathscr{U}$ is an ultrafilter on X, then*

$$f(\mathscr{U}) = \{f(U) : U \in \mathscr{U}\}$$

is an ultrafilter on Y.

PROOF: We check the clauses of Definition 2.2.3.

(1) If $U \in \mathscr{U}$, then $U \neq \varnothing$, so $f(U) \neq \varnothing$; so $\varnothing \notin f(U)$.

$Y = f(X) \in f(\mathscr{U})$, since $X \in \mathscr{U}$.

(3) If $F = f(U) \in f(\mathscr{U})$ and $G \supseteq F$, then $f^{-1}(G) \supseteq f^{-1}(F) \supseteq U$, so $f^{-1}(G) \in \mathscr{U}$. So $G = f\big(f^{-1}(G)\big) \in f(\mathscr{U})$.

(2) If $F_1 = f(U_1)$ and $F_2 = f(U_2)$, and $U_1, U_2 \in \mathscr{U}$, then $U_1 \cap U_2 \in \mathscr{U}$, so

$$\begin{aligned} f(U_1 \cap U_2) &= \{f(x) \mid x \in U_1 \cap U_2\} \\ &\subseteq \{y \mid y \in f(U_1) \cap f(U_2)\} \\ &= f(U_1) \cap f(U_2). \end{aligned}$$

So $f(U_1) \cap f(U_2) \in f(\mathscr{U})$.

Finally, $f(\mathscr{U})$ is an ultrafilter, because if $A \subseteq Y$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$; since $\mathscr{U}$ is an ultrafilter, one of $f^{-1}(A)$ and $X \setminus f^{-1}(A)$ belongs to $\mathscr{U}$; and now since f is onto, $f(f^{-1}(A)) = A$ and $f(f^{-1}(Y \setminus A)) = Y \setminus A$, so one of A and $Y \setminus A$ belongs to $f(\mathscr{U})$. □

THEOREM 2.2.17. *If $f : X \to Y$ is onto, then f is continuous at $x \in X$ iff whenever $\mathscr{U}$ is an ultrafilter on X converging to x, $f(\mathscr{U}) \to f(x)$.*

PROOF: Problem sheets. □

2.3. Infinite Products and Tychonoff's Theorem

Recall that the product topology on a product $X \times Y$ is generated by products $U \times V$, where U is open in X and V is open in Y.

Equivalently, by products $U \times Y$ and $X \times V$, for U open in X and V open in Y.

Equivalently, by ${\pi_Y}^{-1}(U)$, ${\pi_X}^{-1}(V)$.

Equivalently, it is the coarsest topology such that π_X and π_Y are continuous.

Recall also that if X and Y are compact, then so is $X \times Y$.

We seek to generalise the above to infinite products.

Definition 2.3.1. *Let $\langle X_\lambda : \lambda \in \Lambda \rangle$ be a family of sets. The* Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$ *is the set of all functions f with domain Λ such that $f(\lambda)$ ("the λth coordinate of f") is in X_λ.*

If $\lambda \in \Lambda$, the λth projection mapping π_λ *is the function $\pi_\lambda : \prod_{\lambda \in \Lambda} X_\lambda$ such that $\pi_\lambda(f) = f(\lambda)$—picking out the λth coordinate.*

Definition 2.3.2. *Let $\big\langle \langle X_\lambda, \mathscr{T}_\lambda \rangle : \lambda \in \Lambda \big\rangle$ be a family of topological spaces. We define their* Tychonoff product $\langle X, \mathscr{T} \rangle$ *such that*

1. $X = \prod_{\lambda \in \Lambda} X_\lambda$,

2. $\mathscr{T}$ is the coarsest topology such that all π_λ are continuous.

Equivalently, *$\mathscr{T}$ is generated by all ${\pi_\lambda}^{-1}(U_\lambda)$, such that $\lambda \in \Lambda$ and U_λ is open in X_λ. Note that*

$${\pi_\lambda}^{-1}(U_\lambda) = U_\lambda \times \prod_{\mu \neq \lambda} X_\mu;$$

these sets form a subbasis.

Equivalently, *$\mathscr{T}$ is generated by the basis of all sets $\prod_{\lambda \in \Lambda} U_\lambda$, where*

a) each U_λ is open in X_λ,

b) for all but finitely many λ, $U_\lambda = X_\lambda$.

It is not the case that every subset of a product $X \times Y$ is of the form $A \times B$.

It is also not the case that every *open* subset of a product $X \times Y$ is of the form $U \times V$.

It is very much not the case that every *closed* subset of a product $X \times Y$ is an intersection of rectangles of the form $C \times D$.

It is a nice exercise to find counterexamples to all these from $\mathbb{R}^2$.

Theorem 2.3.3. *If $i \leq 3\frac{1}{2}$ and each X is T_i, then their Tychonoff product is T_i.*

Proof: Omitted. (See problem sheet.) □

Theorem 2.3.4. *(Tychonoff's Theorem) Suppose $\langle X_\lambda : \lambda \in \Lambda \rangle$ is a family of non-empty spaces. Then $\prod_{\lambda \in \Lambda} X_\lambda$ is compact iff X_λ is compact for every λ.* *

Proof: $\Rightarrow$) Since π_λ is continuous, and onto, X_λ is the continuous image of a compact space and is therefore compact.

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* Tychonoff's Theorem is in fact equivalent to the Axiom of Choice. It is a nice exercise to try to prove this.

$\Leftarrow$) Suppose all X_λ are compact. We show that $\prod_{\lambda\in\Lambda} X_\lambda$ is compact by showing that every ultrafilter converges, and appealing to Theorem 2.2.15.

Let $\mathscr{U}$ be an ultrafilter on $\prod_{\lambda\in\Lambda} X_\lambda$.

Then, for each λ, $\pi_\lambda(\mathscr{U})$ is an ultrafilter on X_λ. X_λ is compact, so $\pi_\lambda(\mathscr{U})$ converges to some point. We define a function f on Λ such that $f(\lambda)$ is some element of X_λ to which $\pi_\lambda(\mathscr{U})$ converges.

So $f \in \prod_{\lambda\in\Lambda} X_\lambda$.

We show that $\mathscr{U} \to f$.

We need to show that $\mathscr{N}_f \subseteq \mathscr{U}$.

So, let N be a neighbourhood of f; and let $\prod_{\lambda\in\Lambda} U_\lambda$ be a basic open set such that $f \in \prod_{\lambda\in\Lambda} U_\lambda \subseteq N$.

By Definition 2.3.2, $U_\lambda = X_\lambda$ for all but finitely many λ. Say $U_\lambda = X_\lambda$ unless $\lambda = \lambda_1, \dots, \lambda_n$.

Now $U_\lambda = \pi_\lambda(\prod_{\lambda\in\Lambda} U_\lambda)$, and so for each $i = 1, \dots, n$, since $\pi_{\lambda_i}(\mathscr{U}) \to f(\lambda_i)$, U_{λ_i}, which is a neighbourhood of $f(\lambda_i)$, belongs to $\pi_{\lambda_i}(\mathscr{U})$.

Let us say $U_{\lambda_i} = \pi_{\lambda_i}(V_{\lambda_i})$, where $V_{\lambda_i} \in \mathscr{U}$.

Then $V_{\lambda_i} \subseteq {\pi_{\lambda_i}}^{-1}(U_{\lambda_i}) = U_{\lambda_i} \times \prod\limits_{\mu\neq\lambda_i} X_\mu$.

By clause (3) in the definition of a filter at Definition 2.2.3,

$$U_{\lambda_i} \times \prod_{\mu\neq\lambda_i} X_\mu \in \mathscr{U}.$$

By clause (2),

$$\begin{aligned} \mathscr{U} \ni \bigcap_{i=1}^{n} \left(U_{\lambda_i} \times \prod_{\mu\neq\lambda_i} X_\mu \right) \\ = \prod_{i=1}^{n} U_{\lambda_i} \times \prod_{\mu\neq\lambda_1,\dots,\lambda_n} X_\mu \\ = \prod_{\lambda\in\Lambda} U_\lambda. \end{aligned}$$

So now, since $\prod_\lambda U_\lambda \subseteq N$, by Definition 2.2.3 clause (3), $N \in \mathscr{U}$ as required. □

2.4. Compactifications

Compact T_2 spaces are nice. But nearly as nice are subspaces of such.

Recall (Theorems 1.2.1 and 1.2.2) that any compact T_2 space is T_4; and (Theorem 1.3.3) that any subspace of a T_4 space is $T_{3\frac{1}{2}}$.

So only $T_{3\frac{1}{2}}$ spaces have a hope of being embeddable in a compact T_2 space. How far can we go in the other direction?

We formalise the concept that we are after.

Definition 2.4.1. *Let X be a space. A* Hausdorff compactification *of X is a pair $\langle h, Y\rangle$ such that*

1. *Y is a compact T_2 space,*
2. *$h : X \to Y$ has the following properties:*
   a) h is one-to-one,
   b) h is a homeomorphism from X to $h(X)$,
   c) $h(X)$ is dense in Y (ie. $\overline{h(X)} = Y$).

Often one identifies X with its image under h, so simply imagines X as sitting *inside* a compactification.

We now define a condition designed to guarantee the existence of a compactification.

Definition 2.4.2. *A topological space is* locally compact *iff for all $x \in X$ and open $U \ni x$, there exists open V and compact K such that $x \in V \subseteq K \subseteq U$.*

To make life slightly easier:

Proposition 2.4.3. *A T_2 space X is locally compact iff for all $x \in X$, there exists open $W \ni x$ such that $\overline{W}$ is compact.*

Proof: $\Rightarrow$) Use local compactness to find V open and K compact such that $x \in V \subseteq K$. Since X is Hausdorff, K is closed. So $\overline{V} \subseteq K$. But K is compact, so $\overline{V}$ closed implies that $\overline{V}$ is compact.

$\Leftarrow$) Trivial. □

Definition 2.4.4. *Let X be a Hausdorff locally compact non-compact space. Then the* Alexandroff one-point compactification *of X is the pair $\langle h, \alpha X\rangle$ defined as follows:*

1. *αX is a topological space with points $X \cup \{*\}$, such that U is open iff*
   a) U is open in X, or
   b) $* \in U$, and $X \setminus U$ is compact.
2. *$h : X \to X \cup \{*\}$ is the identity.*

Example 2.4.5. $X = \mathbb{R}^2$.

Theorem 2.4.6. *Suppose X is a Hausdorff locally compact non-compact space. Then the Alexandroff one-point compactification is a Hausdorff compactification.*

Proof: Quite obviously, h is one-to-one. onto its image.

X is a subspace, since if $X \setminus U$ is compact, then $X \setminus U$ is closed in X because X is Hausdorff, so $U \cap X$ is open in X. Hence h is a homeomorphism onto its image.

X is dense in αX, because if U is any non-empty open set, *either* $U \subseteq X$—so $U \cap X \neq \varnothing$,—*or* $X \setminus U$ is compact. But X is not compact, so $X \neq X \setminus U$, so $X \cap U \neq \varnothing$.

Now we show that αX is a Hausdorff compactification.

αX is T_2 Let x and y be different points of αX.

Case 1: x, $y \in X$. Well X is T_2: find U and V disjoint open in X such that $x \in U$ and $y \in V$. Then U and V are also open in αX.

Case 2: $x \in X$, $y = *$. Find U open in X such that $x \in U$, and $\overline{U}$ is compact. Let $V = \alpha X \setminus \overline{U}$. Then U and V are open in αX, $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

αX is compact Let $\mathscr{U}$ be an open cover. We find a finite subcover.
First find $U_0 \in \mathscr{U}$ such that $* \in U_0$.
By definition of the topology, $X \setminus U_0$ is compact.
Now find $U_1, \ldots, U_n \in \mathscr{U}$ covering $X \setminus U_0$.
Then $\{U_0, \ldots, U_n\}$ is a finite subcover of αX. □

Corollary 2.4.7. *Any Hausdorff locally compact space is* $T_{3\frac{1}{2}}$.

Often there are many compactifications. We can define an order on them as follows:

Definition 2.4.8. *Suppose* $\langle h, \gamma X\rangle$, $\langle k, \delta X\rangle$ *are Hausdorff compactifications of a* $T_{3\frac{1}{2}}$ *space* X.

We say $\gamma X \leq \delta X$ *iff there exists* $f : \delta X \to \gamma X$ *such that*

$$\begin{array}{ccc} \gamma X & \overset{f}{\longleftarrow} & \delta X \\ h\uparrow & \nearrow k & \\ X & & \end{array}$$

—so if h, k *are inclusion maps, then* $f \restriction X = id$.

Note that $f(k(X)) = h(X)$, which is dense. δX is compact, so $f(\delta X)$ is compact, so closed. So $h(X) \subseteq f(\delta X)$, so $\gamma X = \overline{h(X)} \subseteq f(\delta X)$. So f is onto.

Lemma 2.4.9. *If* Y *is a subset of a compact* T_2 *space* X*, then* Y *is locally compact iff it can be expressed in the form* $V \cap F$*, where* V *is open and* F *is closed.*

Proof: Problem sheets. □

Corollary 2.4.10. *If* X *is locally compact and* $\langle k, \delta X\rangle$ *is a Hausdorff compactification of* X*, then* $k(X)$ *is open in* δX.

Proof: Note that if X is locally compact, then $k(X)$ can be written $V \cap F$, where F is closed and V is open in δX. Since F is closed, it must include $\overline{k(X)} = \delta X$, so of course $F = \delta X$. Thus $k(X) = V$, and is open in δX. □

Theorem 2.4.11. *Suppose* X *is a Hausdorff locally compact non-compact space. Then* αX *is the minimal compactification of* X.

Proof: Given $\langle h, \alpha X\rangle$, let $\langle k, \delta X\rangle$ be another compactification. Define $f : \delta X \to \alpha X$ as follows :

1. $f(k(x)) = h(x)$ for all $x \in X$,
2. If $y \in \delta X \setminus k(X)$, then $f(y) = *$.

We show that f is continuous, by showing that if U is open in αX, then $f^{-1}(U)$ is open in δX.

Case 1. $U \subseteq X$. Then $f^{-1}(U) = k(U) \subseteq k(X)$. But $k(X)$ is open in δX and $k(U)$ is open in $k(X)$, so $k(U)$ is open in δX as required.

Case 2. $* \in U$, and $X \setminus U$ is compact. Then $f^{-1}(U) = \delta X \setminus k(X \setminus U)$.

Now $k(X \setminus U)$ is compact, so closed since δX is T_2.

So $f^{-1}(U)$ is open, as required. □

We can find similar relationships between compactifications of different spaces. We first define some more terminology.

Definition 2.4.12. *Let $f : X \to Y$ be continuous and onto. Then f is* proper *iff f is closed, and has compact fibres.*

Lemma 2.4.13. *Let $f : X \to Y$ be proper. Then*

1. *X is locally compact iff Y is locally compact;*
2. *If X is T_2, so is Y;*
3. *If Y is compact, so is X.*

Proof: Problem sheets. □

Proposition 2.4.14. *If X is a Hausdorff locally compact space, and $f : X \to Y$ is proper, then we can extend f to a proper map $g : \alpha X \to \alpha Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \hookrightarrow & \alpha X \\ f\downarrow & & \downarrow g \\ Y & \hookrightarrow & \alpha Y \end{array}$$

Proof: Define $g(*_X) = *_Y$.

We require to show that g is proper.

1. g is continuous: let $U \subseteq \alpha Y$ be open. We show that $g^{-1}(U)$ is open.

<u>Case 1</u> $U \subseteq Y$. Then $g^{-1}(U) = f^{-1}(U)$ which is open in X since f is continuous. By Corollary 2.4.10, X is open in αX; so $f^{-1}(U)$ is open in αX.

<u>Case 2</u> $*_Y \in U$, and $Y \setminus U$ is compact.

Then $g^{-1}(U) \ni *_X$, and $X \setminus g^{-1}(U) = X \setminus f^{-1}(U) = f^{-1}(Y \setminus U$.

Now by Lemma 2.4.13, $f^{-1}(Y \setminus U)$ is compact.

So $X \setminus g^{-1}(U)$ is compact, so $g^{-1}(U)$ is open.

2. g is onto: this is clear.

3. g is proper: g is closed, because if C is closed in αX, then C is compact because αX is compact; g is continuous, so $g(C)$ is compact; αY is Hausdorff so $g(C)$ is closed.

g has compact fibres, because if $x \in \alpha Y$, then $\{x\}$ is closed, because αY is Hausdorff, so $g^{-1}(\{x\})$ is closed, because g is continuous, and so since αX is compact, $g^{-1}(\{x\})$ is compact. □

Corollary 2.4.15. *If X is Hausdorff and locally compact, $f : X \to Y$ is proper, and $\langle k, \delta X\rangle$ is a Hausdorff compactification of X, then f can be extended to a proper map g as follows:*

$$\begin{array}{ccc} X & \xrightarrow{k} & \delta X \\ f\downarrow & & \downarrow g \\ Y & \hookrightarrow & \alpha Y \end{array}$$

Proof: Exercise, using the diagram:

$$\begin{array}{ccl} X & \xrightarrow{k} & \delta X \\ \| & & \downarrow g_1 \text{ (Theorem 2.4.11)} \\ X & \hookrightarrow & \alpha Y \\ \text{f}\downarrow & & \downarrow g_2 \text{ (Prop 2.4.14)} \\ Y & \hookrightarrow & \alpha Y \end{array}$$

□

Definition 2.4.16. *(The Stone-Čech compactification) Let X be a $T_{3\frac{1}{2}}$ space.*

Let $\langle f_\lambda : \lambda \in \Lambda\rangle$ be all bounded functions from X to $\mathbb{R}$; for each λ, let I_λ be the smallest closed interval including the range of f_λ (ie $I_\lambda = [\inf \operatorname{ran} f_\lambda, \sup \operatorname{ran} f_\lambda]$).

Let Y be the Tychonoff product $\prod_{\lambda\in\Lambda} I_\lambda$.

Define $h : X \to Y$ by

$$h(x)(\lambda) = f_\lambda(x) \qquad \text{for all } \lambda.$$

Let $\beta X = \overline{h(X)}^Y$.

Then $\langle h, \beta X\rangle$ is the Stone-Čech compactification *of X.*

Theorem 2.4.17. *$\langle h, \beta X\rangle$ is a Hausdorff compactification.*

Proof: We show

1. h is one-to-one,
2. h is continuous,
3. h^{-1} is continuous (so h is a homeomorphism from X to $h(X)$),
4. βX is compact T_2 and $h(X)$ is dense in it.

1. h is one-to-one: let $x \neq y$ be elements of X.

X is T_1, so $\{y\}$ is closed. X is $T_{3\frac{1}{2}}$, so $\{x\}$ can be functionally separated from $\{y\}$ (Definition 1.2.5).

So, let $f : X \to [0, 1]$ be a continuous function such that $f(x) = 0$ and $f(y) = 1$. Then f is bounded, so $f = f_\lambda$ for some λ.

Now $h(x)(\lambda) = f_\lambda(x) = 0$, and $h(y)(\lambda) = f_\lambda(y) = 1$. So $h(x) \neq h(y)$, as required.

2. h is continuous: we show that for all U in some *subbasis,* $h^{-1}(U)$ is open. (See Lemma 1.2.9.)

Note that the following sets yield and subbasis for $Y = \prod_{\lambda\in\Lambda} I_\lambda$:

$$U_\lambda \times \prod_{\mu\neq\lambda} I_\mu, \quad \text{where } U_\lambda \subseteq I_\lambda \text{ is open.}$$

Let $U = U_\lambda \times \prod_{\mu\neq\lambda} I_\mu$.

Then

$$\begin{aligned} h^{-1}(U) &= \{x \mid h(x) \in U\} \\ &= \{x \mid h(x)(\lambda) \in U_\lambda\} \\ &= \{x \mid f_\lambda(x) \in U_\lambda\} \\ &= {f_\lambda}^{-1}(U_\lambda), \end{aligned}$$

which is open, as f_λ is continuous.

3. h^{-1} is continuous on $h(X)$. Let $U \subseteq X$ be open. We attempt to show that $h(U)$ is open in $h(X)$. We do this by showing that for each $h(x) \in h(U)$, there is an open set V in Y such that $h(x) \in V \cap h(X) \subseteq h(U)$.

For, let $C = X \setminus U$. C is closed and X is $T_{3\frac{1}{2}}$, so let $f : X \to [0,1]$ be a continuous function with $f(x) = 0$ and $f(C) \subseteq \{1\}$.

Then $f^{-1}(-\infty, 1) \subseteq U$.

For some λ, $f = f_\lambda$.

Now let

$$V = \big((-\infty, 1) \cap I_\lambda\big) \times \prod_{\mu \neq \lambda} I_\mu.$$

Then V is open in Y.

Also, if $p \in V \cap h(X)$, say $p = h(y)$, then $f_\lambda(y) = h(y)(\lambda) \in (-\infty, 1)$. So $y \in U$, and $p = h(y) \in h(U)$.

So, as required, $h(x) \in V \cap h(X) \subseteq h(U)$, and $V \cap h(X)$ is open in $h(X)$.

4. βX is compact Hausdorff and $h(X)$ is dense:

Y is a product of T_2 spaces and is therefore T_2; so βX, as a subspace of a Hausdorff space, is Hausdorff.

Finally, by Tychonoff's Theorem (Theorem 2.3.4), Y is compact. Hence $\beta X = \overline{h(X)}^Y$ is compact.

Obviously $h(X)$ is dense in $\overline{h(X)}^Y$.

We have now completed the proof that $\langle h, \beta X \rangle$ is a Hausdorff compactification. □

THEOREM 2.4.18. *Let $f : X \to \mathbb{R}$ be any bounded continuous function. Then there exists a continuous function $\beta f : \beta X \to \mathbb{R}$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ h\downarrow & \nearrow \beta f & \\ \beta X & & \end{array}$$

PROOF: Let $f = f_\lambda$.

Define βf as follows: $\beta f(p) = p(\lambda)$, for each $p \in \beta X$.

This function, being the restriction of the projection function π_λ to βX, is continuous. And if $x \in X$,

$$\beta f(h(x)) = h(x)(\lambda) = f_\lambda(x) = f(x),$$

as required. □

COROLLARY 2.4.19. *Let $Z = \prod_{\mu \in M} I_\mu$, where each I_μ is a compact interval in $\mathbb{R}$. Then whenever $f : X \to Z$ is continuous, there exists a continuous function $\beta f : \beta X \to Z$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ h\downarrow & \nearrow \beta f & \\ \beta X & & \end{array}$$

Proof: Simply do this coordinatewise.

Let $f^\mu : X \to I_\mu$ be defined so that $f^\mu(x) = f(x)(\mu)$; so f^μ is the composition of f with the μ'th projection map on Z.

Then f^μ is continuous. So, let βf^μ be as in the conclusion of Theorem 2.4.18.

Then, define $\beta f : X \to Z$ by

$$\beta f(x)(\mu) = \beta f^\mu(x).$$

We show that βf is continuous.

A subbasic open set in Z has the form

$$U\mu \times \prod_{\nu \neq \mu} I_\nu,$$

where U_μ is open in I_μ. Then

$$\begin{aligned}\beta f^{-1}\Big(U_\mu \times \prod_{\nu\neq\mu} I_\nu\Big) &= \Big\{x \;\Big|\; \beta f(x) \in U_\mu \times \prod_{\nu\neq\mu} I_\nu\Big\} \\ &= \{x \mid \beta f(x)(\mu) \in U_\mu\} \\ &= \{x \mid \beta f^\mu(x) \in U_\mu\} \\ &= (\beta f^\mu)^{-1}(U_\mu),\end{aligned}$$

which is open, as βf^μ is continuous. □

Lemma 2.4.20. *Any $T_{3\frac{1}{2}}$ space is homeomorphic to a subspace of a product of closed intervals.*

Proof: Examine the proof of Theorem 2.4.17: if X is $T_{3\frac{1}{2}}$ and $\langle h, \beta X\rangle$ is its Stone-Čech compactification, then $h : X \to \beta X$ is an embedding of X in a product of closed intervals. □

Theorem 2.4.21. *(The Stone-Čech Property) Let X be $T_{3\frac{1}{2}}$, K a compact T_2 space (and therefore $T_{3\frac{1}{2}}$). Let $f : X \to K$ be continuous. Then there exists a continuous function $\beta f : \beta X \to K$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & K \\ h\downarrow & \nearrow \beta f & \\ \beta X & & \end{array}$$

Proof: Embed K in a product Z of closed intervals. Then by Corollary 2.4.19, βf can be defined as a continuous function into Z such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ h\downarrow & \nearrow \beta f & \\ \beta X & & \end{array}$$

commutes. Now note that $\beta f^{-1}(K)$ is a closed set containing $h(X)$. Therefore, $\beta f^{-1}(K) = \beta X$, and so $\beta f : \beta X \to K$, and the proof is complete. $\square$

Theorem 2.4.22. *Suppose $\langle k, \gamma X\rangle$ is a compactification with the Stone-Čech property. Then $\gamma X \leq \beta X \leq \gamma X$; and the two compactifications are homeomorphic.*

Thus, $\langle h, \beta X\rangle$ is the unique compactification with the Stone-Čech property.

Proof: We use the Stone-Čech property on βX and γX to define continuous functions βk and γh to make the following diagram commute:

$$\begin{array}{ccc} \beta X & \overset{\beta k}{\underset{\gamma h}{\rightleftharpoons}} & \gamma X \\ h\uparrow & \nearrow k & \\ X & & \end{array}$$

Now $\gamma h \circ \beta k$ is a continuous function from βX to itself, and is equal to the identity on a dense set, namely $h(X)$; so $\gamma h \circ \beta k$ is the identity. Likewise $\beta k \circ \gamma h$ is the identity on γX. So the maps βk and γh are each other's inverses.

Thus $\gamma X \leq \beta X \leq \gamma X$, and βX and γX are homeomorphic. $\square$

Theorem 2.4.23. *$\langle h, \beta X\rangle$ is the maximal compactification of a $T_{3\frac{1}{2}}$ space X.*

Proof: Exercise, using the Stone-Čech property. $\square$

So, NEVER use the definition of the Stone-Čech property. While it's not incorrect to do so, it is virtually ALWAYS simpler to use the Stone-Čech property.

As an example of an application of the Stone-Čech compactification (neither the statement of the theorem nor the proof is on the syllabus):

Theorem 2.4.24. *(Finite Sums Theorem) Suppose $\mathbb{N}$ is the disjoint union of sets $A_1, \ldots, A_n$. Then there exists $i \in \{1, \ldots, n\}$, and there exists an infinite subset B of A_i, such that any sum of distinct elements of B lies in A_i. (That is, if $m_1, \ldots, m_k \in B$, then $m_1 + \cdots + m_k \in A_i$.)*

Sketch proof: (Details on a handout) Extend the operation of addition on $\mathbb{N}$ to the Stone-Čech compactification $\beta\mathbb{N}$.*

Then there exists $p \in \beta\mathbb{N} \setminus h(\mathbb{N})$ such that $p + p = p$.

Now there exists unique $i \in \{1, \ldots, n\}$ such that $p \in \overline{h(A_i)}$, which is a clopen set.

Using the fact that $p+p = p$, carefully choose a sequence B from A_i having the desired property; intuitively, if the elements of B are close enough to p, then their sum is inside the clopen neighbourhood $\overline{h(A_i)}$, and this is enough. $\square$

* There is an irritating technical hitch. The function $P : \langle a, b\rangle \mapsto a + b$ is a *binary* function (a function of two variables). So one can define βP, but it turns out not to be continuous, but to have the weaker property of being continuous on one side only (ie. as a function of the second argument). One needs a certain amount of care to get around this.

2.5. Connectedness and local connectedness

First, an important notion for reasoning about connectedness.

Definition 2.5.1. *Define an equivalence relation on a topological space X by: $x \sim y$ iff there exists a connected subset C of X such that $x, y \in C$.*

We define the component *of x to be the equivalence class of x under this relation.*

Proposition 2.5.2. *The component of a point x in a topological space is the largest connected set containing x.*

All components are closed.

Definition 2.5.3. *X is locally connected iff for all $x \in X$, for all open $U \ni x$, there exists connected C and open V such that $x \in V \subseteq C \subseteq U$.*

Theorem 2.5.4. *X is locally connected iff every component of every open set is open.*

Proof: $\Leftarrow$) Trivial.

$\Rightarrow$) Let U be open, C a component of U. Let $x \in C$. Then there exists open V and connected D such that $x \in V \subseteq D \subseteq U$. But if C is a component of U, then $D \subseteq C$. So $x \in V \subseteq C$—so C is open. $\Box$

Corollary 2.5.5. *If X is locally connected, then each component of X is clopen.*

Proof: Apply Theorem 2.5.4 to X; components are closed. $\Box$

Example 2.5.6. *The topologist's sine curve, which is connected but not locally connected.*

Recall that in general, components are not clopen. (Consider $\mathbb{Q}$.) Also connectedness is defined in terms of clopen sets. So the following seems plausible:

Guess 2.5.7. *Each component is an intersection of clopen sets.*

Is this true?

Definition 2.5.8. *The* quasi-component *of x in a space X is the intersection of all clopen sets containing x.*

Proposition 2.5.9. *Quasi-components are closed.*

Proof: A quasi-component is an intersection of closed sets. $\Box$

Theorem 2.5.10. *The quasi-components of X partition X.*

Proof: We require to show that if $Q(x) \neq Q(y)$, then $Q(x) \cap Q(y) = \varnothing$.

Suppose $Q(y) \nsubseteq Q(x)$. Say $z \in Q(y) \setminus Q(x)$.

Then there exists a clopen set C containing x such that $z \notin C$. Now if $y \in C$, then $z \notin Q(y)$. So $y \notin C$; so $X \setminus C \supseteq Q(y)$ (since $X \setminus U$ is a clopen set containing y).

Now $C \supseteq Q(x)$; so $Q(x) \cap Q(y) = \varnothing$. $\Box$

Theorem 2.5.11. *If x is an element of a topological space X, then the component of x is contained in the quasi-component of x.*

Proof: Suppose $y \notin Q(x)$. We show that there does not exist a connected set C containing both x and y.

For, there exists U clopen such that $x \in U$ and $y \notin U$. So, if C contains x and y, $U \cap C$ is clopen in C and splits x apart from y, disconnecting C.

So y is not in the component of x. □

EXAMPLE 2.5.12. *There exists a space in which components and quasi-components are different. (Problem sheets)*

THEOREM 2.5.13. *(Šura-Bura Lemma) In any compact Hausdorff space, components and quasi-components coincide.*

PROOF: It is necessary and sufficient to show that quasi-components are connected.

So, suppose that D is a quasi-component that is not connected.

Then D can be partitioned into sets E and F which are clopen in D.

Now, E and F closed in D implies that E and F are closed.

Since X is compact and Hausdorff, it is normal. So, let U and V be disjoint open sets in X such that $E \subseteq U$ and $F \subseteq V$.

So $D \subseteq U \cup V$.

Now let

$$\mathscr{C} = \{C \mid C \text{ is clopen, } D \subseteq C\}.$$

Let

$$\mathscr{U} = \{X \setminus C \mid C \in \mathscr{C}\} = \{U \text{ clopen} \mid U \cap D = \varnothing\}.$$

Since $D = \bigcap \mathscr{C}$, $X \setminus D = \bigcup \mathscr{U}$.

So $\mathscr{U} \cup \{U, V\}$ is an open cover of X.

Let $U_1, \ldots, U_n \in \mathscr{U}$ be such that $\{U, V, U_1, \ldots, U_n\}$ covers X.

Now let $C_i = X \setminus U_i$; then C_i is a clopen set including D.

So $D \subseteq G \subseteq U \cup V$, where $G = \bigcap_{i=1}^n C_i$; and G is clopen.

Now $U \cap V = \varnothing$, so $U \cap G$ and $V \cap G$ partition G and so are clopen in G.

So, $U \cap G$ and $V \cap G$ are clopen.

Now $U \cap G$ is a clopen set splitting D. So D cannot after all be a quasi-component, ⁘. □

3. Metric spaces

3.1. Metrisation

What conditions are sufficient to ensure that a topological space possesses a compatible metric?

DEFINITION 3.1.1. *A topological space $\langle X, \mathscr{T}\rangle$ is* metrisable *iff there exists a metric d on X such that $\mathscr{T}$ is the metric topology of X.*

THEOREM 3.1.2. *Let X be a Lindelöf T_3. Then X is normal (so T_4).*

PROOF: Let C and D be disjoint and closed.

For each $y \in D$, let A_y and V_y be disjoint open sets such that $C \subseteq A_y$, $y \in V_y$.

Let $\{V_{y_i} : i \in \mathbb{N}\}$ be a countable subcover of D (exercise: every closed subset of a Lindelöf space is Lindelöf).

Similarly, for $x \in C$, let U_x and B_x be disjoint and open sets such that $x \in U_x$ and $D \subseteq B_x$; let $\{U_{x_i} : i \in \mathbb{N}\}$ be a countable subcover of C.

Notice that since $U_{x_i} \cap B_{x_i} = \varnothing$, $\overline{U_{x_i}} \cap B_{x_i} = \varnothing$ also; so $\overline{U_{x_i}} \cap D = \varnothing$. Similarly $\overline{V_{y_i}} \cap C = \varnothing$.

Now, we recursively construct open sets S_n and T_n with the following properties:

1. For all i, $S_i \subseteq U_{x_i}$;
2. For all j, $T_j \subseteq V_{y_j}$;
3. For all i, $\overline{S_i} \cap D = \varnothing$;
4. For all j, $\overline{T_j} \cap C = \varnothing$;
5. If $i \leq j$, then $\overline{S_i} \cap T_j = \varnothing$;
6. If $j < i$, then $\overline{T_j} \cap S_i = \varnothing$;
7. For all i, $S_i \cap C = U_{x_i} \cap C$;
8. For all j, $T_j \cap D = V_{y_j} \cap D$.

In this list, 3. and 4. follow from 1. and 2. We perform the recursion but proceeding as follows. Suppose S_i and T_i have been defined for all $i < n$.

Then we let

$$S_n = U_{x_n} \setminus \bigcup_{j=1}^{n-1} \overline{T_j};$$

and

$$T_n = V_{y_n} \setminus \bigcup_{i=1}^{n} \overline{S_i}.$$

Properties 1.–8. are now easy to check; and we observe that they trivially imply that for all i and j, S_i and T_j are disjoint.

Now let $U = \bigcup_{i=1}^{\infty} S_i$, and $V = \bigcup_{j=1}^{\infty} T_j$.

Then U and V are open and disjoint, $C \subseteq U$ and $D \subseteq V$. □

Corollary 3.1.3. *Every Lindelöf T_3 space is $T_{3\frac{1}{2}}$.*

Definition 3.1.4. *X is* first countable, *written* $1°$, *iff for each $x \in X$, there exists a sequence $\langle U_n \mid n \in \mathbb{N}\rangle$ of open sets such that for all open $U \ni x$, there exists n such that $x \in U_n \subseteq U$.*

Definition 3.1.5. *X is* second countable, *written* $2°$, *iff X has a countable basis.*

Theorem 3.1.6. *Every second countable space is Lindelöf.*

Proof: Trivial. Let $\mathscr{B} = \langle B_n \mid n \in \mathbb{N}\rangle$ be a countable basis. Let $\mathscr{U}$ be an open cover.

For each n, define $U_n \in \mathscr{U}$ to be some open set in $\mathscr{U}$ such that $B_n \subseteq U_n$, if such exists.

Let $\mathscr{V} = \{U_n \mid U_n \text{ exists}\}$. $\mathscr{V}$ is a countable subfamily of $\mathscr{U}$.

We show that it is a subcover. Suppose $x \in X$. Then there exists $U \in \mathscr{U}$ such that $x \in U$. Since $\mathscr{B}$ is a basis, there exists n such that $x \in B_n \subseteq U$. So U_n exists. Then $x \in U_n \in \mathscr{V}$.

So $\mathscr{V}$ is a countable subcover. □

Lemma 3.1.7. *The Tychonoff product* $\prod_{n\in\mathbb{N}}[0,1]$ *is a metric space.*

Proof: Problem sheets. □

Theorem 3.1.8. *(Urysohn's Metrisation Theorem) If X is T_3 and second countable, then X is separable and metrisable.*

Proof: Let $\mathscr{B}$ be a countable basis. Then $\mathscr{B}\times\mathscr{B}$ is countable, and thus so is

$$E=\left\{\langle B,B'\rangle\in\mathscr{B}\times\mathscr{B}\;\middle|\;\overline{B}\subseteq B'\right\}.$$

Enumerate E as $E=\{\langle B_n,B'_n\rangle\mid n\in\mathbb{N}\}$.

We embed X homeomorphically in $\prod_{n\in\mathbb{N}}[0,1]$ as follows, and deduce that it is a metric space:

X is T_4, so by Urysohn's Lemma, $\overline{B_n}$ and B'_n can be functionally separated by continuous function $f_n : X\to[0,1]$. Now define $\Phi : X\to\prod_{n\in\mathbb{N}}[0,1]$ thus:

$$\Phi(x)(n)=f_n(x).$$

We wish to show that Φ is a homeomorphism.

<u>Φ is one-to-one</u>: Suppose $x\neq y$. Then since X is T_1, there exists open U such that $x\in U$ but $y\notin U$.

Since X is T_3, there exists open V and basic open B and B' such that

$$x\in B\subseteq V\subseteq\overline{V}\subseteq B'\subseteq U.$$

Then $\overline{B}\subseteq B'$, so $\langle B,B'\rangle\in E$.

Let $\langle B,B'\rangle=\langle B_n,B'_n\rangle$.

Then f_n functionally separates B_n from $X\backslash B'_n$; in particular, $f_n(x)=0$ and $f_n(y)=1$, so that $\Phi(x)(n)\neq\Phi(y)(n)$, so $\Phi(x)\neq\Phi(y)$.

<u>Φ is continuous</u>: Let $U_n\times\prod_{m\neq n}[0,1]$ be a subbasic open set.

Then

$$\begin{aligned}\Phi^{-1}\Bigl(U_n\times\prod_{m\neq n}[0,1]\Bigr)&=\Bigl\{x\;\Big|\;\Phi(x)\in U_n\times\prod_{m\neq n}[0,1]\Bigr\}\\&=\{x\mid\Phi(x)(n)\in U_n\}\\&=\{x\mid f_n(x)\in U_n\}\\&={f_n}^{-1}(U_n).\end{aligned}$$

But f_n is continuous, so this is open.

<u>Φ^{-1} is continuous</u>: Let U be an open set in X. We show that $\Phi(U)$ is open in $\Phi(X)$.

Let $x\in U$. We show that there is an open set V such that $x\in V\subseteq U$ and $\Phi(V)$ is open.

Since X is T_3, there exist open W and basic open B and B' such that

$$x\in B\subseteq W\subseteq\overline{W}\subseteq B'\subseteq U.$$

Then $\langle B, B' \rangle \in E$. Let $\langle B, B' \rangle = \langle B_n, B'_n \rangle$.

Then f_n functionally separates $\overline{B_n}$ from B'_n.

Let $V = {f_n}^{-1}([0,1))$.

Then $f_n(x) = 0$, so $x \in V$, and if $y \notin U$, then $y \notin B'_n$, so $f_n(y) = 1$, so $y \notin V$; hence $V \subseteq U$.

Now

$$\begin{aligned}\Phi(V) &= \{\Phi(x) \mid x \in V\}\\ &= \{\Phi(x) \mid f_n(x) \in [0,1)\}\\ &= \{\Phi(x) \mid \Phi(x)(n) \in [0,1)\}\\ &= \Phi(X) \cap \left(U_n \times \prod_{m \neq n} [0,1]\right),\end{aligned}$$

where $U_n = [0,1)$; and so $\Phi(V)$ is open in $\Phi(X)$. □

3.2. Stone's Theorem

We discuss a particularly nice property possessed by metric spaces.

Definition 3.2.1. *A family $\mathscr{U}$ of subsets of a space X is* locally finite *iff for all $x \in X$, there exists open $V \ni x$ such that $V \cap U \neq \varnothing$ for only finitely many $U \in \mathscr{U}$.*

Definition 3.2.2. *If $\mathscr{U}$ and $\mathscr{V}$ are covers of X, then $\mathscr{U}$* refines *$\mathscr{V}$, or is a* refinement *of it, written $\mathscr{U} \prec \mathscr{V}$, iff $\forall U \in \mathscr{U} \exists V \in \mathscr{V}\, U \subseteq V$.*

Definition 3.2.3. *A space X is said to be* paracompact *iff every open cover has a locally finite open refinement.*

Definition 3.2.4. *A collection $\mathscr{U}$ of sets is* closure-preserving *iff for every $\mathscr{V} \subseteq \mathscr{U}$,*

$$\overline{\bigcup \mathscr{V}} = \bigcup_{V \in \mathscr{V}} \overline{V}.$$

Lemma 3.2.5. *Suppose $\mathscr{U}$ is locally finite. Then $\mathscr{U}$ is closure preserving.*

Proof: Let $\mathscr{V} \subseteq \mathscr{U}$. Let $x \notin \bigcup_{V \in \mathscr{V}} \overline{V}$.

We show $x \notin \overline{\bigcup \mathscr{V}}$.

Let $A \ni x$ witness local finiteness of $\mathscr{U}$: that is, let it be open such that $A \cap U \neq \varnothing$ for just finitely many $U \in \mathscr{U}$.

Then there are just finitely many elements $V_1, \ldots, V_k$ of $\mathscr{V}$ such that $A \cap V_i \neq \varnothing$.

For all i $x \notin \overline{V_i}$; so $x \notin \bigcup_i \overline{V_i}$, which is closed; so

$$B = A \setminus \bigcup_i \overline{V_i}$$

is open and $x \in B$.

Also for all $V \in \mathscr{V}$, *either* $V = V_i$ for some $i \le k$, so that $B \cap V = \varnothing$; *or* $A \cap V = \varnothing$, so $B \cap V = \varnothing$ also.

So $x \in B$, B is open, and $B \cap \bigcup \mathscr{V} = \varnothing$. So $x \notin \overline{\bigcup \mathscr{V}}$ as required. $\square$

THEOREM 3.2.6. *Every paracompact Hausdorff space X is regular.*

PROOF: Let $x \in X$ and let $C \subseteq X$ be closed.

For $y \in C$, find $U_y \ni x$ and $V_y \ni y$ such that $U_y \cap V_y = \varnothing$.

Let

$$\mathscr{U} = \{V_y : y \in C\} \cup \{X \setminus C\}.$$

Then $\mathscr{U}$ is an open cover of X.

Let $\mathscr{V}$ be a locally finite open refinement; let

$$\mathscr{V}' = \{V \in \mathscr{V} \mid V \cap C \neq \varnothing\}.$$

Now if $V \in \mathscr{V}$, then for some $U \in \mathscr{U}$, $V \subseteq U$. Clearly $V \nsubseteq X \setminus C$.

So $V \subseteq V_y$ for some y. Hence $V \cap U_y = \varnothing$; $U_y \ni X$ open implies that $x \notin \overline{V}$.

So $x \notin \bigcup_{V \in \mathscr{V}'} \overline{V}$; so $x \notin \overline{\bigcup_{V \in \mathscr{V}'} V}$.

Let $W = \bigcup_{V \in \mathscr{V}'} V$. Then W is open. Also, $W \supseteq C$.

Let $U = X \setminus \overline{W}$; then $x \in U$, and U and W are disjoint. $\square$

THEOREM 3.2.7. *Every paracompact T_3 space is normal.*

PROOF: Problem sheets. $\square$

THEOREM 3.2.8. *The following are equivalent for a regular space X:*

1. *X is paracompact;*
2. *Every open cover of X has a locally finite refinement;*
3. *Every open cover of X has a locally finite closed refinement.*

PROOF: Problem sheets. $\square$

THEOREM 3.2.9. *A regular space X is paracompact iff every open cover has a σ-locally finite open refinement; that is, for every open cover $\mathscr{U}$, there exists a refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\mathscr{V}$ can be expressed as a union $\bigcup_{n \in \mathbb{N}} \mathscr{V}_n$, where each $\mathscr{V}_n$ is locally finite.*

PROOF: $\Rightarrow$) Trivial.

$\Leftarrow$) Let $\mathscr{U}$ be an open cover, $\mathscr{V} = \bigcup_{n \in \mathbb{N}} \mathscr{V}_n$ a σ-locally finite open refinement. We find a locally finite refinement $\mathscr{W}$ (not necessarily open), and use Theorem 3.2.8.

Let $A_n = X \setminus \bigcup \bigcup_{m<n} \mathscr{V}_m$.

Then $\{A_n \mid n \in \mathbb{N}\}$ is locally finite, since for all x, there exists n such that $x \in \bigcup \bigcup_{m<n} \mathscr{V}_m$, and this open set witnesses local finiteness of the family $\{A_n \mid n \in \mathbb{N}\}$.

Also, $A_0 = X$, and $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$.

Now let

$$\mathscr{W} = \{V \cap A_n \mid V \in \mathscr{V}_n,\ n \in \mathbb{N}\}.$$

Then $\mathscr{W}$ is a cover, for if $x \in X$, let n be least such that $x \in \bigcup \mathscr{V}_n$. Then there exists $V \in \mathscr{V}_n$ such that $x \in V$; also, $x \in A_n$. So $x \in V \cap A_n$, which is an element of $\mathscr{W}$.

Also, if $x \in V = \bigcup\bigcup_{m<n} \mathscr{V}_m$, then for each $m < n$, let the open neighbourhood F_m of x witness local finiteness of $\mathscr{V}_m$ at x. Let us say that the only elements of $\mathscr{V}_m$ that F_m meets are $V_{m,1}, \ldots, V_{m,k_m}$.

Then $V \cap \bigcap_{m<n} F_m$ witnesses local finiteness of $\mathscr{W}$ near x, since the only elements of $\mathscr{W}$ that it meets are the elements

$$V_{m,i} \cap A_m,$$

for $m < n$ and $i < k_m$. □

DEFINITION 3.2.10. *Let Y be a set. Then the relation $\leq$ is a* well-ordering *of Y iff*

1. *$\leq$ is a total order of Y, and*
2. *Every non-empty subset of Y has a $\leq$-least element.*

Define seg_y to be $\{z \in Y \mid z < y\}$.

THEOREM 3.2.11. *(Recursion on a well-ordering) Suppose $\leq$ is a well-ordering on Y, A is a set, and, for each $y \in Y$, $\Phi_y : A^{seg_y} \to A$.*

Then there exists a unique function $f : Y \to A$ such that for all $y \in Y$, $f(y) = \Phi_y(f \restriction seg_y)$.

PROOF: Standard theorem of set theory. □

FACT 3.2.12. *(Well-ordering principle) For every set Y, there exists a relation $\leq$ which is a well-ordering of Y.*

THEOREM 3.2.13. *(Stone) Every metric space is paracompact.*

PROOF: (Proof by M. E. Rudin) Let $\langle X, d\rangle$ be a metric space. We show that every open cover of X has a σ-locally finite open refinement, and then appeal to Theorem 3.2.9.

Let $\mathscr{U}$ be an open cover. Let $\leq$ be a well-ordering of $\mathscr{U}$.

For each $n \in \mathbb{N}$, $U \in \mathscr{U}$, we construct a set $V_{n,U}$ by recursion such that

1 $D(V_{n,U}, X \setminus U) = \dfrac{1}{2^n}$, and

2 If $U \neq U'$, then $D(V_{n,U}, V_{n,U'}) \geq \dfrac{1}{2^{n+1}}$,

where if A, B are any disjoint subsets of X,

$$D(A, B) = \inf_{\substack{x \in A,\\ y \in B}} d(x, y).$$

Suppose we have constructed $V_{n,U'}$ for $U' \leq U$. Then we let

$$V_{n,U} = \left\{ y \in U \;\middle|\; D(\{y\}, X \setminus U) > \frac{1}{2^n},\ D\Bigg(\{y\}, \overline{\bigcup_{U' < U} V_{n,U'}}\Bigg) < \frac{1}{2^{n+1}} \right\}.$$

We have a number of things to prove.

1. $V_{n,U}$ is open. This is because D is continuous.

2. $\mathscr{V}_n = \{V_{n,U} \mid U \in \mathscr{U}\}$ is locally finite.

But for all y, $B_{\frac{1}{2^{n+2}}}(y)$ meets at most one $V_{n,U}$, since if $U' < U$ and $z \in V_{n,U}$,

$$\frac{1}{2^{n+1}} < D\left(\{z\}, \overline{\bigcup_{U'<U} V_{n,U'}}\right) \leq D(\{z\}, X \setminus V_{n,U'}).$$

Hence $B_{\frac{1}{2^{n+2}}}(y)$ cannot meet both $V_{n,U}$ and $V_{n,U'}$.

3. $\mathscr{V} = \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ is a cover.

Let $x \in X$. Let $\mathscr{W} = \{U \in \mathscr{U} \mid x \in U\}$. $\mathscr{W}$ is non-empty; let U be the $\leq$-least element. Choose n such that $B_n(x) \subseteq U$; so $D(\{x\}, X \setminus U) > \frac{1}{2^n}$. Also, for $U' < U$, $x \notin U'$ by minimality of U', so $D\left(y\overline{\bigcup_{U'<U} V_{n,U'}}\right) \geq \frac{1}{2^n}$, since $x \notin \overline{X \setminus \bigcup_{U'<U} U'}$. Hence $x \in V_{n,U'}$.

4. $\mathscr{V}$ is a refinement of $\mathscr{U}$. This is trivial.

So $\mathscr{V}$ is a σ-locally finite open refinement of $\mathscr{U}$, and so we are done. □

4. Stone duality

4.1. Boolean algebras

Definition 4.1.1. *A* Boolean algebra *is a tuple* $\langle \mathbb{B}, \leq, \wedge, \vee, \neg, \mathbb{O}, \mathbb{1} \rangle$ *such that:*

1. $\langle \mathbb{B}, \leq \rangle$ *is a partial order,*
2. $\wedge$ *and* $\vee$ *are binary operations on* $\mathbb{B}$, $\neg$ *is a unary operation, and* $\mathbb{O}, \mathbb{1} \in \mathbb{B}$,
3. $\mathbb{O}$ *is the least element of* $\mathbb{B}$ *and* $\mathbb{1}$ *is the greatest,*
4. *for all* $a, b \in \mathbb{B}$, $a \wedge b = \max\{c \in \mathbb{B} : c \leq a, b\}$,
5. *for all* $a, b \in \mathbb{B}$, $a \vee b = \min\{c \in \mathbb{B} : c \geq a, b\}$,
6. $\wedge$ *and* $\vee$ *are distributive over each other,*
7. *for all* $a \in \mathbb{B}$, $a \wedge \neg a = \mathbb{O}$ *and* $a \vee \neg a = \mathbb{1}$.

Examples 4.1.2. *1. Let X be any set. Then* $\langle \wp X, \subseteq, \cap, \cup, X \setminus \cdot, \varnothing, X \rangle$ *is a Boolean algebra.*

2. The two-element Boolean algebra: $\mathbb{B} = \{\mathbb{O}, \mathbb{1}\}$, *with* $\mathbb{O} < \mathbb{1}$ *and* $\neg\mathbb{O} = \mathbb{1}$ *and* $\neg\mathbb{1} = \mathbb{O}$. *This is isomorphic to the powerset of a one-element set.*

3. The finite and co-finite subsets of X form a subalgebra of $\wp X$.

4. Let X be a topological space. Then the set of clopen subsets of X is a subalgebra of $\wp X$.

A homomorphism of Boolean algebras is, as usual, a structure-preserving function.

Definition 4.1.3. *Suppose* $\mathbb{B}$ *and* $\mathbb{B}'$ *are Boolean algebras. Then a function* $\phi : \mathbb{B} \to \mathbb{B}'$ *is a* homomorphism *if and only if*

1. $\phi(\mathbb{O}) = \mathbb{O}$ *and* $\phi(\mathbb{1}) = \mathbb{1}$,
2. $\phi(\neg a) = \neg\phi(a)$,
3. $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$,
4. $\phi(a \vee b) = \phi(a) \vee \phi(b)$,
5. *if* $a \leq b$, *then* $\phi(a) \leq \phi(b)$.

Exercise 4.1.4. *The clauses in the definitions above are not independent of each other. Which can safely be dropped?*

Theorem 4.1.5. *Let $\mathbb{B}$ be a Boolean algebra. The following hold, for all $a, b, c \in \mathbb{B}$:*

1. $a \leq b$ *if and only if* $a \vee b = b$ *if and only if* $a \wedge b = a$,
2. $b = \neg a$ *if and only if* $b \wedge a = \mathbb{O}$ *and* $b \vee a = \mathbb{1}$,
3. $a \leq b$ *if and only if* $\neg a \vee b = \mathbb{1}$ *if and only if* $a \wedge \neg b = \mathbb{O}$,
4. $\neg(a \vee b) = (\neg a) \wedge (\neg b)$,
5. $\neg(a \wedge b) = (\neg a) \vee (\neg b)$,

Proof: Exercise. □

4.2. Dual spaces of Boolean algebras

Definition 4.2.1. *Let $\mathbb{B}$ be a Boolean algebra. A subset F of $\mathbb{B}$ is a* filter *if and only if*

1. $\mathbb{1} \in F$,
2. $\mathbb{O} \notin F$,
3. *if* $a, b \in F$, *then* $a \wedge b \in F$,
4. *if* $a \in F$ *and* $a \leq b$, *then* $b \in F$.

If in addition, for all $a \in \mathbb{B}$, either $a \in F$ or $\neg a \in F$, then F is an ultrafilter.

From here on, we refer to the thing defined in Definition 2.2.3. as a *filter on $\wp X$*.

The set of complements of elements of a filter is an *ideal.*

Proposition 4.2.2. *A subset $\mathscr{U}$ of a Boolean algebra $\mathbb{B}$ is an ultrafilter if and only if it is a maximal filter.*

Proof: Exercise. □

Theorem 4.2.3. *Any filter on a Boolean algebra can be extended to an ultrafilter.*

Proof: Exercise. □

This is the Boolean Prime Ideal Theorem, which is a weakening of the Axiom of Choice.

Definition 4.2.4. *Let $\mathbb{B}$ be a Boolean algebra. We define $\mathscr{S}\mathbb{B}$ to be the set of ultrafilters on $\mathbb{B}$, equipped with the topology generated by a basis of clopen sets of the form*

$$[a] = \{p \in \mathscr{S}\mathbb{B} : a \in p\}.$$

Definition 4.2.5. *A* Stone space *is a compact Hausdorff space with a basis of clopen sets.*

Proposition 4.2.6. *For any Boolean Algebra, $\mathscr{S}\mathbb{B}$ is a Stone space.*

Proof: Every ultrafilter belongs to $[\mathbb{1}]$, so our proposed basis covers $\mathscr{S}\mathbb{B}$. Also, the clopen sets are closed under finite intersection. So they do form a basis for a topology.

Hausdorffness: if $p \neq q$, then there must be some element of $\mathbb{B}$ which is contained in one but not the other. Suppose that $a \in p \setminus q$. Now q is an ultrafilter, so either $a \in q$ or

$\neg a \in q$. Hence $\neg a \in q$. Now $p \in [a]$ and $q \in [\neg a]$. These two sets are disjoint, because if $r \in [a] \cap [\neg a]$, then a, $\neg a \in r$, and then since r is a filter, $a \neg a = \mathbb{O} \in r$, contradicting the statement that r is a filter.

As for compactness, it is sufficient to prove that any cover by basic open sets has a finite subcover. [Exercise: why?]

Suppose that $A \subseteq \mathbb{B}$, and that $\{[a] : a \in A\}$ is an open cover of $\mathscr{S}\mathbb{B}$ with no finite subcover. Note that in this case $\mathscr{S}\mathbb{B}$, and therefore $\mathbb{B}$ itself, is infinite.

Let F be the set of elements c of $\mathbb{B}$ such that there exists a finite subset B of A such that $c \vee \bigvee B = \mathbb{1}$.

Then, firstly, $\mathbb{1}$ itself belongs to F. However $\mathbb{O}$ does not, because otherwise if we have a finite subset B of A such that $\bigvee B = \mathbb{1}$, then $\{[a] : a \in B\}$ is a finite subcover because if $B = \{a_i : i < n\}$, if $p \notin [a_0]$, then $a_0 \notin p$, so $\neg a_0 \in p$. Similarly for all the other a_i. If p is not covered, then all the $\neg a_i \in p$. Hence $\bigwedge \neg a_i \in p$. But $\bigvee a_i = \mathbb{1}$. Hence $\bigwedge \neg a_i = \mathbb{O}$, contradicting the assumption that p is a filter.

Now if a, $b \in F$, then let B and C be subsets of A such that $a \vee \bigvee B = \mathbb{1}$ and $b \vee \bigvee C = \mathbb{1}$. Then $\neg a \leq \bigvee B$ and $\neg b \leq \bigvee C$. Hence $\neg a \vee \neg b \leq \bigvee(B \cup C)$. Hence $a \wedge b \vee \bigvee(B \cup C) = \mathbb{1}$.

Clearly if $a \in F$ and $a \leq b$ then $b \in F$.

So F is a filter.

Extend F to an ultrafilter p.

Suppose that a is an element of A such that $p \in [a]$, or equivalently, $a \in p$.

Then $\neg a$ belongs to F and hence to p, because if $B = \{a\}$, then $\neg a \vee \bigvee B = \neg a \vee a = \mathbb{1}$.

So no such a can exist, and this contradicts the statement that $\mathscr{U}$ is a cover.

Zero-dimensionality is simply the statement that there is a basis of clopen sets. □

4.3. Dual algebras of topological spaces

Now define the reverse.

Definition 4.3.1. *If X is a compact Hausdorff zero-dimensional space, define $\mathscr{B}X$ to be its Boolean algebra of clopen subsets.*

4.4. Duality

$\mathscr{B}$ and $\mathscr{S}$ are mutually inverse, in that there are natural isomorphisms between $\mathscr{B}\mathscr{S}\mathbb{B}$ and $\mathbb{B}$, and between $\mathscr{S}\mathscr{B}X$ and X.

Definition 4.4.1. *If X is a compact zero-dimensional Hausdorff space, define $\eta_X : X \to \mathscr{S}\mathscr{B}X$ so that*

$$\eta_X(x) = \{U : U \in \mathscr{B}X,\, x \in U\}.$$

Proposition 4.4.2. *η_X is well-defined, and is a homeomorphism.*

Proof: It is easy to check that $\eta_X(x)$ is an ultrafilter on $\mathscr{B}X$.

η_X is one-to-one because X is Hausdorff.

To see that η_X is onto, let p be any ultrafilter on $\mathscr{B}X$.

Let $\mathscr{F}$ be p, considered as a subset of $\wp X$, and, noting that $\mathscr{F}$ generates a filter on $\wp X$, extend $\mathscr{F}$ to an ultrafilter $\mathscr{U}$ on $\wp X$.

Because X is compact, $\mathscr{U}$ converges to some point x; thus the neighbourhood filter at x is a subset of $\mathscr{U}$.

Therefore $\mathscr{U}$ contains the family $\eta_X(x)$, considered as a subset of $\wp X$, and because $\eta_X(x) \subseteq \mathscr{B}X$ and $p = \mathscr{U} \cap \mathscr{B}X$, $\eta_X(x) \subseteq p$. Since $\eta_X(x)$ is an ultrafilter, $\eta_X(x) = p$, as required.

As for continuity, let $[U]$ be a basic open set in $\mathscr{S}\mathscr{B}X$. Then $\eta_X(x) \in [U]$ iff $U \in \eta_X(x)$ iff $x \in U$, so ${\eta_X}^{-1}([U]) = U$.

Now η_X is a continuous bijection from a compact space (namely X) to a Hausdorff space (namely $\mathscr{S}\mathscr{B}X$), so it is a homeomorphism. □

DEFINITION 4.4.3. *Suppose $\mathbb{B}$ is a Boolean algebra. Define $\eta_{\mathbb{B}} : \mathbb{B} \to \mathscr{B}\mathscr{S}\mathbb{B}$ by:*

$$\eta_{\mathbb{B}}(a) = [a].$$

PROPOSITION 4.4.4. *$\eta_{\mathbb{B}}$ is well-defined, and is an isomorphism.*

PROOF: Exercise. □

It follows trivially that every Boolean algebra is isomorphic to a subalgebra of a powerset.

4.5. Duals of Boolean algebra homomorphisms

DEFINITION 4.5.1. *If $\phi : \mathbb{A} \to \mathbb{B}$ is a homomorphism of Boolean algebras, then define $\mathscr{S}\phi : \mathscr{S}\mathbb{B} \to \mathscr{S}\mathbb{A}$ by*

$$\mathscr{S}\phi(p) = \{a \in \mathbb{A} : \phi(a) \in p\}.$$

Note the reversal of the arrow.

THEOREM 4.5.2. *$\mathscr{S}\phi$ is well-defined and continuous.*

PROOF: We show that $\mathscr{S}\phi(p)$ is an ultrafilter on $\mathbb{A}$. This follows easily from the fact that ϕ is a homomorphism.

In detail:

$\phi(\mathbb{1}) = \mathbb{1}$, since ϕ is a homomorphism; since p is a filter, $\mathbb{1} \in p$ in $\mathbb{B}$, so $\mathbb{1} \in \mathscr{S}\phi(p)$.

Suppose that $\mathbb{O} \in \mathscr{S}\phi(p)$. Then $\phi(\mathbb{O}) = \mathbb{O} \in p$, which is impossible. So $\mathbb{O} \notin \mathscr{S}\phi(p)$.

Suppose that $a, b \in \mathscr{S}\phi(p)$. Then $\phi(a), \phi(b) \in p$. Since p is a filter, $\phi(a) \wedge \phi(b) \in p$; that is, $\phi(a \wedge b) \in p$. Hence $a \wedge b \in p$, as required.

Suppose $a \in \mathscr{S}\phi(p)$ and $a \leq b$. Then $\phi(a) \in p$, and $\phi(a) \leq \phi(b)$. Hence since p is a filter, $\phi(b) \in p$. Hence $b \in \mathscr{S}\phi(p)$.

Suppose that $a \in \mathscr{A}$. Then $\phi(a) \in \mathscr{B}$. Since p is an ultrafilter, one of $\phi(a)$ and $\neg\phi(a) = \phi(\neg a)$ is in p. Hence one of a or $\neg a$ belongs to $\mathscr{S}\phi(p)$.

Now we show that $\mathscr{S}\phi$ is continuous. Suppose that $[a]$ is a basic open set in $\mathscr{A}$, so that $a \in \mathscr{A}$. Now $\mathscr{S}\phi(p) \in [a]$ if and only if $a \in \mathscr{S}\phi(p)$ if and only if $\phi(a) \in p$, if and only if $p \in [\phi(a)]$. So $(\mathscr{S}\phi)^{-1}\big([a]\big) = [\phi(a)]$, which is open, as required. □

THEOREM 4.5.3. *$\mathscr{S}\mathrm{id} = \mathrm{id}$, and $\mathscr{S}(\psi \circ \phi) = \mathscr{S}\phi \circ \mathscr{S}\psi$.*

PROOF: That $\mathscr{S}\mathrm{id} = \mathrm{id}$ is obvious.

Suppose $\phi : \mathbb{A} \to \mathbb{B}$ and $\psi : \mathbb{B} \to \mathbb{C}$ are Boolean algebra homomorphisms. Suppose that $p \in \mathscr{B}\mathbb{C}$.

Then $a \in (\mathscr{S}\phi)((\mathscr{S}\psi)(p))$ if and only if $\phi(a) \in \mathscr{S}\psi(p)$ if and only if $\psi(\phi(a)) \in p$ if and only if $(\psi \circ \phi)(a) \in p$ if and only if $a \in \mathscr{S}(\psi \circ \phi)(p)$.

So $\mathscr{S}(\psi \circ \phi) = \mathscr{S}\phi \circ \mathscr{S}\psi$ as required. □

Proposition 4.5.4. *$\mathscr{S}\phi$ is one-to-one iff ϕ is onto and onto iff ϕ is one-to-one.*

Proof: Exercise. □

4.6. Duals of continuous functions

Definition 4.6.1. *If X and Y are two such, and $f : X \to Y$ is continuous, define $\mathscr{B}f : \mathscr{B}Y \to \mathscr{B}X$ so that $\mathscr{B}f(U) = f^{-1}[U]$.*

Proposition 4.6.2. *$\mathscr{B}f$ is a Boolean algebra homomorphism.*

Proof: Elementary. □

Theorem 4.6.3. *$\mathscr{B}\mathrm{id} = \mathrm{id}$, and $\mathscr{B}(f \circ g) = \mathscr{B}g \circ \mathscr{B}f$.*

Proof: Elementary. □

4.7. Duality of functions

Theorem 4.7.1. *If X, Y are compact Hausdorff zero-dimensional spaces, and $f : X \to Y$ is continuous, then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta_X\downarrow & & \downarrow\eta_Y \\ \mathscr{S}\mathscr{B}X & \underset{\mathscr{S}\mathscr{B}f}{\longrightarrow} & \mathscr{S}\mathscr{B}Y \end{array}$$

Proof: Suppose that $x \in X$.

Then $U \in (\mathscr{S}\mathscr{B}f)(\eta_X(x))$ iff $(\mathscr{B}f)(U) \in \eta_X(x)$ iff $x \in (\mathscr{B}f)(U)$ iff $x \in f^{-1}(U)$ iff $f(x) \in U$.

So $(\mathscr{S}\mathscr{B}f)(\eta_X(x)) = \{U : f(x) \in U\} = \eta_Y(f(x))$, as required. □

Theorem 4.7.2. *If $\mathbb{A}, \mathbb{B}$ are compact Hausdorff zero-dimensional spaces, and $\phi : \mathbb{A} \to \mathbb{B}$ is continuous, then the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\phi} & \mathbb{B} \\ \eta_{\mathbb{A}}\downarrow & & \downarrow\eta_{\mathbb{B}} \\ \mathscr{B}\mathscr{S}\mathbb{A} & \underset{\mathscr{B}\mathscr{S}\phi}{\longrightarrow} & \mathscr{B}\mathscr{S}\mathbb{B} \end{array}$$

Proof: Suppose that $a \in \mathbb{A}$.

Then $(\mathscr{B}\mathscr{S}\phi)(\eta_{\mathbb{A}})(a)$ is a clopen set in $\mathscr{S}\mathbb{B}$. We try to identify which one.

$q \in (\mathscr{B}\mathscr{S}\phi)(\eta_{\mathbb{A}})(a)$ if and only if $(\mathscr{S}\phi)(q) \in \eta_{\mathbb{A}}(a)$ if and only if $a \in (\mathscr{S}\phi)(q)$ if and only if $\phi(a) \in q$ if and only if $q \in \eta_{\mathbb{B}}(\phi(a))$.

So $(\mathscr{B}\mathscr{S}\phi)(\eta_{\mathbb{A}})(a) = \eta_{\mathbb{B}}(\phi(a))$, as required. □

If, as mathematicians, we are less interested in what objects are than in their behaviour, then the operators $\mathscr{S}\mathscr{B}$ and $\mathscr{B}\mathscr{S}$ are very close to being identity operators; so close that we tend to describe $\mathscr{B}$ and $\mathscr{S}$ as being mutually inverse.