C3.3 Differentiable Manifolds Lecture Notes. MT22. Prof Joyce. Lecture 1

1. Manifolds

Reading: Hitchin notes §2.

1.1 Topological manifolds

Definition A topological space X is a topological manifold of dimension $n \in \mathbb{N}$, if:

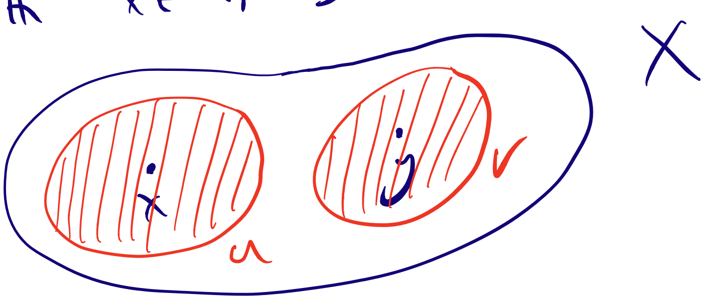
(i) X is Hausdorff;

(ii) X is second countable;

(iii) for all $x \in X$, there is an open neighborhood $x \in V \subseteq X$, an open set $U \subseteq \mathbb{R}^n$, and a homeomorphism $\phi: U \longrightarrow V$.

That is, X is locally homeomorphic to $\mathbb{R}^n$.

Hausdorff and second countable are global topological assumptions. X is Hausdorff if $\forall\, x \neq y$ in X, there exist open $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.



X is <u>second countable</u> if there exists a countable set $\mathcal{B} = \{ U_n : n = 1, 2, \dots \}$ of open sets in X, such that every open set in X is a union of sets in $\mathcal{B}$.

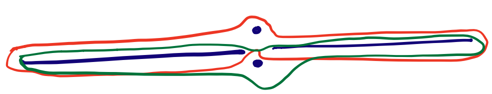
X second countable means X is not "too big".

For instance, we need X second countable to show that every manifold is a submanifold of $\mathbb{R}^n$, $n \gg 0$.

Some authors use X <u>paracompact</u> (weaker), instead.

The only sensible notion of "morphisms" of topological manifolds is continuous map.

<u>Examples</u> (a) $\mathbb{R}^n$ and $S^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \dots + x_n^2 = 1 \}$ are topological manifolds of dimension n.

(b)  Line with 2 origins satisfies (ii), (iii), but is not a topological manifold, as not Hausdorff.

(c) Let S be any set. Make it into a topological space with the discrete topology. Then S is a topological manifold, of dimension 0, iff S is countable (needed for second countable).

Topological manifolds can only have countably many connected components.

1.2. Smooth manifolds

Want to do *calculus* on manifolds — differentiation and integration. But on topological manifolds there is no meaningful notion of differentiable function. A *smooth structure* is an additional structure on a topological manifold, which tells us which functions are differentiable. We express this in terms of an *atlas of charts*.

Definition Let X be a topological space. A *chart* on X, of *dimension* $n \in \mathbb{N}$, is a pair (U, ϕ) with $U \subseteq \mathbb{R}^n$ open, and $\phi : U \to X$ a continuous map, such that $\phi(U) \subseteq X$ is open, and $\phi : U \to \phi(U)$ is a homeomorphism.

Definition Two charts (U, ϕ), (V, ψ) on X are *compatible* if

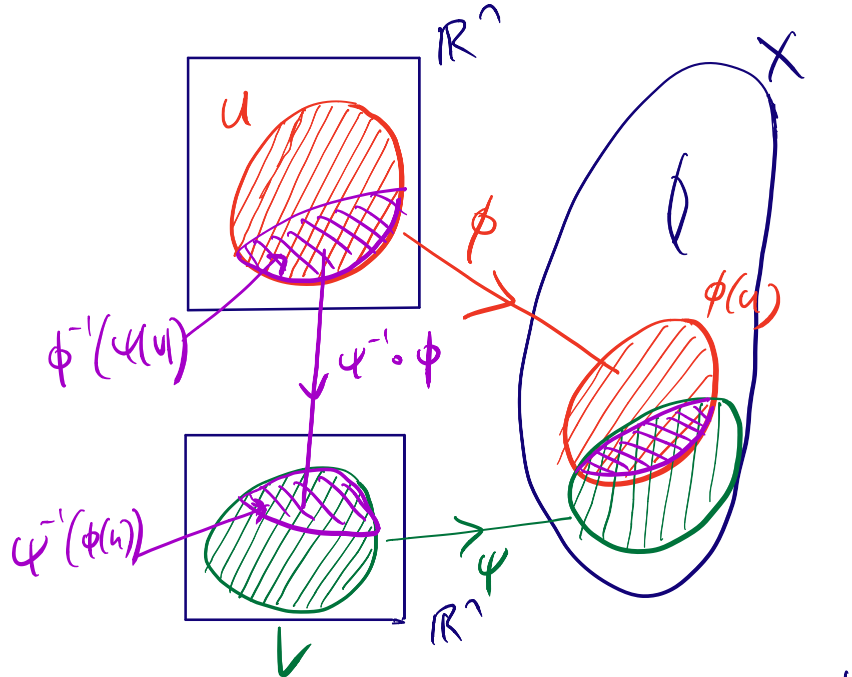
$$\psi^{-1} \circ \phi : \phi^{-1}(\psi(V)) \to \psi^{-1}(\phi(U))$$

is a *smooth* map of open sets in $\mathbb{R}^n$, with smooth inverse. (*Smooth* = has infinitely many derivatives, C^∞.)

It is automatic that $\psi^{-1} \circ \phi$ is continuous, with continuous inverse. We want smooth as well.

Picture:

$$\begin{tikzcd}
\phi^{-1}(\psi(V)) \subseteq U \subseteq \mathbb{R}^n \arrow[dr, "\phi"] \arrow[dd, "\psi^{-1}\circ\phi"'] & \\
& X \arrow[dl, dotted, shift left, "\psi^{-1}"] \\
\psi^{-1}(\phi(U)) \subseteq V \subseteq \mathbb{R}^n \arrow[ur, "\psi"] &
\end{tikzcd}$$



Definition An <u>atlas</u> on X of <u>dimension</u> $n \in \mathbb{N}$ is a family $\{(U_i, \phi_i) : i \in I\}$ of charts of dimension n on X, such that (U_i, ϕ_i), (U_j, ϕ_j) are compatible for all $i, j \in I$, and $X = \bigcup_{i \in I} \phi_i(U_i)$.

An atlas is called <u>maximal</u> if it is not a proper subset of any atlas.

If $\{(U_i, \phi_i) : i \in I\}$ is an atlas on X, then the set of all charts (U, ϕ) on X which are compatible with (U_i, ϕ_i) for all $i \in I$ is a maximal atlas on X, and it is the <u>unique</u> maximal atlas containing $\{(U_i, \phi_i) : i \in I\}$.

Definition A (smooth) manifold $(X, \mathcal{A})$ of dimension $n \in \mathbb{N}$ is a Hausdorff, second countable topological space X together with a maximal atlas $\mathcal{A}$ of dimension n.

Then X is a topological manifold.

Usually we just call X the manifold, leaving $\mathcal{A}$ implicit. A chart on X is an element (U, ϕ) of $\mathcal{A}$. Then $V = \phi(U)$ is open in X, and $\phi^{-1} = (x_1, \dots, x_n) : V \to \underset{\underset{U}{\cup\!\shortmid}}{\mathbb{R}^n}$ is a local coordinate system on X.

Example $X = \mathbb{R}^n$ has an atlas $\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$ with only one chart. This extends to a unique maximal atlas, making $\mathbb{R}^n$ into an n-dimensional manifold.

Example $X = S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \dots + x_n^2 = 1\}$ is an n-dimensional manifold. It has an atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ with two charts, where $U_1 = U_2 = \mathbb{R}^n$, $\phi_1(U_1) = S^n \setminus \{(-1, 0, \dots, 0)\}$, $\phi_2(U_2) = S^n \setminus \{(1, 0, \dots, 0)\}$, and ϕ_1, ϕ_2 are the inverses of

$$\phi_1^{-1} : (x_0, \dots, x_n) \longmapsto \frac{1}{1+x_0}(x_1, \dots, x_n) = (y_1, \dots, y_n)$$

$$\phi_2^{-1} : (x_0, \dots, x_n) \longmapsto \frac{1}{1-x_0}(x_1, \dots, x_n) = (z_1, \dots, z_n)$$

As $x_0^2 + \dots + x_n^2 = 1$ we have

$$(y_1^2 + \dots + y_n^2)(z_1^2 + \dots + z_n^2) = \frac{1}{(1+x_0)^2(1-x_0)^2}(x_1^2 + \dots + x_n^2)^2 = 1.$$

So $\phi_2^{-1} \circ \phi_1 : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ maps

$$(y_1, \dots, y_n) \longmapsto \frac{1}{y_1^2 + \dots + y_n^2}(y_1, \dots, y_n) = (z_1, \dots, z_n).$$

This is smooth with smooth inverse, so (U_1, ϕ_1), (U_2, ϕ_2) are compatible, and $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is an atlas, which extends to a unique maximal atlas.

Example $X = T^n = \mathbb{R}^n / \mathbb{Z}^n$ is an n-manifold. It has an atlas $\{(U_{\underline{y}}, \phi_{\underline{y}}) : \underline{y} \in \mathcal{Y}\}$, where $\mathcal{Y} = \{\underline{y} = (y_1, \dots, y_n) : y_i \in \{0, \frac{1}{2}\}\}$

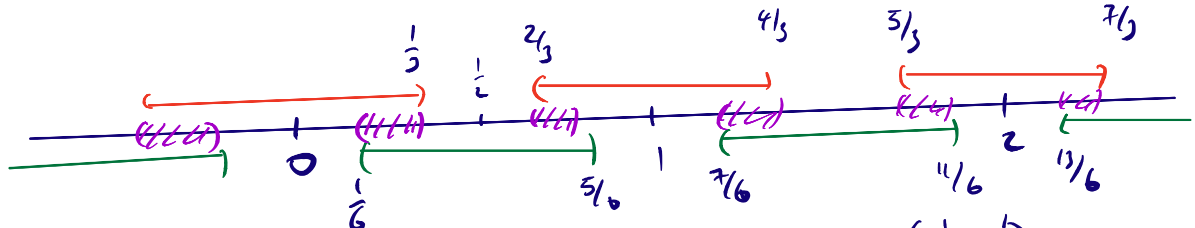
and $U_{\underline{y}} = (-\frac{1}{3}, \frac{1}{3})^n$, and

$$\phi_{\underline{y}} : (x_1, \dots, x_n) \longmapsto (x_1 + y_1 + \mathbb{Z}, \dots, x_n + y_n + \mathbb{Z}).$$

The transition maps $\phi_{\underline{y}_2}^{-1} \circ \phi_{\underline{y}_1}$ map

$$x_i \longmapsto \begin{cases} x_i + \frac{1}{2} & \text{or} \\ x_i & \text{or} \\ x_i - \frac{1}{2} \end{cases} \quad \text{for } i = 1, \dots, n, \text{ locally.}$$

e.g. for $n = 1$, $X = S^1$.



$$\phi_{\frac{1}{2}} \circ \phi_0 : (-\tfrac{1}{3}, -\tfrac{1}{6}) \amalg (\tfrac{1}{6}, \tfrac{1}{3}) \longrightarrow (-\tfrac{1}{3}, -\tfrac{1}{6}) \amalg (\tfrac{1}{6}, \tfrac{1}{3})$$

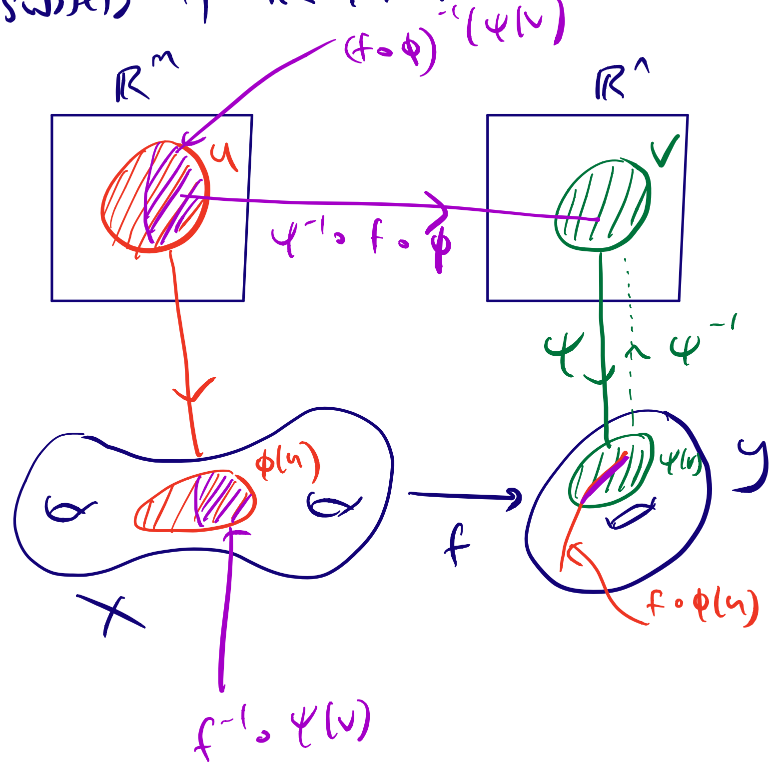
$$x \longmapsto \begin{cases} x + \frac{1}{2} & x \in (-\frac{1}{3}, -\frac{1}{6}) \\ x - \frac{1}{2} & x \in (\frac{1}{3}, \frac{1}{6}). \end{cases}$$

1.3. Smooth maps between manifolds.

Definition. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be manifolds of dimensions m, n, and $f: X \to Y$ be a continuous map. We say that f is <u>smooth</u> if whenever $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, then $\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \to V$ is a smooth map between open subsets of $\mathbb{R}^m, \mathbb{R}^n$.

<u>Picture:</u>

$$\begin{tikzcd}
U \supseteq (f\circ\phi)^{-1}(\psi(V)) \arrow[r, dashed, "\psi^{-1}\circ f\circ\phi"] \arrow[d, "\phi"'] \arrow[d, "\phi", shift left=3] & V \arrow[d, "\psi"'] \\
X \arrow[r] & Y \arrow[u, dotted, bend right, "\psi"']
\end{tikzcd}$$



A <u>diffeomorphism</u> $f: X \to Y$ is a smooth map with smooth inverse. This is the natural notion of isomorphism of manifolds.

Lemma. If $f: X \to Y$ and $g: Y \to Z$ are smooth maps of manifolds then $g\circ f: X \to Z$ is smooth. Identities $\mathrm{id}_X : X \to X$ are smooth. Thus, manifolds and smooth maps form a <u>category</u>.

Proof. To show $g\circ f$ smooth, let $(U,\phi), (V,\psi), (W,\chi)$ be charts on X, Y, Z. Then we have:

$$\begin{tikzcd}[column sep=large]
(f\circ\phi)^{-1}(\psi(V)) \cap (g\circ f\circ\phi)^{-1}(\chi(W)) \arrow[r, "\underset{\text{smooth as } f \text{ is}}{\psi^{-1}\circ f\circ\phi}"'] \arrow[rr, bend left, "\chi^{-1}\circ(g\circ f)\circ\phi \leftarrow \text{smooth}"] \arrow[d, "\phi"] & (g\circ\psi)^{-1}(\chi(W)) \arrow[r, "\underset{\text{smooth as } g \text{ is}}{\chi^{-1}\circ g\circ\psi}"'] \arrow[d, "\psi"] & W \arrow[d, "\chi"] \\
X \arrow[r, "f"'] & Y \arrow[u, dotted, bend left, "\psi^{-1}"] \arrow[r, "g"] & Z . \arrow[u, dotted, bend left, "\chi^{-1}"]
\end{tikzcd}$$

Hence $\chi^{-1}\circ(g\circ f)\circ\phi$ is smooth on $(g\circ f\circ\phi)^{-1}(\chi(W)) \cap (f\circ\phi)^{-1}(\psi(V))$. As the for (V,ψ) covering Y, it is smooth on $(g\circ f\circ\phi)^{-1}(\chi(W))$. The rest is easy. $\square$

Manifolds behave nicely under <u>products</u>.

If X, Y are manifolds, then $X \times Y$ has a unique manifold structure such that if (U,ϕ), (V,ψ) are charts on X, Y then $(U\times V, \phi\times\psi)$

is a chart on $X \times Y$.

If $f: X \to Y$, $g: X \to Z$ are smooth, the <u>direct product</u> $(f,g): X \to Y \times Z$, $(f,g)(x) = (f(x), g(x))$, is smooth.

If $f: W \to Y$, $g: X \to Z$ are smooth, the <u>product</u> $f \times g: W \times X \to Y \times Z$, $(f \times g)(w,x) = (f(w), g(x))$, is smooth.

2. Tangent and cotangent bundles

2.1 The algebra $C^\infty(X)$ of a manifold X.

<u>Definition</u> Let X be a manifold. Write $C^\infty(X)$ for the set of smooth functions $f: X \to \mathbb{R}$. Then $C^\infty(X)$ is an <u>$\mathbb{R}$-algebra</u>, under pointwise addition, multiplication, and scalar multiplication.

If $\dim X > 0$ then $C^\infty(X)$ is infinite-dimensional.

We can recover X completely, up to diffeomorphism, from the $\mathbb{R}$-algebra $C^\infty(X)$.

Points $x \in X$ $\longleftrightarrow$ $\mathbb{R}$-algebra morphisms $x_*: C^\infty(X) \to \mathbb{R}$ mapping $x_*: f \longmapsto f(x)$.

This determines X as a set.

The topology on X is the weakest such that $f: X \longrightarrow \mathbb{R}$ is continuous for all $f \in C^\infty(X)$.

There is then a unique manifold structure on X such that $f: X \longrightarrow \mathbb{R}$ is smooth for all $f \in C^\infty(X)$.

Let $g: X \longrightarrow Y$ be a smooth map. Then $g^*: C^\infty(Y) \longrightarrow C^\infty(X)$ is an $\mathbb{R}$-algebra morphism.

Conversely, any $\mathbb{R}$-algebra morphism $\gamma: C^\infty(Y) \to C^\infty(X)$ is $\gamma = g^*$ for a unique smooth map $g: X \to Y$.

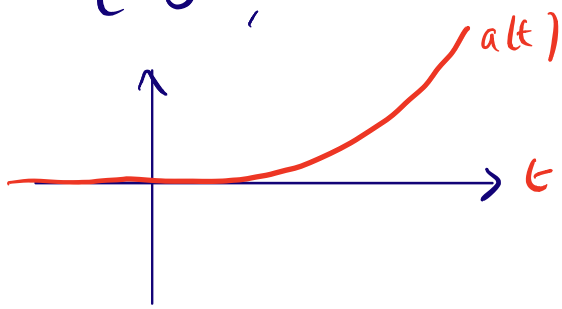
<u>Moral:</u> the $\mathbb{R}$-algebra $C^\infty(X)$ knows everything about the manifold X.

We will show that there are many smooth functions on X.

<u>Example 2.1.</u> Define $a: \mathbb{R} \longrightarrow \mathbb{R}$,

$$a(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

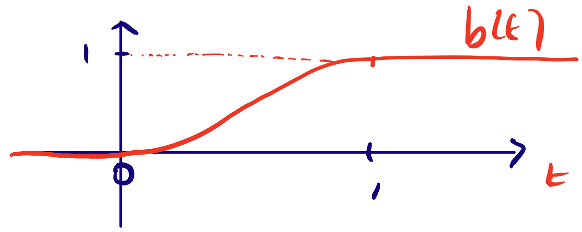
Then a is smooth.



Define $b: \mathbb{R} \to \mathbb{R}$, $\quad b(t) = \dfrac{a(t)}{a(t) + a(1-t)}$

Then b is smooth with

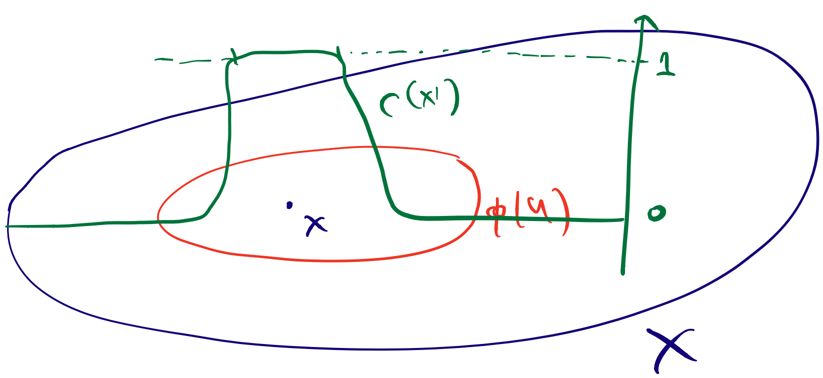
$$b(t) = \begin{cases} 0 & t \le 0 \\ 1 & t \geq 1 \end{cases}$$



Let X be a manifold, and $x \in X$. Choose a chart (U, ϕ) on X with $0 \in U \subseteq \mathbb{R}^n$, $\phi(0) = x$. Choose $\varepsilon > 0$ with $\overline{B_{\sqrt{2}\varepsilon}(0)} \subseteq U \subseteq \mathbb{R}^n$.

Define $c: X \to \mathbb{R}$ by

$$c(x') = \begin{cases} b\left(2 - \dfrac{x_1^2 + \dots + x_n^2}{\varepsilon^2}\right), & \text{if } x' = \phi(x_1, \dots, x_n),\ (x_1, \dots, x_n) \in U, \\ 0, & \text{otherwise.} \end{cases}$$



Then c is smooth, and is 1 in $\phi(B_\varepsilon(0)) \ni x$, and is 0 outside $\phi(B_{\sqrt{2}\varepsilon}(x))$.

We call c a <u>bump function</u>.

Also define $d_i : X \longrightarrow \mathbb{R}$ for $i=1,\dots,n$ by

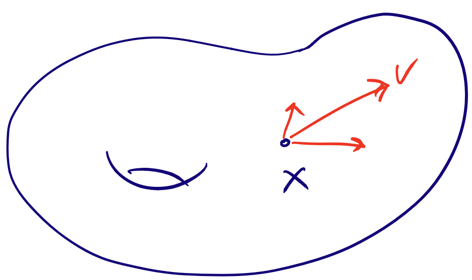
$$d_i(x') = \begin{cases} x_i \cdot b\left(2 - \frac{x_1^2+\dots+x_n^2}{\varepsilon^2}\right), & \text{if } x' = \phi(x_1,\dots,x_n), \\ & (x_1,\dots,x_n) \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Then $d_1,\dots,d_n$ are smooth on X, and in $\phi(B_\varepsilon(0)) \ni x$, $d_1,\dots,d_n$ agree with the local coordinates $(x_1,\dots,x_n) = \phi^{-1}$ on X.

So on any manifold, there are enough <u>global</u> smooth functions to define <u>local</u> coordinate systems near any point.

2.2. Tangent vectors and tangent spaces

Let X be a manifold and $x \in X$. We will define a vector space $T_x X$ called the <u>tangent space</u> to X at x. Elements $v \in T_x X$ are <u>tangent vectors</u>.



Heuristically they point in some direction in X at x. Think of them as the velocity of a point moving in X.

<u>Definition.</u> Let X be a manifold and $x \in X$. A <u>tangent vector</u> at x is a linear map $v : C^\infty(X) \to \mathbb{R}$ satisfying

$$\underbrace{v(ab) = a(x) \cdot v(b) + b(x) \cdot v(a)}_{\text{"Leibnitz rule": Compare product rule for differentiation.}} \quad \forall a, b \in C^\infty(X)$$

Tangent vectors form a vector space $T_x X$, the tangent space to X at x, a vector subspace of $C^\infty(X)^*$.

Proposition 2.2. Let X be an n-manifold, (U, ϕ) be a chart on X, and $(u_1, \dots, u_n) \in U$ with $\phi(u_1, \dots, u_n) = x \in X$. Then $v : C^\infty(X) \longrightarrow \mathbb{R}$ is a tangent vector iff it is of the form

$$v(a) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}(a \circ \phi)\Big|_{(u_1, \dots, u_n)},$$

for unique $v_1, \dots, v_n \in \mathbb{R}$. Hence $T_x X \cong \mathbb{R}^n$. This gives a basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ for $T_x X$, where $(x_1, \dots, x_n)$ are local coordinates on X near x.

Proof. For the 'if' part, let $v_1, \dots, v_n \in \mathbb{R}$ and set

$$v(a) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}(a \circ \phi)\Big|_{(u_1, \dots, u_n)}$$

for $a \in C^\infty(X)$. Then $v(ab) = a(x)v(b) + b(x)v(a)$ follows from the product rule, so v is a tangent vector. For the 'only if' part, by Example 2.1 we can define smooth $d_1, \dots, d_n : X \longrightarrow \mathbb{R}$ with $d_i \circ \phi(x_1, \dots, x_n) = x_i - u_i$ in an open neighbourhood of x in X.

Let $v \in T_x X$, and set $v_i = v(d_i)$ for $i = 1, \dots, n$.

Using Taylor's Theorem for $a \circ \phi : U \longrightarrow \mathbb{R}$ at $(u_1, \dots, u_n)$, we can write

$$a = a(x) \cdot 1 + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a \circ \phi)\Big|_{(u_1, \dots, u_n)} \cdot d_i$$

$$+ \sum_{i,j=1}^{n} F_{ij} \cdot d_i \cdot d_j + g,$$

where $F_{ij} : X \longrightarrow \mathbb{R}$ and $g : X \longrightarrow \mathbb{R}$ are smooth with $g = 0$ in a neighbourhood of X.

Using Example 2.1 we write $g = g \cdot (1 - c)$ where $c = 1$ near x. So

$$v(a) = a(x) \cdot v(1) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a \circ \phi)\Big|_{(u_1, \dots, u_n)} \cdot v_i$$

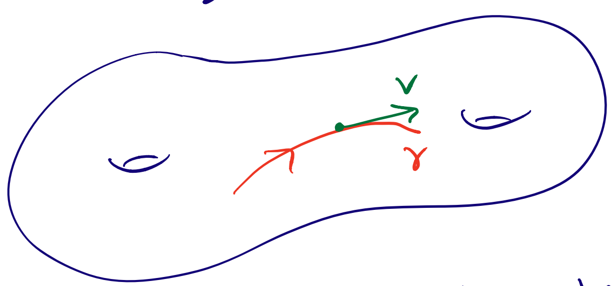
$$+ \sum_{i,j=1}^{n} v((F_{ij}\, d_i)\, d_j) + v(g(1-c))$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x} (a \circ \phi)\Big|_{(u_1, \dots, u_n)} \cdot v_i,$$

since $v(1) = 0$ as $v(1 \cdot 1) = 1(x) \cdot v(1) + v(1) \cdot 1(x) = 2v(1)$, and $v((F_{ij}\, d_i)\, d_j) = v(g(1-c)) = 0$ as $F_{ij}\, d_i\big|_x = d_j\big|_x = g\big|_x = (1-c)\big|_x = 0$.

$\square$

Example Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow X$ be smooth with $\gamma(0) = x$. Define $v : C^\infty(X) \longrightarrow \mathbb{R}$ by $v(a) = \frac{d}{dt}(a \circ \gamma(t))\Big|_{t=0}$. Then the product rule shows that $v \in T_x X$. So the velocity of a moving point $\gamma(t)$ in X is a tangent vector at $\gamma(t)$.



Definition Let $f : X \longrightarrow Y$ be a smooth map of manifolds, and $x \in X$ with $f(x) = y$. Define $T_x f : T_x X \longrightarrow T_y Y$ by $(T_x f)(v) : a \longmapsto v(a \circ f)$, for $v \in T_x X$ and $a \in C^\infty(Y)$. This is a well-defined linear map. If $g : Y \to Z$ is smooth with $g(y) = z$ then $T_x(g \circ f) = T_y g \circ T_x f : T_x X \to T_z Z$. So tangent spaces are covariantly functorial.

2.3. Cotangent spaces and 1-forms

Let X be a manifold and $x \in X$.

<u>Definition</u> Define the <u>cotangent space</u> $T_x^* X$ to be the dual vector space $(T_x X)^*$. Elements of $T_x^* X$ are called <u>1-forms</u>.

If $(x_1, \dots, x_n)$ are local coordinates on X near x then $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are a basis for $T_x X$. We write $dx_1, \dots, dx_n$ for the dual basis for $T_x^* X$. If $f: X \to Y$ is smooth and $x \in X$ with $f(x) = y$, we write $T_x^* f: T_y^* Y \to T_x^* X$ for the linear map dual to $T_x f: T_x X \to T_y Y$.

For $g: Y \to Z$ smooth with $g(y) = z$ we have $T_x^*(g \circ f) = T_x^* f \circ T_y^* g$, so cotangent spaces are <u>contravariantly functorial</u>.

Proposition 2.3 Let X be a manifold and $x \in X$. Write $I_x = \{a \in C^\infty(X) : a(x) = 0\}$, an ideal in $C^\infty(X)$. Write I_x^2 for the vector subspace of $C^\infty(X)$ generated by ab for $a, b \in I_x$, also an ideal in $C^\infty(X)$. Then there is a canonical isomorphism $T_x^*X \cong C^\infty(X)/(\langle 1\rangle_{\mathbb{R}} \oplus I_x^2)$.

If $(x_1, \ldots, x_n)$ are local coordinates on X near x, then $\langle 1\rangle_{\mathbb{R}} \oplus I_x^2$ is the kernel of the surjective linear map $C^\infty(X) \to \mathbb{R}^n$ mapping $a \longmapsto \left(\frac{\partial a}{\partial x_1}\Big|_x, \ldots, \frac{\partial a}{\partial x_n}\Big|_x\right)$.

Proof. By definition $T_xX \subset C^\infty(X)^*$. Thus there is a natural isomorphism $T_x^*X \cong C^\infty(X)/W$, where $W \subset C^\infty(X)$ is the vector subspace of $a \in C^\infty(X)$ with $v(a) = 0$ for all $v \in T_xX$, and the dual pairing $T_x^*X \times T_xX \to \mathbb{R}$ maps $(a + W, v) \longmapsto v(a)$.

If $(x_1, \ldots, x_n)$ are local coordinates at x, then $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are a basis for T_xX, so W is the kernel of $C^\infty(X) \to \mathbb{R}^n$ mapping $a \longmapsto \left(\frac{\partial a}{\partial x_1}\Big|_x, \ldots, \frac{\partial a}{\partial x_n}\Big|_x\right)$. Using Taylor's Theorem, we can also see that $W = \langle 1\rangle_{\mathbb{R}} \oplus I_x^2$. $\square$

<u>Definition</u> Let X be a manifold, $x \in X$, and $a \in C^\infty(X)$.
Define $d_x a \in T_x^* X$ to be the linear map $T_x X \longrightarrow \mathbb{R}$
mapping $v \longmapsto v(a)$. Equivalently, under the isomorphism
$T_x^* X \cong C^\infty(X) / (\langle 1 \rangle_{\mathbb{R}} \oplus I_x^2)$, $d_x a$ is $a + (\langle 1 \rangle_{\mathbb{R}} \oplus I_x^2)$.
We call $d_x a$ the <u>derivative</u> of a.
If $(x_1, \dots, x_n)$ are local coordinates on X near x,
and $dx_1, \dots, dx_n$ are the corresponding basis for $T_x^* X$,
then $d_x a = \left.\frac{\partial a}{\partial x_1}\right|_x \cdot dx_1 + \dots + \left.\frac{\partial a}{\partial x_n}\right|_x \cdot dx_n$.

But $d_x a$ makes sense without choosing coordinates.

2.4. Vector bundles

If X is a manifold it has vector spaces T_xX, T_x^*X which 'vary smoothly with x'. Vector bundles make sense of this idea.

Definition. Let X be a manifold of dimension n. A **vector bundle** $E \longrightarrow X$, of **rank** k, consists of:

(a) A manifold E, of dimension $n+k$.

(b) A smooth map $\pi : E \longrightarrow X$.

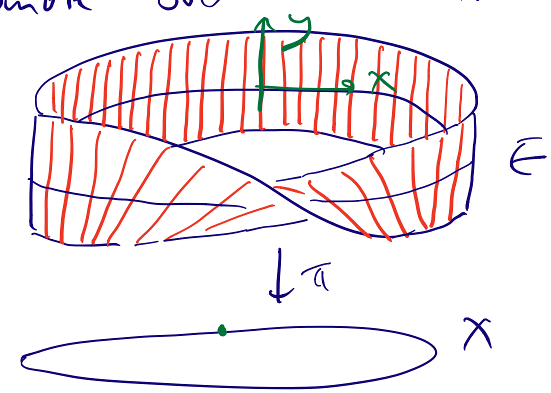
(c) For each $x \in X$ the structure of a vector space on the fibre $E_x := \pi^{-1}(x) \subset E$, such that for every $x \in X$ there is an open neighbourhood $x \in V \subseteq X$ and a diffeomorphism $\pi^{-1}(V) \cong V \times \mathbb{R}^k$ which identifies $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \longrightarrow V$ with $\pi_V : V \times \mathbb{R}^k \longrightarrow V$, and identifies the vector space structure on $\pi^{-1}(v)$ for $v \in V \subseteq X$ with that on $\{v\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

$$\begin{tikzcd}
E \supseteq \arrow[d, "\pi"] & \pi^{-1}(V) \arrow[r, "\text{diffeomorphism}"] \arrow[d, "\pi|_{\pi^{-1}(V)}"] & V \times \mathbb{R}^k \arrow[d] \\
X \supseteq & V \arrow[r, equal] & V
\end{tikzcd}$$

"locally trivial".

__Example__ (i) The __trivial vector bundle__ is $\pi_X : X \times \mathbb{R}^k \longrightarrow X$.

(ii) Set $X = \mathbb{R}/\mathbb{Z}$, the circle. Define $E = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts by $n : (x,y) \longmapsto (x+n, (-1)^n y)$. Define $\pi : E \longrightarrow X$ by $\pi : (x,y)\mathbb{Z} \longmapsto x\mathbb{Z}$. Then E is a nontrivial vector bundle over S^1 with fibre $\mathbb{R}$, the __Möbius strip__.



__Definition__ Let X be a manifold, and $\pi : E \longrightarrow X$ a vector bundle. A __section__ of E is a smooth map $s : X \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X : X \longrightarrow X$. Then $s(x) \in E_x$ for all $x \in X$. Write $C^\infty(E)$ or $\Gamma^\infty(E)$ for the set of sections of E.

Using the vector space structures on E_x, can add and scalar multiply sections: $(\lambda s + \mu t)(x) = \lambda s(x) + \mu t(x) \in E_x$ for $s, t \in \Gamma^\infty(E)$, $\lambda, \mu \in \mathbb{R}$. Local triviality implies $\lambda s + \mu t : X \longrightarrow E$ is smooth, so $\lambda s + \mu t \in \Gamma^\infty(E)$. This makes $\Gamma^\infty(E)$ into a __real vector space__ (usually infinite-dimensional).

2.5 Tangent and cotangent bundles

Definition Let X be an n-manifold. As sets, define

$$TX = \{(x,v) : x \in X,\ v \in T_xX\},$$

$$T^*X = \{(x,\alpha) : x \in X,\ \alpha \in T_x^*X\},$$ with maps

$\pi : TX \to X$, $\pi : (x,v) \mapsto x$, $\pi : T^*X \to X$, $\pi : (x,\alpha) \mapsto x$.

Let (U,ϕ) be a chart on X, giving local coordinates $(x_1,\ldots,x_n) = \phi^{-1}$ on $\phi(U) \subseteq X$. Then we have bases $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ for T_xX and $dx_1,\ldots,dx_n$ for T_x^*X, all $x \in \phi(U)$.

Define $T\phi : U \times \mathbb{R}^n \to TX$, $T^*\phi : U \times \mathbb{R}^n \to T^*X$ by

$$T\phi(u_1,\ldots,u_n,v_1,\ldots,v_n) = \left(\phi(u_1,\ldots,u_n),\ v_1\frac{\partial}{\partial x_1} + \cdots + v_n\frac{\partial}{\partial x_n}\right),$$

$$T^*\phi(u_1,\ldots,u_n,w_1,\ldots,w_n) = \left(\phi(u_1,\ldots,u_n),\ w_1\,dx_1 + \cdots + w_n\,dx_n\right).$$

$(U \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ open.$)$

Then (for suitable topologies on TX, T^*X), $(U \times \mathbb{R}^n, T\phi)$ and $(U \times \mathbb{R}^n, T^*\phi)$ are charts on TX, T^*X.

If (U,ϕ) and (V,ψ) are (compatible) charts on X, then

$\psi^{-1}\circ\phi = \left(f_1(u_1,\dots,u_n),\dots,f_n(u_1,\dots,u_n)\right)$ is smooth with smooth inverse.

Calculation shows that

$$(T\psi)^{-1}\circ(T\phi) : (u_1,\dots,u_n,v_1,\dots,v_n) \longmapsto$$

$$\left(f_1(u_1,\dots,u_n),\dots,f_n(u_1,\dots,u_n),\ \sum_{j=1}^{n}\frac{\partial f_1}{\partial u_j}(u_1,\dots,u_n)v_j,\dots,\sum_{j=1}^{n}\frac{\partial f_n}{\partial u_j}(u_1,\dots,u_n)v_j\right),$$

where $\left(\frac{\partial f_i}{\partial u_j}\right)_{i,j=1}^{n}$ is an invertible matrix of smooth functions of $u_1,\dots,u_n$.

Also $(T^*\psi)^{-1}\circ(T^*\phi)$ is the same, but with the transpose inverse matrix $\left(\left(\frac{\partial f_i}{\partial u_j}\right)_{i,j}^{-1}\right)^t$. So $(U\times\mathbb{R}^n, T\phi)$, $(V\times\mathbb{R}^n, T\psi)$ are compatible charts on TX, and similarly for T^*X.

Hence, if $\{(U_i,\phi_i): i\in I\}$ is the maximal atlas on X, then $\{(U_i\times\mathbb{R}^n, T\phi_i): i\in I\}$, $\{(U_i\times\phi_i), T^*\phi_i : i\in I\}$ are atlases on TX, T^*X, which extend to unique maximal atlases, so TX, T^*X are manifolds.

Clearly $\pi: TX\to X$, $\pi: T^*X\to X$ are smooth, and make TX, T^*X into rank n vector bundles.

Sections $v\in\Gamma^\infty(TX)$ are called <u>vector fields</u>.

Sections $\alpha\in\Gamma^\infty(T^*X)$ are called <u>1-forms</u>.

Now let $f : X \to Y$ be a smooth map of manifolds. For $x \in X$ with $f(x) = y$, we defined a linear map $T_x f : T_x X \to T_y Y$.

Define $Tf : TX \to TY$ by $Tf : (x, v) \longmapsto (f(x), T_x f(v))$.

Then Tf is smooth. If $g : Y \to Z$ is smooth then $T(g \circ f) = Tg \circ Tf$.

<u>Warning</u>: there is no natural map $T^*f : T^*X \to T^*Y$ or $T^*Y \to T^*X$.

<u>Aside, not used later:</u>

Another way to express the functoriality of (co)tangent bundles is via <u>pull back vector bundles</u>. If $f : X \to Y$ is smooth and $\pi : E \to Y$ is a vector bundle, we can define the pull back $f^*(E) \to X$, such that points of $f^*(E)$ are (x, e) for $x \in X$, $e \in E$ with $f(x) = \pi(e)$.

Then there are bundle-linear maps of vector bundles on X:

$$Df : TX \to f^*(TY), \quad D^*f : f^*(T^*Y) \to T^*X.$$

2.6 Partitions of unity

Definition. Let X be a manifold, and $\{U_i : i \in I\}$ an open cover of X. A <u>partition of unity of X subordinate to $\{U_i : i \in I\}$</u> is $\{\eta_i : i \in I\}$ with $\eta_i : X \longrightarrow \mathbb{R}$ smooth for $i \in I$, satisfying:

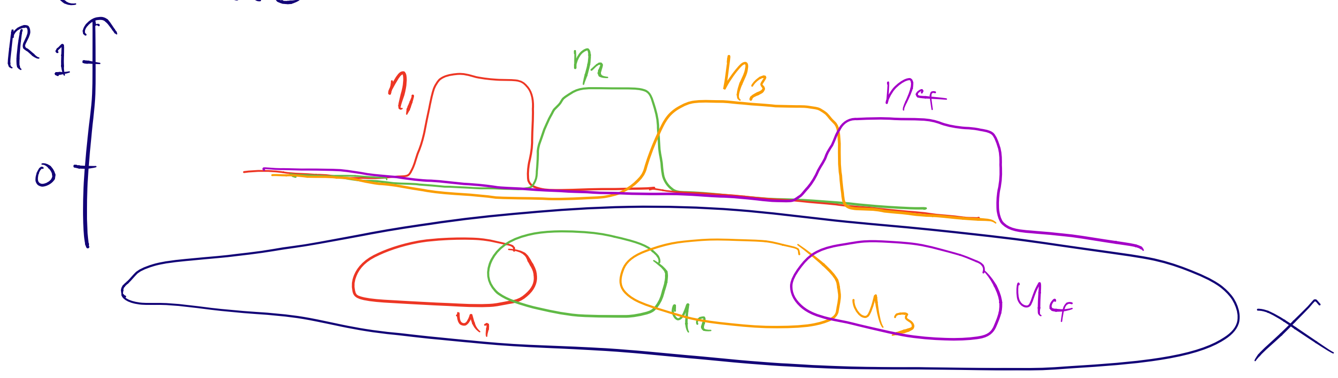
(i) $\eta_i(X) \subseteq (0,1) \subseteq \mathbb{R}$.

Take closure in X

(ii) The (closed) <u>support</u> $\operatorname{supp} \eta_i = \overline{\{x \in X : \eta_i(x) \neq 0\}} \subseteq U_i$ for $i \in I$.

(iii) Each $x \in X$ has an open neighbourhood V in X with $\eta_i|_V = 0$ for all but finitely many $i \in I$ ("locally finite").

(iv) $\sum_{i \in I} \eta_i = 1$, where the sum makes sense by (iii).



Theorem 2.4 A partition of unity exists subordinate to any open cover on X.

– For X compact, proved in Sheet 1, q. 6. For general X, see Hitchin notes.

Partitions of unity are often used to define (global) sections of vector bundles.

For example, if $E \to X$ is a vector bundle, and $\{U_i : i \in I\}$ an open cover of X, and $e_i \in \Gamma^\infty(E|_{U_i})$ for $i \in I$, then we can choose a subordinate partition of unity $\{\eta_i : i \in I\}$, and write $e = \sum_{i \in I} \eta_i e_i$ in $\Gamma^\infty(E)$.

Here $\eta_i e_i$ extends to a smooth section of E on $X \supseteq U_i$, zero outside U_i, as $\operatorname{supp} \eta_i \subseteq U_i$, and the sum makes sense as $\{\eta_i : i \in I\}$ is locally finite.

We will use this method later to show that every manifold has Riemannian metrics.

3. Submersions, immersions and embeddings

(Note: parts of this lecture are not on the syllabus, but will make you into a better person.)

(On the syllabus.)

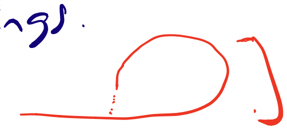
Definition Let $f: X \longrightarrow Y$ be a smooth map of manifolds.

(a) We call f a submersion if $T_xf: T_xX \to T_yY$ is surjective for all $x \in X$ with $f(x)=y$ in Y. This possible only if $\dim X \geq \dim Y$.

(b) We call f an immersion if $T_xf: T_xX \to T_yY$ is injective for all $x \in X$ with $f(x)=y$ in Y. This is only possible if $\dim X \leq \dim Y$.

(c) We call f an embedding if it is an immersion and $f: X \to f(X)$ is a homeomorphism of topological spaces, where $f(X) \subseteq Y$ has the subspace topology.

Embeddings are injective immersions, but not all injective immersions are embeddings.

[Consider $f: \mathbb{R} \longrightarrow \mathbb{R}^2$ ]

An <u>immersed</u> or <u>embedded submanifold</u> X in Y is an immersion or embedding $i : X \to Y$, though often we leave the map i implicit.

3.1. Fibres of submersions are submanifolds

(On the syllabus.)

Theorem 3.1. Let $f : X \to Y$ be a submersion of manifolds. Then for each $y \in Y$, the fibre $X_y = f^{-1}(y) = \{x \in X : f(x) = y\}$ has the unique structure of a manifold, of dimension $\dim X - \dim Y$, such that the inclusion $i : X_y \hookrightarrow X$ is an embedding.

Proof. Let X_y have the subspace topology. It is Hausdorff and second countable, as X is. We must construct an atlas of charts on X_y. Write $\dim X = m+n$, $\dim Y = n$.

Let $x \in X_y$. Choose charts (U, ϕ) on X with $0 \in U$ and $\phi(0) = x$, and (V, ψ) on Y with $0 \in V$ and $\psi(0) = y$, and write

$\psi^{-1} \circ f \circ \phi = (g_1, \dots, g_n)$, for $g_j = g_j(x_1, \dots, x_{m+n})$ defined on an open neighbourhood R of 0 in $\mathbb{R}^{m+n}$. Then $T_x f : T_x X \to T_y Y$ maps $\frac{\partial}{\partial x_i} \mapsto \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i}(0) \frac{\partial}{\partial y_j}$. As f is a submersion, $T_x f$ is surjective, and $\left(\frac{\partial g_j}{\partial x_i}(0)\right)_{i,j}$ has rank n. Applying a linear transformation to $(x_1, \dots, x_{m+n})$ we can suppose

$$\frac{\partial g_j}{\partial x_i}(0) = \begin{cases} 1 & i = j+m, \ j = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Apply the Implicit Function Theorem to

$$(g_1, \dots, g_n) : \underset{\mathbb{R}^m \times \mathbb{R}^n}{\overset{0 \in}{R}} \longrightarrow \mathbb{R}^n.$$

Since $\partial g|_0 : 0 \times \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, this says that there exist open neighbourhoods S, T of 0 in $\mathbb{R}^m, \mathbb{R}^n$ with $S \times T \subseteq R \subseteq \mathbb{R}^{m+n}$ and a smooth function $h = (h_1, \dots, h_n) : S \to T$ such that

$$\left\{ \underset{\in S \times T}{(x_1, \dots, x_{m+n})} : \underset{j=1,\dots,n}{g_j(x_1, \dots, x_{m+n}) = 0,} \right\} = \left\{ (x_1, \dots, x_m, h_1(x_1, \dots, x_m), \dots, h_n(x_1, \dots, x_m)) : (x_1, \dots, x_m) \in S \right\}.$$

Define $\chi : S \to X_y$ by

$$\chi(x_1, \dots, x_m) = \phi(x_1, \dots, x_m, h_1(x_1, \dots, x_m), \dots, h_n(x_1, \dots, x_m)).$$

Then (S, χ) is a chart on X_y, with $\chi(0) = x$. The family of such charts cover X_y. To see they are compatible, note that if $a : X \to \mathbb{R}$ is smooth then $a \circ \chi : S \to \mathbb{R}$ is smooth, and so deduce transition functions are smooth. Thus all such charts form an atlas on X_y, making X_y into a manifold, of dimension $m = \dim X - \dim Y$.

From the chart presentation it is clear that $T_x i : \underset{\mathbb{R}^m}{T_x X_y} \to \underset{\mathbb{R}^{m+n}}{T_x X}$ is injective, and $i : X_y \to i(X_y)$ is a homeomorphism by definition, so i is an embedding. $\square$

To show X_y is a manifold, the proof only needs $T_x f$ surjective for all $x \in X_y$, not all $x \in X$. Using this when $Y = \mathbb{R}^n$ and $y = 0$ gives:

<u>Corollary 3.2</u> Suppose X is a manifold and $f_1, \dots, f_n : X \longrightarrow \mathbb{R}$ are smooth such that $d_x f_1, \dots, d_x f_n$ are linearly independent in $T_x^* X$ for all $x \in X$ with $f_1(x) = \dots = f_n(x) = 0$.

Then $Y = \{x \in X : f_1(x) = \dots = f_n(x) = 0\}$ is an embedded submanifold of X, of dimension $\dim X - n$.

This allows us to define many examples of manifolds.

<u>Example</u>. Define $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ by $f(x_0, \dots, x_n) = x_0^2 + \dots + x_n^2 - 1$. As $d_x f \neq 0$ whenever $f(x) = 0$, Corollary 3.2 says that $f^{-1}(0) = S^n$ is an n-manifold. We have no need to write down an atlas for S^n.

Sard's Theorem. Let $f: X \to Y$ be a smooth map. Then for a dense subset of points $y \in Y$ (in fact, for $y \in Y \setminus S$ with S of measure zero), for all $x \in X$ with $f(x) = y$, we have $T_x f : T_x X \to T_y Y$ surjective, so that $X_y = f^{-1}(y)$ is an embedded submanifold of X, of dimension $\dim X - \dim Y$.

$\square$

Proof: Omitted.

So, for any smooth map $f: X \to Y$, $f^{-1}(y)$ is a submanifold of X for almost all y in Y.

3.2. Embeddings and embedded submanifolds.

Proposition 3.3. Let $f: X \to Y$ be an embedding. Write $X' = f(X)$ and $i: X' \to Y$ for the inclusion. Then X' has a unique manifold structure, depending only on X', i, Y and not on X, f, such that $i: X' \hookrightarrow Y$ is an embedding, and then $f: X \to X'$ is a diffeomorphism.

$\square$

Proof. Omitted, but easy.

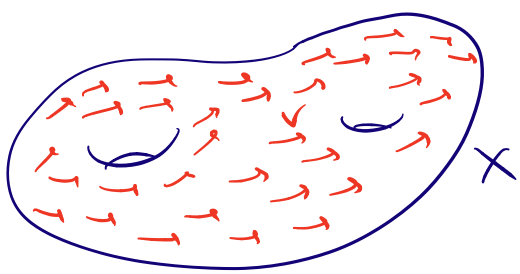
The moral is that we can think of embedded submanifolds $X \hookrightarrow Y$ just as subsets of Y. The analogue is false for immersed submanifolds.

Theorem 3.4. (Whitney Embedding Theorem.) Let X be a manifold of dimension m, and $n > 2m$. Then there exist embeddings $f : X \hookrightarrow \mathbb{R}^n$ (in fact, a generic smooth $f : X \to \mathbb{R}^n$ is an embedding). We can choose f with $f(X)$ closed in $\mathbb{R}^n$. Hence, every manifold is diffeomorphic to a (closed) submanifold of $\mathbb{R}^n$ for $n \gg 0$.

Proof: Omitted, but see Sheet 2 q.4 when X is compact. Needs X second countable. $\square$

4. Vector fields

Let X be a manifold, with tangent bundle TX. A vector field is a smooth section v of TX, $v \in \Gamma^\infty(TX)$. It gives a vector $v_x \in T_x X$ for each $x \in X$, varying smoothly with $x \in X$.



Think of v as the velocity of a fluid in motion on X, e.g. $X = S^2$ could be the surface of the earth, and v the wind velocity.

4.1. Vector fields as derivations, the Lie bracket.

Let X be a manifold.

Proposition 4.1 Then there is a natural 1-1 correspondence between vector fields $v \in \Gamma^\infty(TX)$, and linear maps $\delta: C^\infty(X) \to C^\infty(X)$ satisfying

$$(*) \qquad \delta(ab) = a \cdot \delta(b) + b \cdot \delta(a) \quad \forall a, b \in C^\infty(X),$$

such that $v_x(a) = (\delta(a))(x)$ for all $x \in X$ and $a \in C^\infty(X)$.

Such maps are called <u>derivations</u>.

<u>Proof.</u> Recall that a vector $v_x \in T_x X$ is a linear map $v_x : C^\infty(X) \to \mathbb{R}$ satisfying

(**) $\quad v_x(ab) = a(x)\, v_x(b) + b(x)\, v_x(a) \quad \forall a, b \in C^\infty(X).$

If $\delta : C^\infty(X) \to C^\infty(X)$ is a derivation then restricting (*) to $x \in X$ gives

$$\delta(ab)|_x = a(x)\, \delta(b)|_x + b(x)\, \delta(a)|_x, \quad \text{so}$$

$v_x : C^\infty(X) \to \mathbb{R}$, $\quad v_x(a) = \delta(a)|_x$, lies in $T_x X$.

Hence $v : X \to TX$, $\; v : x \mapsto (x, v_x)$ is a map with $\pi \circ v = \mathrm{id}_X$. Working in coordinates we see v is smooth. So $v \in \Gamma^\infty(TX)$.

Conversely, if $v \in \Gamma^\infty(TX)$ we define $\delta : C^\infty(X) \to C^\infty(X)$ by $\delta(a)(x) = v_x(a)$. Working in coordinates we see $\delta(a) : X \to \mathbb{R}$ is smooth, so $\delta(a) \in C^\infty(X)$, and (**) for each $x \in X$ implies (*). $\square$

Now let $\delta, \varepsilon : C^\infty(X) \to C^\infty(X)$ be derivations, and $a, b \in C^\infty(X)$. Then

$$\delta \circ \varepsilon(ab) = \delta[a \cdot \varepsilon(b) + b \cdot \varepsilon(a)]$$
$$= \delta(a)\varepsilon(b) + a \cdot (\delta \circ \varepsilon)(b) + \delta(b) \cdot \varepsilon(a) + b \cdot (\delta \circ \varepsilon)(a),$$
$$\varepsilon \circ \delta(ab) = \varepsilon[a \cdot \delta(b) + b \cdot \delta(a)]$$
$$= \varepsilon(a) \cdot \delta(b) + a \cdot (\varepsilon \circ \delta)(b) + \varepsilon(b) \cdot \delta(a) + b \cdot (\varepsilon \circ \delta)(a).$$

Subtracting, half the terms cancel:

$$[\delta, \varepsilon](ab) = (\delta \circ \varepsilon)(ab) - (\varepsilon \circ \delta)(ab)$$
$$= a \cdot [\delta, \varepsilon](b) + b \cdot [\delta, \varepsilon](a).$$

Thus, the commutator $[\delta, \varepsilon] = \delta \circ \varepsilon - \varepsilon \circ \delta : C^\infty(X) \to C^\infty(X)$

is also a derivation.

__Definition__ Let X be a manifold and $v, w \in \Gamma^\infty(TX)$ be vector fields. Then v, w correspond to derivations $\delta, \varepsilon : C^\infty(X) \to C^\infty(X)$ by Proposition 4.1. So $[\delta, \varepsilon] = \delta \circ \varepsilon - \varepsilon \circ \delta$ is also a derivation. Define the __Lie bracket__ $[v, w] \in \Gamma^\infty(TX)$ to be the vector field corresponding to δ, ε under Proposition 4.1.

If $(x_1, \dots, x_n)$ are local coordinates on $U \subseteq X$ then we may write

$$v = v_1 \frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n}, \quad w = w_1 \frac{\partial}{\partial x_1} + \dots + w_n \frac{\partial}{\partial x_n}$$

for $v_i, w_j : U \to \mathbb{R}$ smooth.

Then δ, ε act locally by

$$\delta(a) = v_1 \frac{\partial a}{\partial x_1} + \dots + v_n \frac{\partial a}{\partial x_n}, \quad \varepsilon(a) = w_1 \frac{\partial a}{\partial x_1} + \dots + w_n \frac{\partial a}{\partial x_n}.$$

So computing $(\delta \circ \varepsilon)(a) - (\varepsilon \circ \delta)(a)$ shows that

$$[v,w] = \sum_{i,j=1}^{n} \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

in local coordinates.

<u>Proposition 4.2.</u> Let u, v, w be vector fields on X. Then the Lie brackets satisfy the <u>Jacobi identity</u>

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0. \quad (***)$$

<u>Proof.</u> Let $\gamma, \delta, \varepsilon$ be the derivations corresponding to u, v, w. Then $(***)$ corresponds to the equation

$$[\gamma,[\delta,\varepsilon]] + [\delta,[\varepsilon,\gamma]] + [\varepsilon,[\gamma,\delta]] =$$

$$\gamma(\underset{1}{\cancel{\delta\varepsilon}} - \underset{2}{\cancel{\varepsilon\delta}}) - (\underset{3}{\cancel{\delta\varepsilon}} - \underset{4}{\cancel{\varepsilon\delta}})\gamma + \delta(\underset{3}{\cancel{\varepsilon\gamma}} - \underset{5}{\cancel{\gamma\varepsilon}}) - (\underset{6}{\cancel{\varepsilon\gamma}} - \underset{2}{\cancel{\gamma\varepsilon}})\delta + \varepsilon(\underset{6}{\cancel{\gamma\delta}} - \underset{4}{\cancel{\delta\gamma}}) - (\underset{1}{\cancel{\gamma\delta}} - \underset{5}{\cancel{\delta\gamma}})\varepsilon = 0.$$

$\square$

4.2. Flowing along a vector field.

Definition Let X be a manifold. A 1-parameter group of diffeomorphisms of X is a smooth map $\varphi : \mathbb{R} \times X \to X$ such that, writing $\varphi_t : X \to X$, $\varphi_t(x) = \varphi(t,x)$, then

(i) $\varphi_t : X \to X$ is a diffeomorphism;

(ii) $\varphi_0 = \mathrm{id}_X$; and

(iii) $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s,t \in \mathbb{R}$.

Then $t \mapsto \varphi_t$ is a group morphism $\mathbb{R} \to \mathrm{Diff}(X)$.

The group of diffeomorphisms of X.

Given such φ, define $\delta : C^\infty(X) \to C^\infty(X)$ by
$$\delta(a) = \frac{d}{dt}\left(a \circ \varphi_t\right)\Big|_{t=0}.$$

We have
$$\delta(ab) = \frac{d}{dt}\left((a \circ \varphi_t)\cdot(b \circ \varphi_t)\right)\Big|_{t=0}$$

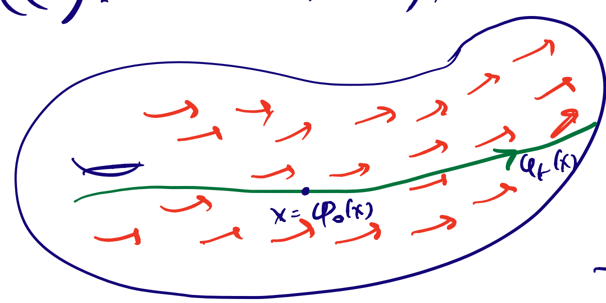
$$= (a\circ\varphi_t)\big|_{t=0}\cdot \frac{d}{dt}(b\circ\varphi_t)\big|_{t=0} + (b\circ\varphi_t)\big|_{t=0}\cdot\frac{d}{dt}(a\circ\varphi_t)\big|_{t=0}$$

$$= a\cdot\delta(b) + b\cdot\delta(a), \quad \text{since } \varphi_0 = \mathrm{id}.$$

Hence δ is a derivation, and corresponds to a vector field $v \in \Gamma^\infty(TX)$ by Proposition 4.1.

We have $\quad v_x = \frac{d}{dt}\left(\varphi_t(x)\right)\big|_{t=0}$ for all $x \in X$.

Thus, each 1-parameter group of diffeomorphisms φ of X gives a vector field $v \in \Gamma^\infty(TX)$. We will show that under additional conditions (e.g. X compact), each v corresponds to a unique φ.



We can understand the relationship of φ and v like this: for each fixed $x \in X$, $t \mapsto \varphi_t(x)$ is a "flow line" of the vector field v, so that $\frac{d}{dt}\left(\varphi_t(x)\right) = v\big|_{\varphi_t(x)}$ for all $t \in \mathbb{R}$, with $\varphi_0(x) = x$ at $t = 0$.

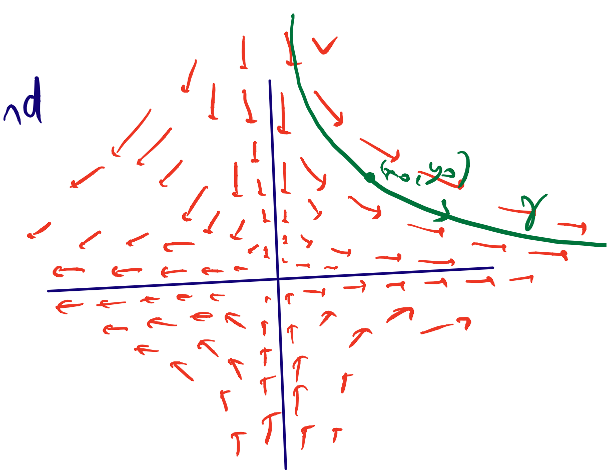
Think of v as the velocity of the wind on the earth (supposed time independent). Then

$t \longmapsto \varphi_t(x)$ is the path of a particle of air, moving in time, starting at x when $t=0$.
This tells us how to reconstruct φ from v: we have to find all the flow lines of v.

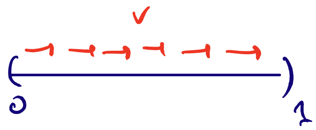
Recap: $\varphi : \mathbb{R} \times X \to X$ 1-parameter group of diffeomorphisms.
Define a vector field $v \in \Gamma^\infty(TX)$ by $v_x = \frac{d}{dx}(\varphi_t(x))|_{t=0}$.
Then $t \mapsto \varphi_t(x)$ is a "flow line" of v, with $\frac{d}{dt}(\varphi_t(x)) = v|_{\varphi_t(x)}$.
We will explain how to reconstruct φ from v.

Definition. Let X be a manifold and $v \in \Gamma^\infty(TX)$.
An integral curve of v is a smooth map $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is an open interval, such that $(T_t\gamma)(\frac{\partial}{\partial t}) = v_{\gamma(t)}$ for all $t \in I$.
An integral curve γ is maximal if it cannot be extended to another integral curve $\gamma' : I' \to X$ with $I \subsetneq I' \subseteq \mathbb{R}$, i.e. I is as large as possible.

Example Take $X = \mathbb{R}^2$ and $v(x,y) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.
Then for every $(x_0, y_0) \in \mathbb{R}^2$, $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (x_0 e^t, y_0 e^{-t})$ is a maximal integral curve.



Example Let $X = (0,1)$ and $v(x) = \frac{\partial}{\partial x}$.



Then for any $x \in X$,

$\gamma : (-x, 1-x) \longrightarrow X$, $\gamma(t) = t + x$,

is a maximal integral curve for v with $\gamma(0) = x$.

Note that in this case the maximal interval is $I = (-x, 1-x) \neq \mathbb{R}$.

Theorem 4.3 Let X be a manifold, $v \in \Gamma^\infty(TX)$, and $x \in X$. Then there exists a unique maximal integral curve $\gamma : I \longrightarrow X$ of v with $\gamma(0) = x$, for $0 \in I \subseteq \mathbb{R}$ an open interval. If X is compact then $I = \mathbb{R}$.

— This is proved using existence and uniqueness of solutions of o.d.e.s.

— It is well known that equations of the form

$$\frac{dx_i}{dt} = f_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n$$

$$x_i(0) = \tilde{x}_i, \quad i = 1, \dots, n$$

for $f_1, \dots, f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth and $\tilde{x}_1, \dots, \tilde{x}_n \in \mathbb{R}$ have unique solutions for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ small.

— In local coordinates on X, the equations for an integral curve are o.d.e.s of this form.

— The maximal interval $I \subseteq \mathbb{R}$ can have $I \neq \mathbb{R}$ only if the curve "falls off the edge of X" / "goes to infinity in X". If X is compact, this cannot happen.

<u>Definition</u> (not standard.) A vector field v on a manifold X is called <u>complete</u> if in Theorem 4.3 we have $I = \mathbb{R}$ for all $x \in X$.

If X is compact then any $v \in \Gamma^\infty(TX)$ is complete.

<u>Theorem 4.4.</u> Let X be a manifold, and $v \in \Gamma^\infty(X)$ a complete vector field. Define $\varphi : \mathbb{R} \times X \longrightarrow X$ such that for each $x \in X$, $t \longmapsto \varphi(t,x)$ is the maximal integral curve $\gamma : \mathbb{R} \to X$ of v with $\gamma(0) = x$ given by Theorem 4.3. Then φ is smooth, and a 1-parameter group of diffeomorphisms of X.

We can recover v from φ by $v_x = \frac{\partial}{\partial t}(\varphi(t,x))\big|_{t=0}$, $\forall x \in X$. This gives a 1-1 correspondence between complete $v \in \Gamma^\infty(X)$ and 1-parameter groups φ of diffeomorphisms of X.

Proof. That φ is smooth follows from the results on o.d.e.s used to prove Theorem 4.3. Writing $\varphi_t(x) = \varphi(t,x)$, consider the function $s \longmapsto \varphi_{s+t}(x)$. This is a flow-line of v, mapping $0 \mapsto \varphi_t(x)$. Thus by uniqueness in Theorem 4.3, it is

$$\varphi(s, \varphi_t(x)) = \varphi_s \circ \varphi_t(x).$$

Hence $\varphi_{s+t} = \varphi_s \circ \varphi_t$, for all $s,t \in \mathbb{R}$, as we want. Also $\varphi_0(x) = x$ by definition, so $\varphi_0 = \mathrm{id}$. Thus $\mathrm{id} = \varphi_{(-t)+t} = \varphi_{-t} \circ \varphi_t$

$= \varphi_t \circ \varphi_{-t}$, and $\varphi_{-t} = (\varphi_t)^{-1}$, so $\varphi_t : X \to X$

is a diffeomorphism.

This proves φ is a 1-parameter group of diffeomorphisms, and $v_x = \frac{\partial}{\partial t}(\varphi(t,x))\big|_{t=0}$ follows from the integral curve of v o.d.e.

To show this is a 1-1 correspondence, given φ, define v as above, and then show that $\gamma : t \longmapsto \varphi(t,x)$ is an integral curve for $x \in X$, of v. So uniqueness in Theorem 4.3 shows that

φ is what we constructed from v, and also v is complete. $\square$

4.3. The Lie derivative

Let X be a manifold, $v \in \Gamma^\infty(TX)$ a complete vector field, and $\varphi : \mathbb{R} \times X \longrightarrow X$, $\varphi_t(x) = \varphi(t,x)$ the corresponding 1-parameter group of diffeomorphisms from Theorem 4.4.

We can use this to define a notion of "differentiation along v" called the Lie derivative.

Suppose α is some geometric quantity we want to differentiate along v. Then we set

$$\mathcal{L}_v(\alpha) = \frac{d}{dt}\left[\varphi_t^*(\alpha)\right)\Big|_{t=0},$$

where $\mathcal{L}_v$ means Lie derivative along v.

This only makes sense if:

(i) α is linear (lies in a vector space)

(ii) α can be pulled back by diffeomorphisms φ_t

<u>Example</u> Let $a \in C^\infty(X)$ be a smooth function. Then $\varphi_t^*(a) = a \circ \varphi_t$, and

$$\mathcal{L}_v a = \frac{d}{dt}(a \circ \varphi_t)\Big|_{t=0} \in C^\infty(X), \text{ and}$$

$\delta : a \longmapsto \mathcal{L}_v a$ is the derivation $\delta : C^\infty(X) \to C^\infty(X)$ corresponding to v in Proposition 4.1.

Let $v \in \Gamma^\infty(TX)$ be a vector field.

<u>Example</u> Let $v \in \Gamma^\infty(TX)$ be a vector field. Then $\mathcal{L}_v w = \frac{d}{dt}(\varphi_t^*(w))\Big|_{t=0} = [v, w]$, the Lie bracket of vector fields.

In coordinates $[v,w]_j = \sum_i \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right)$.

So $\mathcal{L}_v w$ does involve derivatives of v, not just w.

In particular, $\mathcal{L}_v w$ is <u>not pointwise linear</u> in v:

$$\mathcal{L}_{\lambda v_1 + \mu v_2} w \neq \lambda \mathcal{L}_{v_1} w + \mu \mathcal{L}_{v_2} w$$ if λ, μ are non constant functions.

Lie derivatives also make sense for "tensors" and "exterior forms" (next week).

In fact the Lie derivative $\mathcal{L}_v$ is well defined for any vector field v, not just v complete, since only need $\varphi : \mathbb{R}\times X \to X$ to be defined in an open neighbourhood of $0\times X$ in $\mathbb{R}\times X$.

5. Tensor products and exterior algebras

5.1. Tensor products of vector spaces

Definition Let V, W be finite-dimensional vector spaces over $\mathbb{R}$. We will define a vector space $V \otimes W$ called the tensor product of V, W, with the properties:

(i) if $v \in V$ and $w \in W$, there is a product $v \otimes w \in V \otimes W$.

(ii) This is bilinear:
$(\lambda v_1 + \mu v_2) \otimes w = \lambda (v_1 \otimes w) + \mu (v_2 \otimes w)$,
$v \otimes (\lambda w_1 + \mu w_2) = \lambda (v \otimes w_1) + \mu (v \otimes w_2)$.

(iii) $\dim(V \otimes W) = \dim V \cdot \dim W$.

If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases for V, W then $v_i \otimes v_j : i = 1, \ldots, m, \ j = 1, \ldots, n$ is a basis for $V \otimes W$.

(iv) $V \otimes W$ has the universal property that if $B : V \times W \to U$ is a bilinear map to a vector space U, then there is a unique linear map $\beta : V \otimes W \longrightarrow U$ with $B(v,w) = \beta(v \otimes w)$.

One way to define $V \otimes W$ is as the dual vector space $V \otimes W = B^*_{V,W}$, where

$B_{V,W} = \{ B : V \times W \longrightarrow \mathbb{R}$ bilinear map $\}$.

If $v \in V$, $w \in W$ then $v \otimes w : B_{V,W} \longrightarrow \mathbb{R}$,

$v \otimes w : B \longmapsto B(v,w)$ defines $v \otimes w \in V \otimes W$.

Then (i)–(iv) are easy to check.

Tensor products can also be defined for infinite-dimensional vector spaces, but not using this dual space definition, as then $V \neq (V^*)^*$.

Tensor product is <u>associative</u>,

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W,$$

<u>commutative</u>, $U \otimes V \cong V \otimes U$, and

<u>distributive over direct sums</u>,

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$

5.2. Tensor algebras and exterior algebras.

Definition. Let V be a finite-dimensional $\mathbb{R}$-vector space. Then we can form V, $V \otimes V$, ..., $\bigotimes^k V$, ..., where $\bigotimes^k V = \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}}$.

By convention $\bigotimes^0 V = \mathbb{R}$.

The tensor algebra is $T(V) = \bigoplus_{k=0}^{\infty} \bigotimes^k V$.

It is an associative $\mathbb{R}$-algebra, with product $\otimes$ by

$$\overbrace{(v_1 \otimes \cdots \otimes v_k)}^{\text{in } \bigotimes^k V} \otimes \overbrace{(w_1 \otimes \cdots \otimes w_\ell)}^{\text{in } \bigotimes^\ell V}$$

$$= \underbrace{v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell}_{\text{in } \bigotimes^{k+\ell} V},$$

with identity $1 \in \bigotimes^0 V = \mathbb{R}$.

The symmetric group S_k of permutations of $\{1, 2, \ldots, k\}$ acts on $\bigotimes^k V$ by permuting the k factors of V, so that $\sigma \in S_k$ acts by $\rho_k(\sigma) : \bigotimes^k V \to \bigotimes^k V$,

$$\rho_k(\sigma) : v_1 \otimes \cdots \otimes v_k \longmapsto v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)},$$

$\rho_k : S_k \to \mathrm{Aut}(\bigotimes^k V)$ the representation.

Define $\Lambda^k V$ to be the vector subspace of $\bigotimes^k V$ on which S_k acts antisymmetrically, that is,

$$\Lambda^k V = \{ \alpha \in \bigotimes^k V : \rho_k(\sigma)\alpha = \operatorname{sign}\sigma \cdot \alpha \quad \forall \sigma \in S_k \},$$

where $\operatorname{sign} : S_k \to \{\pm 1\}$ is the usual group morphism.

There is a projection $\pi : \bigotimes^k V \to \Lambda^k V$ by $\pi : \alpha \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sign}\sigma) \cdot \rho_k(\sigma)\alpha$.

It is surjective, with $\pi \circ \pi = \pi$.

We can also consider $\Lambda^k V$ as the quotient space $\Lambda^k V \cong \bigotimes^k V / \operatorname{Ker}\pi$, rather than the subspace $\Lambda^k V \subset \bigotimes^k V$.

The <u>exterior product</u> "wedge" $\wedge : \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V$ is the composition

$$\begin{tikzcd} \Lambda^k V \times \Lambda^l V \arrow[r, hook] \arrow[rrr, bend right, "\wedge"'] & \bigotimes^k V \times \bigotimes^l V \arrow[r, "\otimes"] & \bigotimes^{k+l} V \arrow[r, "\pi"] & \Lambda^{k+l} V \end{tikzcd}$$

It is associative, since $\otimes$ is associative.

We have $\Lambda^0 V = \mathbb{R} = \otimes^0 V$ and $\Lambda^1 V = V$.

If $v_1, \dots, v_n$ is a basis for V, $n = \dim V$, then $\{ v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n \}$ is a basis for $\Lambda^k V$. Hence $\dim \Lambda^k V = \binom{n}{k}$.

In particular, $\Lambda^n V \cong \mathbb{R}$, and $\Lambda^k V = 0$ for $k > n$.

If $\alpha \in \Lambda^k V$, $\beta \in \Lambda^l V$ then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \text{in } \Lambda^{k+l} V.$$

The <u>exterior algebra</u> is $\Lambda^* V = \bigoplus_{k=0}^{\dim V} \Lambda^k V$.

It is an associative algebra under $\wedge$, with identity $1 \in \mathbb{R} = \otimes^0 V = \Lambda^0 V$.

It has dimension $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Tensor products and exterior products are also functorial under linear maps. That is, if $\alpha : T \to V$, $\beta : U \to W$ are linear maps then we have a linear map $\alpha \otimes \beta : T \otimes U \to V \otimes W$ with $(\alpha \otimes \beta)(t \otimes u) = \alpha(t) \otimes \beta(u)$ for all $t \in T$, $u \in U$.

If $\alpha : V \to W$ is linear we have

$\bigotimes^k \alpha : \bigotimes^k V \to \bigotimes^k W$ and $\Lambda^k \alpha : \Lambda^k V \to \Lambda^k W$,

mapping $v_1 \otimes \cdots \otimes v_k \longmapsto \alpha(v_1) \otimes \cdots \otimes \alpha(v_k)$,

$v_1 \wedge \cdots \wedge v_k \longmapsto \alpha(v_1) \wedge \cdots \wedge \alpha(v_k)$.

<u>Example</u> Let A be an $n \times n$ matrix, thought of as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then $\Lambda^n \mathbb{R} \cong \mathbb{R}$, and $\Lambda^n A : \Lambda^n \mathbb{R}^n \to \Lambda^n \mathbb{R}^n$ is multiplication by $\det A$. (Exercise to prove this.)

5.3 Algebraic operations on vector bundles

Let X be a manifold. Then our operations V^*, $V \oplus W$, $V \otimes W$, $\bigotimes^k V$, $\Lambda^k V$ on vector spaces V, W also make sense for vector bundles on X.

Proposition 5.1. Let $E \to X$, $F \to X$ be vector bundles on a manifold X. Then there are natural vector bundles E^*, $E \oplus F$, $E \otimes F$, $\bigotimes^k E$, $\Lambda^k E$ on X whose fibers satisfy $E^*|_x = (E|_x)^*$,
$$(E \oplus F)\big|_x = E|_x \oplus F|_x, \ldots$$
at each $x \in X$.

Proof. For example, as a set define
$$E \otimes F = \{(x, \alpha) : x \in X,\ \alpha \in E_x \otimes F_x\},$$
with projection $\pi : E \otimes F \to X$, $\pi : (x, \alpha) \mapsto x$.

Then we show there is a canonical manifold structure on $E \otimes F$ making $E \otimes F \to X$ into a vector bundle, such that if $x \in X$ and U is an open neighborhood of x with local trivializations

$$\begin{tikzcd} \pi_E^{-1}(U) \arrow[r, "\cong"] \arrow[d] & U \times \mathbb{R}^k \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd} \qquad \begin{tikzcd} \pi_F^{-1}(U) \arrow[r, "\cong"] \arrow[d] & U \times \mathbb{R}^l \arrow[d] \\ U \arrow[r, equal] & U, \end{tikzcd}$$

then $\pi_{E\otimes F}^{-1}(U) \cong U \times (\mathbb{R}^k \otimes \mathbb{R}^l)$ is the induced local trivialization of $E \otimes F$. $\square$

Tensor and exterior products of elements of vector spaces extend to products of sections of vector bundles. Thus, if $e \in \Gamma^\infty(E)$, $f \in \Gamma^\infty(F)$ then we have

$e \oplus f \in \Gamma^\infty(E \oplus F)$, $e \otimes f \in \Gamma^\infty(E \otimes F)$, etc.

Linear maps of vector bundles induce linear maps of their tensor products, etc.

Basically, lots of linear algebra of vector spaces extends easily to linear algebra of vector bundles.

6. Differential forms

Let X be a manifold. Then we have the (co)tangent bundles TX, T^*X. So as in §5.3 we can form $\bigotimes^j TX$, $\bigotimes^k T^*X$, $\bigotimes^j TX \otimes \bigotimes^k T^*X$, $\Lambda^j TX$, $\Lambda^k T^*X$, all vector bundles on X.

Definition A *tensor* is $T \in \Gamma^\infty(\bigotimes^j TX \otimes \bigotimes^k T^*X)$ for $j,k \geqslant 0$.

An *exterior form*, or *k-form*, is $\alpha \in \Gamma^\infty(\Lambda^k T^*X)$.

Exterior forms are examples of tensors, as
$$\Lambda^k T^*X \subset \bigotimes^k T^*X = \bigotimes^0 TX \otimes \bigotimes^k T^*X.$$

These are important classes of geometric objects on X. Tensors are very general, they include functions $(j=k=0)$, vector fields $(j=1, k=0)$, Riemannian metrics $(j=0, k=2)$, ...

We will write $\Omega^k(X) = \Gamma^\infty(\Lambda^k T^*X)$ for the vector space of k-forms.

We have the <u>wedge product</u> $\wedge : \Omega^k(X) \times \Omega^l(X) \to \Omega^{k+l}(X)$.

It satisfies $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, and

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma.$$

Let $(x_1, \ldots, x_n)$ be local coordinates on an open $U \subseteq X$. Then $dx_1, \ldots, dx_n$ are a local basis of sections of $T^*X|_U$.

Hence $(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n)$ is a basis of sections for $\Lambda^k T^*X|_U$.

So if $\alpha \in \Omega^k(X)$, we may write $\alpha|_U$ uniquely as

$$\alpha|_U = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

for $\alpha_{i_1 \ldots i_k} : U \to \mathbb{R}$ smooth.

6.1. Pullbacks of exterior forms

Let $f: X \to Y$ be a smooth map of manifolds and $x \in X$ with $f(x) = y \in Y$. The derivative of f gives a linear map $T_x^* f: T_y^* Y \to T_x^* X$.

As exterior products are functorial under linear maps, this gives $\Lambda^k T_x^* f : \Lambda^k T_y^* Y \to \Lambda^k T_x^* X$.

Let $\alpha \in \Omega^k(Y)$. Define the <u>pullback</u> $f^*(\alpha) \in \Omega^k(X)$ by

$$f^*(\alpha)|_x = \left(\Lambda^k T_x^* f\right)(\alpha|_y), \quad \text{for all } x \in X.$$

This depends smoothly on $x \in X$ (can show by formula in coordinates), and so is a k-form.

Pullbacks $f^*: \Omega^k(Y) \to \Omega^k(X)$ are linear, and contravariantly functorial. That is, if $g: Y \to Z$ is another smooth map, then

$$(g \circ f)^* = f^* \circ g^* : \Omega^k(Z) \to \Omega^k(X).$$

We have

$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta).$$

So $f^*: \Omega^*(Y) \to \Omega^*(X)$ is an algebra morphism.

6.2. The exterior derivative

Theorem 6.1 On any manifold X there is a natural linear map $d: \Omega^k(X) \to \Omega^{k+1}(X)$ called the exterior derivative, satisfying:

(i) If $f \in \Omega^0(X) = C^\infty(X)$ then $df \in \Omega^1(X) = \Gamma^\infty(T^*X)$ is the derivative of f.

(ii) $d^2 = 0$.

(iii) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ if $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^\ell(X)$.

These properties determine d uniquely.

Proof. Let $(x_1, .., x_n)$ be local coordinates on open $U \subseteq X$, and $\alpha \in \Omega^k(X)$, so that we have

$$\alpha|_U = \sum_{1 \le i_1 < i_2 < .. < i_k \le n} \alpha_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}. \qquad (*)$$

We claim that (i)-(iii) force us to define

$$d\alpha|_U = \sum_{j,\ 1 \le i_1 < .. < i_k \le n} \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x_k} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}, \qquad (**)$$

and that conversely, defining d by $(**)$ on U satisfies (i)–(iii), at least on U.

To see that (i)–(iii) imply $(**)$, note that (ii), (iii) and induction on k give $d[dx_{i_1} \wedge \dots \wedge dx_{i_k}] = 0$, as

$$d[dx_{i_1} \wedge \dots \wedge dx_{i_k}] = \underbrace{\cancel{d[dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}]}}_{0 \text{ by induction}} \wedge dx_{i_k} + (-1)^{k-1} (dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}) \wedge \underbrace{\cancel{d^2 x_{i_k}}}_{0 \text{ by (ii)}}.$$

So (iii) and $(*)$ give

$$d\alpha|_U = \sum_{1 \le i_1 < \dots < i_k \le n} d(\alpha_{i_1 \dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

and (i) gives $d(\alpha_{i_1 \dots i_k}) = \sum_j \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x_j} dx_j$, proving $(**)$.

To see that $(**)$ implies (i)–(iii) on U, (i) is case $k=0$ of $(**)$, (ii) follows using

$$\frac{\partial^2 \alpha_{i_1 \dots i_k}}{\partial x_j \, \partial x_{j'}} = \frac{\partial^2 \alpha_{i_1 \dots i_k}}{\partial x_{j'} \, \partial x_j} \quad \text{and} \quad dx_j \wedge dx_{j'} = -dx_{j'} \wedge dx_j,$$

and (iii) follows from $(**)$, the product rule for differentiation, and $\wedge$ anticommutative.

Suppose now that $(y_1, \dots, y_n)$ are local coordinates on open $V \subseteq X$, so we have

$$\alpha|_V = \sum_{1 \le j_1 < \dots < j_k \le n} \tilde{\alpha}_{j_1 \dots j_k} \, dy_{j_1} \wedge \dots \wedge dy_{j_k}. \qquad (\tilde{*})$$

As above, (i)–(iii) force

$$\tilde{d}\alpha|_V = \sum_{l,\ 1 \le j_1 < \dots < j_k \le n} \frac{\partial \tilde{\alpha}_{j_1 \dots j_k}}{\partial y_l} \, dy_l \wedge dy_{j_1} \wedge \dots \wedge dy_{j_k}. \qquad (\tilde{**})$$

We need to show that $d\alpha|_{U \cap V} = \tilde{d}\alpha|_{U \cap V}$, comparing $(**)$ and $(\tilde{**})$. Can do this by

$$\tilde{d}\alpha|_{U \cap V} = \sum_{\substack{l, j_1 \dots j_k, \\ j, i_1 \dots i_k}} \frac{\partial \tilde{\alpha}_{j_1 \dots j_k}}{\partial y_l} \left(\frac{\partial y_l}{\partial x_j} dx_j \right) \wedge \left(\frac{\partial y_{j_1}}{\partial x_{i_1}} dx_{i_1} \right) \wedge \dots \wedge \left(\frac{\partial y_{j_k}}{\partial x_{i_k}} dx_{i_k} \right)$$

$$= \sum_{\substack{l, j_1 \dots j_k, \\ j, i_1 \dots i_k}} \Big(\underbrace{\frac{\partial y_l}{\partial x_j} \cdot \frac{\partial}{\partial y_l}}_{\frac{\partial}{\partial x_j}} (\tilde{\alpha}_{j_1 \dots j_k}) \Big) \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \cdot dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$= \sum_{\substack{j_1 \dots j_k \\ j, i_1 \dots i_k}} \frac{\partial}{\partial x_j} \left(\tilde{\alpha}_{j_1 \dots j_k} \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \right) dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$0 = \Big[- \sum_{m=1}^{k} \tilde{\alpha}_{j_1 \dots j_k} \cdot \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \underbrace{\frac{\partial^2 y_{j_m}}{\partial x_j \partial x_{i_m}}}_{\text{symmetric in } j, i_m} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \cdot \underbrace{dx_j \wedge dx_{i_1} \dots dx_{i_k}}_{\text{antisymmetric in } dx_j,\ dx_{i_m}}$$

$$= \sum_{j, i_1 \dots i_k} \frac{\partial}{\partial x_j} (\alpha_{i_1 \dots i_k}) \, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$= d\alpha \,|_{U \cap V}.$$

— For this proof, regard $\alpha_{i_1 \dots i_k}, \tilde{\alpha}_{j_1 \dots j_k}$ as defined for all $i_1 \dots i_k$, $j_1 \dots j_k$ and antisymmetric, not as defined for $i_1 < \dots < i_k$, $j_1 < \dots < j_k$.

Hence the expression (*) is independent of coordinate system. Thus we can use it to define $d\alpha$ on all of X, by covering X by coordinate neighbourhoods. So $d\alpha$ is well-defined, and satisfies (i) – (iii). $\square$

Think of d as a kind of derivative of exterior forms. It depends only on the manifold structure of X, not on any additional choices.

Recall: Th. 6.1 defined exterior derivative

$d : \Omega^k(X) \to \Omega^{k+1}(X)$ with

(i) if $f \in \Omega^0(X) = C^\infty(X)$, $df \in \Omega^1(X) = \Gamma^\infty(T^*X)$ is the usual derivative.

(ii) $d^2 = 0$.

(iii) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (d\beta)$.

Lemma 6.2 Let $f : X \to Y$ be a smooth map of manifolds, and $\alpha \in \Omega^k(Y)$. Then

$f^*(d\alpha) = d(f^*(\alpha))$ in $\Omega^{k+1}(X)$.

Proof. Can deduce from coordinate expressions for f^*, d, or use that $f^*(da) = d(f^*(a))$ ~~for~~ for $a \in C^\infty(Y)$, and $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$, and uniqueness under (i)–(iii) in Th. 6.1. $\square$

6.3. Lie derivatives of exterior forms

Let X be a manifold, v a complete vector field on X, $\varphi : \mathbb{R}\times X \to X$ the associated 1-parameter group of diffeomorphisms from Theorem 4.4, and $\alpha \in \Omega^k(X)$. Then as in §4.3, the Lie derivative is
$$\mathcal{L}_v \alpha = \frac{d}{dt}\left(\varphi_t^*(\alpha)\right)\Big|_{t=0},$$
for pullback φ_t^* as in §6.1.

We will give a formula for $\mathcal{L}_v\alpha$. First we define interior products.

Definition. Let X be a manifold, $v \in \Gamma^\infty(TX)$, and $\alpha \in \Omega^k(X)$. Define the interior product $i_v(\alpha)$ in $\Omega^{k-1}(X)$ by using
$$\Lambda^k T^*X \subseteq T^*X \otimes \Lambda^{k-1}T^*X \subseteq T^*X \otimes \bigotimes^{k-1} T^*X = \bigotimes^k T^*X,$$
and using the inner product
$$\underset{\substack{\uparrow\\ v}}{TX} \times \underbrace{T^*X \otimes \Lambda^{k-1}T^*X}_{\substack{\uparrow\\ \alpha}} \to \Lambda^{k-1}T^*X$$
pair

pairing $v \in TX$ with the first T^*X factor of α.

In coordinates $(x_1,\dots,x_n)$ on $U\subseteq X$, if

$$v=\sum_{j=1}^{n} v^j \frac{\partial}{\partial x_j}, \qquad \alpha=\sum_{1\le i_1,\dots,i_k\le n} \alpha_{i_1\dots i_k}\, dx_{i_1}\wedge\dots\wedge dx_{i_k}$$

with $\alpha_{i_1\dots i_k}$ antisymmetric in $i_1,\dots,i_k$, then

$$i_v(\alpha)=\sum_{1\le i_1,\dots,i_k\le n} v^{i_1}\alpha_{i_1 i_2\dots i_k}\, dx_{i_2}\wedge\dots\wedge dx_{i_k}.$$

It satisfies (i) $i_v(da)=v(a)$ for $a\in C^\infty(X)$.

(ii) $i_v(\alpha\wedge\beta)=(i_v\alpha)\wedge\beta+(-1)^k\alpha\wedge i_v(\beta)$

if $\alpha\in\Omega^k(X)$, $\beta\in\Omega^l(X)$.

Theorem 6.3. The Lie derivative $\mathcal{L}_v\alpha$ of a k-form α satisfies $\mathcal{L}_v\alpha=d(i_v\alpha)+i_v(d\alpha)$. (*)

This is Cartan's formula.

Proof. Define $R_v(\alpha)=d(i_v\alpha)+i_v(d\alpha)$, as in the r.h.s. of (*).

Then
$$d\circ R_v(\alpha)=\overset{0}{\cancel{d^2}}(i_v\alpha)+d\circ i_v\circ d(\alpha)$$
$$=d\,i_v(d\alpha)+i_v(\overset{0}{\cancel{d^2\alpha}})=R_v(d\alpha),$$
as $d^2=0$, so d commutes with R_α.

Also, if $\alpha \in \Omega^k(X)$, $\beta \in \Omega^l(X)$ then

$$i_v(\alpha \wedge \beta) = i_v(\alpha) \wedge \beta + (-1)^k \alpha \wedge i_v(\beta)$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta,$$

we have $R_v(\alpha \wedge \beta) = R_v(\alpha) \wedge \beta + \alpha \wedge R_v(\beta)$.

Since $\varphi_t^*(d\alpha) = d(\varphi_t^*\alpha)$, differentiating at $t=0$ gives $\mathcal{L}_v(d\alpha) = d(\mathcal{L}_v\alpha)$. Also $\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*(\alpha) \wedge \varphi_t^*(\beta)$, so differentiating at $t=0$ gives $\mathcal{L}_v(\alpha \wedge \beta) = \mathcal{L}_v(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_v(\beta)$.

So $\mathcal{L}_v, R_v : \Omega^k(X) \to \Omega^k(X)$ both commute with d and satisfy the same Leibnitz rule.

Writing a k-form $\alpha = \sum_{1 \le i_1, \dots, i_k \le n} \alpha_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$, $\alpha_{i_1 \dots i_k}$ antisymmetric in $i_1, \dots, i_k$

in local coordinates, we see that $\mathcal{L}_v = R_v$ if $\mathcal{L}_v = R_v$ on functions. But $R_v a = i_v da$

$$= v(a) = \frac{d}{dt}(a \circ \varphi_t)\Big|_{t=0} = \mathcal{L}_v a,$$

so $\mathcal{L}_v = R_v$. $\square$

6.4. De Rham cohomology

<u>Definition</u> Let X be a manifold. Then we have linear maps $d : \Omega^k(X) \to \Omega^{k+1}(X)$ with $d^2 = 0$. The <u>k^{th} de Rham cohomology group</u> is the quotient vector space

$$H^k(X) = \frac{\operatorname{Ker}(d : \Omega^k(X) \longrightarrow \Omega^{k+1}(X))}{\operatorname{Im}(d : \Omega^{k-1}(X) \longrightarrow \Omega^k(X))},$$

which makes sense as $d^2 = 0$, so $\operatorname{Im}(\dots) \subset \operatorname{Ker}(\dots)$.

<u>Remark 6.4</u> (a) In Algebraic Topology, for a topological space X and a commutative ring R one can define the <u>cohomology groups</u> $H^k(X, R)$ for $k = 0, 1, \dots$. There are many ways to do this (Čech cohomology, singular cohomology, ...), but for nice spaces they all agree.

<u>De Rham's Theorem</u> says that if X is a manifold then $H^k(X) \cong H^k(X, \mathbb{R})$ for all k.

So de Rham cohomology is part of a general story for topological spaces.

(b) We can identify $H^0(X)$:

$$H^0(X) = \mathrm{Ker}\left(d : \Omega^0(X) \to \Omega^1(X)\right) = \{f : X \to \mathbb{R} : f \text{ is locally constant}\}$$

$$= \mathbb{R}^{\{\text{connected components of } X\}}$$

If X is connected then $H^0(X) = \mathbb{R}$.

(c) We call $\alpha \in \Omega^k(X)$ <u>closed</u> if $d\alpha = 0$, and <u>exact</u> if $\alpha = d\beta$, $\beta \in \Omega^{k-1}(X)$. Then

$$H^k(X) = \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-forms}\}.$$

(d) If X is compact, one can show $H^k(X)$ is a <u>finite-dimensional</u> vector space, although the $\Omega^k(X)$ are infinite-dimensional.

The <u>Betti numbers</u> are $b^k(X) = \dim H^k(X)$.

(e) If $\alpha \in \Omega^k(X)$ with $d\alpha = 0$, we write $[\alpha]$ for the class $\alpha + \mathrm{Im}\, d$ in $H^k(X)$.

(f) $H^k(X) = 0$ if $k > \dim X$.

<u>Definition</u> Let X be a manifold. Define the <u>cup product</u> $\cup : H^k(X) \times H^l(X) \to H^{k+l}(X)$

by $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$.

Here $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta = 0$,

so $[\alpha \wedge \beta] \in H^{k+l}(X)$. To see $[\alpha \wedge \beta]$ is independent of the choice of $\alpha \in [\alpha]$, $\beta \in [\beta]$,

note that $(\alpha + d\gamma) \wedge (\beta + d\delta)$

$$= \alpha \wedge \beta + d(\gamma \wedge \beta + (-1)^k \alpha \wedge \beta + \gamma \wedge d\delta),$$

so $[\alpha \wedge \beta] = [(\alpha + d\gamma) \wedge (\beta + d\delta)]$.

By properties of $\wedge : \Omega^k(X) \times \Omega^l(X) \to \Omega^{k+l}(X)$, the cup product satisfies:

(a) $([\alpha] \cup [\beta]) \cup [\gamma] = [\alpha] \cup ([\beta] \cup [\gamma])$.

(b) $[\alpha] \cup [\beta] = (-1)^{kl} [\beta] \cup [\alpha]$ for $[\alpha] \in H^k(X)$, $[\beta] \in H^l(X)$.

(c) $\cup$ is bilinear.

(d) $1_X \in \Omega^0(X)$ with $d1_X = 0$, so $[1_X] \in H^0(X)$, and $[1_X] \cup [\alpha] = [\alpha] \cup [1_X] = [\alpha]$ for all $\alpha \in H^k(X)$.

So $(H^*(X), \cup, [1_X])$ is a <u>supercommutative graded $\mathbb{R}$-algebra</u>.

<u>Recall:</u> X manifold, $H^k(X) = \dfrac{\mathrm{Ker}(d : \Lambda^k(X) \to \Lambda^{k+1}(X))}{\mathrm{Im}(d : \Lambda^{k-1}(X) \to \Lambda^k(X))}$

de Rham cohomology. Supercommutative $\mathbb{R}$-algebra with product $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$.

<u>Definition</u> Let $f : X \to Y$ be a smooth map of manifolds. Then we have pullback maps $f^* : \Lambda^k(Y) \to \Lambda^k(X)$ with $d(f^*(\alpha)) = f^*(d\alpha)$.

Define <u>pullback maps</u> $f^* : H^*(Y) \to H^*(X)$ by $f^*([\alpha]) = [f^*(\alpha)]$ for $\alpha \in \Lambda^k(Y)$ with $d\alpha = 0$.

Then $d(f^*(\alpha)) = f^*(d\alpha) = 0$, so $[f^*(\alpha)] \in H^k(X)$.

Also $f^*(\alpha + d\gamma) = f^*(\alpha) + d(f^*(\gamma))$, so $[f^*(\alpha)]$ is independent of the choice of α in $[\alpha] = \alpha + \mathrm{Im}\, d$.

By properties of f^* on forms, $f^* : H^k(Y) \to H^k(X)$ is linear, with $f^*(a \cup b) = f^*(a) \cup f^*(b)$ and $f^*(1_Y) = 1_X$. So $f^* : H^*(Y) \to H^*(X)$ is a morphism of supercommutative $\mathbb{R}$-algebras.

We will show that $f^*: H^k(Y) \to H^k(X)$ is unchanged under smooth deformation of f.

For cohomology in Algebraic Topology, the analogue holds for continuous deformations of $f: X \to Y$.

As we have not yet defined manifolds with boundary, say that $f: X \times [0,1] \to Y$ is smooth if it extends to smooth $\tilde{f}: X \times (-\varepsilon, 1+\varepsilon) \to Y$ for small $\varepsilon > 0$.

Theorem 6.5. Let X, Y be manifolds and $f: X \times [0,1] \to Y$ be smooth. Set $f_t = f(x,t)$ and consider the pull back maps $f_t^*: H^k(Y) \to H^k(X)$. Then $f_0^* = f_1^*$.

Proof. Represent $a \in H^k(Y)$ by closed $\alpha \in \Omega^k(Y)$, and consider $f^*(\alpha)$ on $X \times [0,1]$ (or better, $\tilde{f}^*(\alpha)$ on $X \times (-\varepsilon, 1+\varepsilon)$).

Write $f^*(\alpha) = \beta_t + dt \wedge \gamma_t$, where t is the coordinate on $[0,1]$, and β_t is a k-form on X depending on $t \in [0,1]$, and γ_t is a $(k-1)$-form on X depending on t. As $d\alpha = 0$, $d\, f^*(\alpha) = 0$,

so $$d_X \beta_t + dt \wedge \frac{\partial \beta_t}{\partial t} - dt \wedge d_X \gamma_t = 0,$$

where d_X is exterior derivative in the X directions.

Hence $$\frac{d\beta_t}{dt} = d_X \gamma_t,$$ so

$$f_1^*(\alpha) - f_0^*(\alpha) = \beta_1 - \beta_0 = \int_0^1 \frac{\partial \beta_t}{dt}\, dt$$
$$= \int_0^1 (d_X \gamma_t)\, dt = d_X\left(\int_0^1 \gamma_t\, dt\right),$$

and therefore $f_0^*(a) = [f_0^*(\alpha)] = [f_1^*(\alpha)] = f_1^*(a)$. $\square$

<u>Proposition 6.6.</u> $H^0(\mathbb{R}^n) = \mathbb{R}$ and $H^k(\mathbb{R}^n) = 0$, $k > 0$.

Proof. When $n=0$ we have $\mathbb{R}^0 = \{0\}$, and $\Omega^k(\{0\}) = \begin{cases} \mathbb{R}, & k=0 \\ 0, & k>0, \end{cases}$

so the proposition is obvious.

Define $\pi: \mathbb{R}^n \longrightarrow \{0\}$, $i: \{0\} \longrightarrow \mathbb{R}^n$ by $\pi: \underline{x} \longmapsto 0$, $i: 0 \longmapsto \underline{0}$. Then π, i are smooth, so we have $\pi^*: H^k(\{0\}) \longrightarrow H^k(\mathbb{R}^n)$, $i^*: H^k(\mathbb{R}^n) \longrightarrow H^k(\{0\})$.

As $\pi \circ i = \mathrm{id}: \{0\} \to \{0\}$,
$i^* \circ \pi^* = \mathrm{id}: H^k(\{0\}) \longrightarrow H^k(\{0\})$.

Define $f: \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$ by $f(\underline{x}, t) = t\underline{x}$.
Then f is smooth with $f_0 = 0 = i \circ \pi: \mathbb{R}^n \to \mathbb{R}^n$ and $f_1 = \mathrm{id}: \mathbb{R}^n \to \mathbb{R}^n$. Thus
$\pi^* \circ i^* = f_0^* = f_1^* = \mathrm{id}^* = \mathrm{id}: H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$.

This shows that $\pi^*: H^k(\{0\}) \to H^k(\mathbb{R}^n)$ and $i^*: H^k(\mathbb{R}^n) \longrightarrow H^k(\{0\})$ are inverse maps,
so $H^k(\mathbb{R}^n) \cong H^k(\{0\}) = \begin{cases} \mathbb{R}, & k=0 \\ 0, & k>0. \end{cases}$ $\square$

By almost the same proof we show:

Proposition 6.7. $H^k(X \times \mathbb{R}^n) \cong H^k(X)$ for any manifold X and all $k \geq 0$.

Proposition 6.8 $H^k(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise,} \end{cases}$ for $n > 0$.

Proof. On Sheet 3. $\square$

Theorem 6.9 (Künneth Theorem). Let X, Y be manifolds. Define a linear map

$$\bigoplus_{\substack{i+j=k, \\ i,j \geq 0}} H^i(X) \otimes H^j(Y) \longrightarrow H^k(X \times Y)$$

to map $a \otimes b \mapsto \pi_X^*(a) \wedge \pi_Y^*(b)$, for $a \in H^i(X)$, $b \in H^j(Y)$.

This map is an isomorphism.

Proof. Omitted. $\square$

<u>Example.</u> The torus T^2 is $S^1 \times S^1$,

where $H^k(S^1) = \begin{cases} \mathbb{R} & k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$

So Künneth gives $H^k(T^2) = \begin{cases} \mathbb{R} & k = 0,2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise.} \end{cases}$

Observe that $H^1(S^2) = 0$, $H^1(T^2) = \mathbb{R}^2$.

Thus there cannot exist a diffeomorphism $f: S^2 \to T^2$, as $f^*: H^1(T^2) \to H^1(S^2)$ would be an isomorphism. So we can use de Rham cohomology to distinguish manifolds up to diffeomorphism.

7. Integration of forms.

7.1. Orientations.

Definition. Let V be a real vector space of dimension n, and $(v_1, \dots, v_n)$, $(v_1', \dots, v_n')$ be two bases for V. Then $v_i' = \sum_{j=1}^{n} A_{ij} v_j$ for $A_{ij} \in \mathbb{R}$, and $(A_{ij})_{i,j=1}^{n}$ is an invertible real matrix, so it has a determinant $\det(A_{ij}) \in \mathbb{R} \setminus 0$.

Define an equivalence relation on such bases by

$$(v_1, \dots, v_n) \sim (v_1', \dots, v_n') \text{ if } \det(A_{ij}) > 0.$$

Write $[v_1, \dots, v_n]$ for the $\sim$-equivalence class of $(v_1, \dots, v_n)$. An orientation O on V is a $\sim$-equivalence class. There are two orientations,

$$[v_1, v_2, \dots, v_n] \text{ and } [-v_1, v_2, \dots, v_n].$$

Given an orientation O, we call a basis $(v_1, \dots, v_n)$ oriented if $(v_1, \dots, v_n) \in O$, and anti-oriented otherwise.

* An orientation on $V \cong \mathbb{R}$ is a direction $\longrightarrow$.
* An orientation on $V \cong \mathbb{R}^2$ gives notions of 'clockwise' and 'anticlockwise'.
* An orientation on $V \cong \mathbb{R}^3$ gives notions of 'right handed' and 'left handed'.

We can write orientations in terms of $\Lambda^n V$.

It has dimension $\binom{\dim V}{n} = 1$, so $\Lambda^n V \cong \mathbb{R}$.

If $v_1, \dots, v_n$ is a basis for V then

$v_1 \wedge \dots \wedge v_n \in \Lambda^n V \setminus 0$. If $v_1', \dots, v_n'$ is another basis with $v_i' = \sum_j A_{ij} v_j$ then

$$v_1' \wedge \dots \wedge v_n' = \det(A_{ij})\, v_1 \wedge \dots \wedge v_n.$$

Thus, an orientation on V corresponds to a choice of connected component of

$$\Lambda^n V \setminus 0 \cong \mathbb{R} \setminus 0 \cong (-\infty, 0) \amalg (0, \infty),$$

and a basis $v_1, \dots, v_n$ is oriented if $v_1 \wedge \dots \wedge v_n$ lies in this connected component.

Recall: An orientation for a real vector space V is an equivalence class $[v_1, \ldots, v_n]$ of bases $v_1, \ldots, v_n$ for V, where bases $v_1, \ldots, v_n$ and $v_1', \ldots, v_n'$ are equivalent if $v_i' = \sum_j A_{ij} v_j$ with $\det(A_{ij}) > 0$.

Note that bases for V correspond to dual bases for V^*, so orientations for V correspond to orientations for V^*.

Definition. Let X be an n-manifold. An orientation for X is a choice of orientation for $T_x X$ (equivalently, for $T_x^* X$) for all $x \in X$, which varies continuously with $x \in X$.

As orientations for $T_x^* X$ correspond to choices of connected component of $(\Lambda^n T_x^* X) \setminus 0$, an alternative definition is:

Definition. An orientation on an n-manifold X is a $\sim$-equivalence class $[\omega]$ of nonvanishing $\omega \in \Omega^n(X)$, where $\omega \sim \omega'$ if $\omega' = f\omega$ for smooth $f : X \to (0, \infty)$.

An <u>oriented manifold</u> is a manifold X with orientation O. The <u>opposite orientation</u> is $-O = [-\omega]$, for $O = [\omega]$.

A manifold X is <u>orientable</u> if it admits an orientation, i.e. if there exist non-vanishing n-forms ω on X.

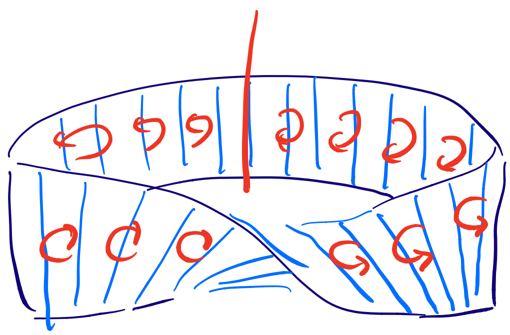
<u>Example.</u> $\mathbb{R}^n$ has two orientations; $[dx_1 \wedge \dots \wedge dx_n]$ and $[-dx_1 \wedge dx_2 \wedge \dots \wedge dx_n]$.

<u>Example</u> Let X be the Möbius strip $\mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts by $n : (x,y) \mapsto (x+n, (-1)^n y)$.

There is a projection $\pi : \mathbb{R}^2 \to X$, $\pi : (x,y) \mapsto (x,y)\mathbb{Z}$.

Suppose ω is a 2-form on X. Then $\pi^*(\omega)$ is a 2-form on $\mathbb{R}^2$. Write $\pi^*(\omega) = f(x,y)\, dx \wedge dy$. Then $\pi^*(\omega)$ $\mathbb{Z}$-invariant implies that $f(x+n, (-1)^n y) = (-1)^n f(x,y)$, since n maps $dx \wedge dy \mapsto (-1)^n dx \wedge dy$.

Since f changes sign it must be zero somewhere in $\mathbb{R}^2$. So $\omega = 0$ at some point of X, and X is <u>not orientable</u>.



— There is no consistent notion of 'clockwise' on the Möbius strip.

<u>Definition</u> Let X be an n-manifold with orientation $[\omega]$. Let (U, ϕ) be a chart on X. We call U <u>oriented</u> if $\phi^*(\omega) = f\, dx_1 \wedge \dots \wedge dx_n$ for $f > 0$. Then in the local coordinates $(x_1, \dots, x_n)$ on $\phi(U) \subset X$, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are an oriented basis for $T_x X$ at $x \in \phi(U)$, and $dx_1, \dots, dx_n$ are an oriented basis for $T_x^* X$.

We can find an atlas for X consisting of oriented charts. For two such charts with local coordinates $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$,

$$\det \left(\frac{\partial y_j}{\partial x_i} \right)_{i,j=1}^{n} > 0 \text{ on the overlap.}$$

7.2. Integration on manifolds.

We are all familiar with integrals like

$$\int_0^1 f(t)\,dt \quad \text{or} \quad \int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy.$$

They happen in subsets $U \subseteq \mathbb{R}^n$, and involve a particular choice of coordinates t, (x,y), ... on U. We also know formulae for how integrals behave under change of coordinates, e.g.

$$\int_a^b \left(f(y(x)) \frac{dy}{dx}(x) \right) dx = \int_c^d f(y)\,dy,$$

if $y: (a,b) \longrightarrow (c,d)$ is smooth and increasing, $y(a)=c$, $y(b)=d$.

You may have been told 'dt', '$dx\,dy$', ... are just notation, and don't mean anything.

In Differential Geometry, choosing coordinates is bad style. So, how can we understand integration in a coordinate-independent way?

Principle. One should write integrals as $\int_X \alpha \in \mathbb{R}$, where X is an <u>oriented</u> n-dimensional manifold (possibly with boundary or corners), and α is an n-form on X, so $\alpha \in \Omega^n(X)$. (Can also allow α non-smooth.)

Definition Let X be an oriented n-manifold, with orientation $[\omega]$, and $\alpha \in \Omega^n(X)$, such that $\operatorname{supp}\alpha = \overline{\{x \in X : \alpha(x) \neq 0\}}$ is compact. We will define $\int_X \alpha \in \mathbb{R}$.

Need $\operatorname{supp}\alpha$ compact to guarantee integral exists — otherwise it could be infinite.

Choose an atlas $\{(U_i, \varphi_i) : i \in I\}$ for X consisting of <u>oriented</u> charts, so that $\phi_i^*(\omega) = f_i \, dx_1 \wedge \dots \wedge dx_n$ for $f_i > 0$.
Set $V_i = \phi_i(U_i)$. Then $\{V_i : i \in I\}$ is an

open cover of X. Choose a subordinate partition of unity $\{\eta_i : i \in I\}$ on X.

Write $\phi_i^*(\eta_i \cdot \alpha) = g_i \, dx_1 \wedge \dots \wedge dx_n$, for $g_i : U_i \longrightarrow \mathbb{R}$ smooth and compactly supported in U_i. Then $\int_{U_i} g_i \, dx_1 \wedge \dots \wedge dx_n$ is well-defined, by ordinary integration.

Define $$\int_X \alpha = \sum_{i \in I} \int_{U_i} g_i \, dx_1 \wedge \dots \wedge dx_n. \quad (*)$$

As $\mathrm{supp}\,\alpha$ is compact, and $\{\eta_i : i \in I\}$ is locally finite, there are only finitely many $i \in I$ with $\mathrm{supp}\,\alpha \cap \mathrm{supp}\,\eta_i \neq \phi$, so $(*)$ has only finitely many nonzero terms, and is well defined.

We claim $\int_X \alpha$ is independent of the choices of atlas $\{(U_i, \phi_i) : i \in I\}$ and $\{\eta_i : i \in I\}$.

That $\int_X \eta_i \alpha = \int_{U_i} g_i \, dx_1 \wedge \dots \wedge dx_n$ is independent of the choice of oriented coordinates $(x_1, \dots, x_n) = \phi_i^{-1}$ on V_i follows from the change of variables formula for integration on $U \subseteq \mathbb{R}^n$.

That $\int_X \alpha = \sum_{i \in I} \int_X \eta_i \alpha$ is independent of the choice of $\{\eta_i : i \in I\}$ follows as $\sum_{i \in I} \eta_i = 1$.

7.3. Manifolds with boundary

The interval $[0,1)$ and the disc $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ are not manifolds, even topologically: near 0 in $[0,1)$, or near $(1,0)$ in D, they are not locally homeomorphic to $\mathbb{R}$, $\mathbb{R}^2$. But they are manifolds with boundary.

Definition Manifolds with boundary X are defined as for manifolds, but using (maximal) atlases $\{(U_i, \varphi_i) : i \in I\}$ in which we allow $U \subseteq \mathbb{R}^n$ open or $U \subseteq [0,\infty) \times \mathbb{R}^{n-1}$ open.

To define compatible charts, we need to define when a map $f : U \longrightarrow V$ smooth, for open $U, V \subseteq [0,\infty) \times \mathbb{R}^{n-1}$. This means that all derivatives

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

exist and are continuous on U, including one-sided derivatives $\frac{\partial}{\partial x_1}$ at $x_1 = 0$.

It is a theorem that this holds iff f extends to smooth $\tilde{f} : \tilde{U} \longrightarrow \tilde{V}$, for $\tilde{U}, \tilde{V}$ open neighbourhoods of U, V in $\mathbb{R}^n$.

<u>Example</u> (a) $[0,1)$ is a manifold with boundary.

(b) $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ is a manifold with boundary.

(c) The square $[0,1]^2$ is not a manifold with boundary (with the obvious smooth structure). But it is a <u>manifold with corners</u> (locally modelled on $[0,\infty)^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$).

<u>Definition.</u> Let X be a manifold with boundary, with maximal atlas $\{(U_i, \varphi_i) : i \in I\}$. The <u>boundary</u> of X is

$$\partial X = \{x \in X : \exists \text{ chart } (U_i, \varphi_i) \text{ with } U_i \subseteq [0,\infty) \times \mathbb{R}^{n-1} \text{ open and } x = \phi_i(0, x_2, \ldots, x_n),\ (0, x_2, \ldots, x_n) \in U_i\}.$$

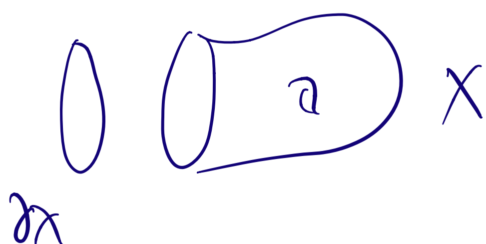
It is a closed subset of X.

Define J to be the subset of $i \in I$ with $U_i \subseteq [0,\infty) \times \mathbb{R}^{n-1}$ open. For $j \in J$ set

$$V_j = \{ (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : (0, y_1, \dots, y_{n-1}) \in U_j \}$$

and $\psi_j : V_j \longrightarrow \partial X$, $\psi_j(y_1, \dots, y_{n-1}) = \phi_j(0, y_1, \dots, y_{n-1})$.

Then $\{ (V_j, \psi_j) : j \in J \}$ is a maximal atlas on ∂X, making ∂X into an $(n-1)$-manifold without boundary.



If X is an oriented manifold with boundary, one can define an orientation on ∂X. This requires an orientation convention. Ours is this: if $(x_1, \dots, x_n) \in [0,\infty) \times \mathbb{R}^{n-1}$ are oriented local coordinates on X, then $(x_2, \dots, x_n)$ are anti-oriented local coordinates on ∂X.

Equivalently, if $dx_1 \wedge \dots \wedge dx_n$ defines the orientation on X, then $-dx_2 \wedge \dots \wedge dx_n$ defines the orientation on ∂X.

<u>Example</u> Let $[0,1] = X$, with orientation $[dx]$. Then $\partial X = \{0\} \sqcup \{1\}$, where 0 has the negative orientation, and 1 has the positive orientation. This is consistent with $\int_0^1 f'(x)\,dx = -f(0) + f(1)$ for smooth $f: [0,1] \to \mathbb{R}$ — an example of Stokes' Theorem.

Exterior forms $\alpha \in \Omega^k(X)$ on a manifold with boundary X can be restricted to $\alpha|_{\partial X} \in \Omega^k(\partial X)$. This can be regarded as a pull back: $\alpha|_{\partial X} = i^*(\alpha)$ for $i: \partial X \hookrightarrow X$ the inclusion.

7.4. Stokes Theorem

Theorem 7.1 (Stokes). Let X be an oriented n-manifold with boundary, so that ∂X is an oriented $(n-1)$-manifold, and $\alpha \in \Omega^{n-1}(X)$ with $\operatorname{supp}\alpha$ compact.

Then $\displaystyle\int_X d\alpha = \int_{\partial X} (\alpha|_{\partial X})$.

Proof. Choose an atlas $\{(U_i, \varphi_i) : i \in I\}$ of oriented charts on X, and a subordinate partition of unity $\{\eta_i \in I\}$. Let J be the subset of I with $U_i \subseteq (0,\infty) \times \mathbb{R}^{n-1}$ open, and set $V_j = \{(y_1, .., y_{n-1}) \in \mathbb{R}^{n-1} : (0, y_1, .., y_{n-1}) \in U_j\}$, $\psi_j : V_j \to \partial X$, $\psi_j : (y_1, .., y_{n-1}) \mapsto \phi_j(0, y_1, .., y_{n-1})$.

Then $\{(V_j, \phi_j) : j \in J\}$ is an atlas of anti-oriented charts on ∂X, and $\{\eta_j|_{\partial X} : j \in J\}$ is a subordinate partition of unity.

Since $\operatorname{supp}\alpha$ is compact,

$\operatorname{supp}\alpha \cap \operatorname{supp}\eta_i \neq \emptyset$ for only finitely many

$i \in I$ as $\{\eta_i : i \in I\}$ is locally finite

Thus $\alpha = \sum_{i \in I} \eta_i \alpha$, with only finitely

many non zero terms. Hence

$$\int_X d\alpha = \sum_{i \in I} \int_X d(\eta_i \alpha) = \sum_{i \in I} \int_{U_i} d\left(\phi_i^*(\eta_i \alpha)\right).$$

Fix $i \in I$. Write

$$\phi_i^*(\eta_i \alpha) = \sum_{k=1}^{n} (-1)^{k-1} a_k \, dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n,$$

where $a_k : U_i \to \mathbb{R}$ is smooth and

compactly-supported. Then

$$d\left(\phi_i^*(\eta_i \alpha)\right) = \left(\sum_{k=1}^{n} \frac{\partial a_k}{\partial x_k}\right) dx_1 \wedge \cdots \wedge dx_n.$$

So $$\int_{U_i} d\left(\phi_i^*(\eta_i \alpha)\right) = \sum_{k=1}^{n} \int_{U_i} \frac{\partial a_k}{\partial x_k} \, dx_1 \cdots dx_n.$$

If $i \in I \setminus J$, so $U_i \subseteq \mathbb{R}^n$ open, then

$$\int_{U_i} \frac{\partial a_k}{\partial x_k}\, dx_1 \dots dx_n = \underbrace{\int \dots \int}_{n-1} \left(\int_{\substack{x_k \ll 0 \\ x_k = -\infty}}^{\substack{x_k = +\infty \\ x_k \gg 0}} \frac{\partial a_k}{\partial x_k}\, dx_k \right) dx_1 \dots \widehat{dx_k} \dots dx_n$$

Fubini

$$= \underbrace{\int \dots \int}_{n-1} \underbrace{\left(a_k \right)_{x_k \ll 0}^{x_k \gg 0}}_{= 0 - 0} dx_1 \dots \widehat{dx_k} \dots dx_n = 0,$$

as a_k compactly-supported.

If $i \in J$ then the case $k = 1$ is different: we get

$$\int_{U_i} \frac{\partial a_1}{\partial x_1}\, dx_1 \dots dx_n = \underbrace{\int \dots \int}_{n-1} \left[a_1 \right]_{x_1 = 0}^{x_1 \gg 0} dx_2 \dots dx_n$$

$$= -\underbrace{\int \dots \int}_{n-1} a_1(0, x_2, \dots, x_n)\, dx_2 \dots dx_n$$

$$= -\int_{V_i} a_1 \big|_{x_1 = 0}\, dx_2 \dots dx_n$$

$$= +\int_{V_i} \varphi_i^*(\eta_i^* \alpha|_{\partial X}),$$

where the sign change is as $(x_2, \dots, x_n)$ are anti-oriented coordinates on $\varphi_i(V_i)$.

Thus
$$\int_X d(\eta_i\alpha) = \int_{U_i} d(\phi_i^*(\eta_i\alpha)) = \begin{cases} 0 & i\in I\setminus J, \\ \int_{V_i} \psi_i^*(\eta_i\alpha|_{\partial X}) & i\in J. \end{cases}$$

Hence
$$\int_X d\alpha = \sum_{i\in I} \int_X d(\eta_i\alpha) = \sum_{i\in J} \int_{\partial X} \eta_i\alpha|_{\partial X} = \int_{\partial X}\alpha|_{\partial X}.$$

$\square$

<u>Example</u> <u>Green's Theorem</u> says that if $D\subset\mathbb{R}^2$ is a compact region in the plane with smooth boundary ∂D, and $P, Q: D\longrightarrow\mathbb{R}$ are smooth, then

$$\oint_{\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$$

This follows from Stokes' with $X=D$, $\alpha = P\,dx + Q\,dy$, so that $d\alpha = \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right)dx\wedge dy$.

Theorem 7.2. Let X be a compact, oriented n-manifold. Then there is a natural, nonzero linear map $\Phi : H^n(X) \to \mathbb{R}$, $\Phi([\alpha]) = \int_X \alpha$. Hence $H^n(X) \neq 0$. If X is connected then Φ is an isomorphism.

Proof (first part only). To show Φ is well-defined, note that if $\alpha, \alpha' \in [\alpha]$ then $\alpha' = \alpha + d\beta$, so $\int_X \alpha' = \int_X \alpha + d\beta = \int_X \alpha + \int_{\partial X} \beta|_{\partial X} = \int_X \alpha$ by Stokes' Theorem, as $\partial X = \emptyset$. Then

Let $[\omega]$ be the orientation on X. Then $[\omega] \in H^n(X)$, $\omega \in \Omega^n(X)$ with $d\omega = 0$, so $\Phi([\omega]) = \int_X \omega$ is the sum of integrals of positive functions, by definition of $\int_X$, so $\Phi([\omega]) > 0$, and Φ is nonzero, so $H^n(X) \neq 0$. For the last part, see Hitchin notes, §8. $\square$

7.5. The degree of a smooth map.

Theorem 7.3 Let X, Y be compact, connected, oriented, smooth n-manifolds, and $f: X \longrightarrow Y$ be a smooth map. Then there is a unique integer $\deg f$ called the degree of f, such that:

(a) $$\int_X f^*(\alpha) = \deg f \cdot \int_Y \alpha \quad \text{for all } \alpha \in \Omega^n(Y).$$

(b) Suppose $y \in Y$ such that for all $x \in X$ with $f(x) = y$, $T_x f : T_x X \longrightarrow T_y Y$ is an isomorphism. (Note that by Sard's Theorem, the set of such y is dense in Y.)

Then $$\deg f = \sum_{\substack{x \in X: \\ f(x) = y}} \begin{cases} 1, & T_x f \text{ orientation-preserving} \\ -1, & T_x f \text{ orientation-reversing,} \end{cases}$$

where the sum is finite.

<u>Proof.</u> We have a diagram

$$
\begin{tikzcd}
{[\alpha]} \arrow[d, mapsto] & H^n(Y) \arrow[r, "f^*", "{[\alpha]} \mapsto {[f^*(\alpha)]}"'] \arrow[d, "\Phi_Y", "\cong"'] & H^n(X) \arrow[d, "\Phi_X", "\cong"'] & {[f^*(\alpha)]} \arrow[d, mapsto] \\
\int_Y \alpha & \mathbb{R} \arrow[r, dashed, "\cdot \deg f"] & \mathbb{R} & \int_X f^*(\alpha)
\end{tikzcd}
$$

with columns isomorphisms by Theorem 7.2.
There is a unique linear map $\mathbb{R} \to \mathbb{R}$ making this commute, which is multiplication by some $\deg f \in \mathbb{R}$.
Hence there exists a unique $\deg f \in \mathbb{R}$ satisfying (a).

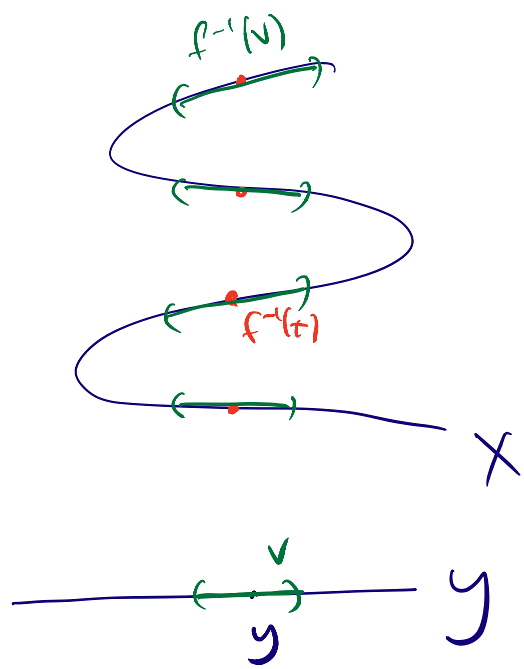
Let $y \in Y$ be as in (b). Then f is a submersion near $f^{-1}(y)$, so $f^{-1}(y)$ is a 0-dimensional manifold by Theorem 3.1. It is compact as it is closed in X which is compact, so $f^{-1}(y)$ is finite.
By the inverse function theorem, f is a local diffeomorphism near each $x \in f^{-1}(y)$.

Hence there exists a connected open neighborhood V of y in Y with

$$U = f^{-1}(V) \cong f^{-1}(y) \times U,$$

with $f|_U : U \to V$ identified with the projection

$$\pi_V : f^{-1}(y) \times V \to V.$$



Choose $\alpha \in \Omega^n(Y)$ supported in V with

$$\int_Y \alpha = \int_V \alpha = 1.$$

Then

$$\int_X f^*(\alpha) = \int_U f^*(\alpha)$$

$$= \sum_{x \in f^{-1}(y)} \int_{\{x\} \times V} f^*(\alpha)$$

$$= \sum_{x \in f^{-1}(y)} (\pm 1) \cdot \int_V \alpha$$

$$= \sum_{x \in f^{-1}(y)} (\pm 1) \cdot \int_Y \alpha,$$

where the sign is $\begin{cases} 1 & T_x f \text{ orientation-preserving} \\ -1 & T_x f \text{ orientation-reversing.} \end{cases}$

Hence $\deg f = \sum_{x \in f^{-1}(y)} \begin{cases} 1 & T_x f \text{ orientation-preserving}, \\ -1 & T_x f \text{ orientation-reversing}. \end{cases}$

Thus $\deg f \in \mathbb{Z}$, since by Sard there exists such y. $\square$

__Corollary 7.4__ In Theorem 7.3, if $f : X \to Y$ is not surjective then $\deg f = 0$.

__Proof.__ Take $y \in Y \setminus f(X)$ in (b). $\square$

__Example.__ Let $X = \mathbb{C} \cup \{\infty\}$ be the extended complex plane, thought of as a compact, oriented 2-manifold diffeomorphic to S^2. Let $k \geq 1$, and consider $f : X \to X$ given by

$$f(z) = \begin{cases} z^k + a_1 z^{k-1} + \cdots + a_k, & z \in \mathbb{C}, \\ \infty, & z = \infty, \end{cases}$$

for $a_1, \ldots, a_k \in \mathbb{C}$. Then f is smooth, and has a degree $\deg f \in \mathbb{Z}$.

Since $\deg f$ is determined by $f^*: H^2(X) \longrightarrow H^2(X)$, it is unchanged by smooth deformation of f by Theorem 6.5. By considering

$$f_t(z) = z^k + t(a_1 z^{k-1} + \cdots + a_0)$$

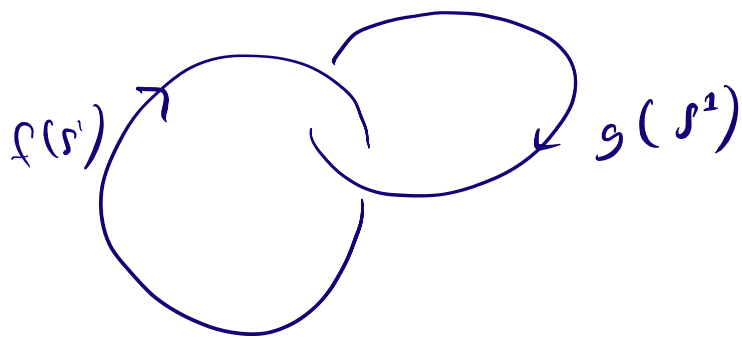
we see that $\deg f = \deg g$, where

$$g(z) = \begin{cases} z^k & z \in \mathbb{C}, \\ \infty & z = \infty \end{cases}$$

From Theorem 7.3 (b) for g with $y \in \mathbb{C} \setminus 0$, we see that $\deg g = k$, since a nonzero complex number has k k^{th} roots. Hence $\deg f = k \geq 1$. So Corollary 7.4 proves that f is surjective. Hence there exists $z \in \mathbb{C}$ with $z^k + a_1 z^{k-1} + \cdots + a_k = 0$. This is the <u>Fundamental Theorem of Algebra</u>.

The degree $\deg f$ counts the number of solutions z of $f(z) = c$ for $c \in \mathbb{C}$, with multiplicity.

Example. Let $f, g : S^1 \to \mathbb{R}^3$ be smooth (e.g. they could be embeddings), giving two circles in $\mathbb{R}^3$. Suppose these are disjoint, $f(S^1) \cap g(S^1) = \emptyset$.



Define $h : S^1 \times S^1 \to S^2$ by

$$h(s,t) = \frac{f(s) - g(t)}{|f(s) - g(t)|} \quad \text{for } s,t \in S^1.$$

Then h is smooth and $S^1 \times S^1$, S^2 are compact oriented 2-manifolds, so we have a degree $\deg h \in \mathbb{Z}$.

This is the <u>linking number</u> of $f(S^1), g(S^1)$ in $\mathbb{R}^3$.

<u>Example</u> Let $X \subset \mathbb{R}^3$ be a compact, oriented surface and $\underline{n}: X \to S^2 \subset \mathbb{R}^3$ be the unit normal map. We call $\underline{n}$ the <u>Gauss map.</u>

Let ω be the standard 2-form (area form) on S^2, with $\int_{S^2} \omega = 4\pi$. Calculation shows that $\underline{n}^*(\omega) = K\sqrt{EG-F^2}\,du\,dv$ on X, where (u,v) are local oriented coordinates, E, F, G the first fundamental form, and K the Gaussian curvature. Hence the Gauss-Bonnet Theorem says that $\int_X \underline{n}^*(\omega) = 2\pi\chi(X)$ $= 4\pi(1-g)$, where g is the genus of X.

But $\int_X \underline{n}^*(\omega) = \deg \underline{n} \cdot \int_{S^2}\omega$, so

$4\pi(1-g) = 4\pi \deg \underline{n}$, and $\deg \underline{n} = 1-g$.

Interpretation: the number of points in X with a given normal direction $\underline{n}_0$, counted with signs, is $1-g$.

8. Riemannian metrics

Riemannian geometry is a huge subject. We only have time to scratch the surface.

In Euclidean geometry on $\mathbb{R}^n$, the distance between points $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ is

$$d_{\mathbb{R}^n}(\underline{x}, \underline{y}) = \left((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 \right)^{1/2}.$$

Note that <u>squares of distances</u>, rather than distances, behave nicely algebraically.

If $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n) : [0,1] \to \mathbb{R}^n$ is a smooth path in $\mathbb{R}^n$, the <u>length</u> of γ is

$$l(\gamma) = \int_0^1 \left(\left(\frac{d\gamma_1}{dt}\right)^2 + \cdots + \left(\frac{d\gamma_n}{dt}\right)^2 \right)^{1/2} dt.$$

This is unchanged by parametrizations of $[0,1]$.

Regarding $\gamma : [0,1] \to \mathbb{R}^n$ as a smooth map of manifolds, we have $l(\gamma) = \int_0^1 g_{\mathbb{R}^n}|_{\gamma(t)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{1/2} dt$, where $\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}\mathbb{R}^n$, and

$$g_{\mathbb{R}^n}|_{\underline{x}} = (dx_1)^2 + \cdots + (dx_n)^2 \text{ in } S^2 T^*_{\underline{x}}\mathbb{R}^n \subset \otimes^2 T^*_x \mathbb{R}^n,$$

so that $g_{\mathbb{R}^n} \in \Gamma^\infty(S^2 T^*\mathbb{R}^n)$.

(Here $S^2T^*\mathbb{R}^n \subseteq \bigotimes^2 T^*\mathbb{R}^n$ is the subbundle of symmetric tensors, symmetric under exchanging the two factors of $T^*\mathbb{R}^n$.)

This is a simple example of a Riemannian metric on a manifold, being used to define lengths of curves. We call $g_{\mathbb{R}^n}$ the <u>Euclidean metric</u> on $\mathbb{R}^n$.

<u>Definition</u> Let X be a manifold. A <u>Riemannian metric</u> (or just <u>metric</u>) g on X is a smooth section of S^2T^*X such that $g|_x \in S^2T_x^*X$ is a positive definite quadratic form on T_xX for all $x \in X$. We call (X,g) a <u>Riemannian manifold</u>.

If $(x_1,\ldots,x_n)$ are local coordinates on open $V \subseteq X$, then $dx_1,\ldots,dx_n$ are a basis for $T^*X|_V$, and

$$g|_V = \sum_{i,j=1}^n g_{ij}\, dx_i \otimes dx_j, \text{ where } (g_{ij})_{i,j=1}^n$$

is a positive definite matrix of smooth functions on V. Often we omit the $\otimes$ and write $\sum_{i,j=1}^n g_{ij}\, dx_i\, dx_j$.

Let $\gamma : [0,1] \longrightarrow X$ be a smooth map, regarded as a curve in X. The __length__ of γ is $\ell(\gamma) = \int_0^1 g|_{\gamma(t)} \left(\frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t) \right)^{1/2} dt$.

If X is (path) connected, we can define a metric d_g on X, in the sense of metric spaces, by

$$d_g(x,y) = \inf_{\substack{\gamma : [0,1] \to X \ C^\infty \\ \gamma(0) = x, \ \gamma(1) = y}} \ell(\gamma).$$

Roughly, $d_g(x,y)$ is the length of the shortest curve from x to y.

__Proposition 8.1.__ Every manifold X admits Riemannian metrics.

__Proof.__ Let $\{(U_i, \phi_i) : i \in I\}$ be an atlas on X. Choose a subordinate partition of unity $\{\eta_i : i \in I\}$. Define $g = \sum_{i \in I} \eta_i \cdot (\phi_i)_* (dx_1^2 + \cdots + dx_n^2)$ in $\Gamma^\infty(S^2 T^* X)$. Here $(\phi_i)_* (dx_1^2 + \cdots + dx_n^2)$ is a smooth section of $S^2 T^* X$ on $\phi_i(U_i) \subseteq X$.

After multiplying by η_i, it extends by 0 on $X \setminus \phi_i(U_i)$ to a smooth section on X. The sum is locally finite, so g is well defined.
At each point it is a positive linear combination of positive definite quadratic forms, so positive definite. Thus g is a Riemannian metric. $\square$

8.1. Riemannian metrics on submanifolds

Definition Let X, Y be manifolds, and $i: X \to Y$ an immersion or embedding, and h a Riemannian metric on Y.
For each $x \in X$ with $i(x) = y \in Y$, we have $T_x i: T_x X \hookrightarrow T_y Y$, injective as i is an immersion. Dualizing gives $T_x^* i: T_y^* Y \to T_x^* X$ and $S^2(T_x^* i): S^2 T_y^* Y \to S^2 T_x^* X$.
Define $g \in \Gamma^\infty(S^2 T^* X)$ by $g|_x = S^2(T_x^* i)(h|_y)$.
This depends smoothly on $x \in X$. Since $T_x i$ is injective, and $h|_y$ is positive definite,

$g|_x$ is positive definite. Thus g is a Riemannian metric. We write $g = i^*(h)$. If we think of $X \subseteq Y$ as a submanifold, we write $g = h|_X$.

Note that if i is not an immersion, then $g = i^*(h) \in \Gamma^\infty(S^2T^*X)$ is well defined, but it is only positive <u>semidefinite</u>, and is not a Riemannian metric.

<u>Example</u>. Let $\mathbb{R}^n$ have its Euclidean metric $g_{\mathbb{R}^n} = dx_1^2 + \cdots + dx_n^2$. Let $X \subset \mathbb{R}^n$ be a submanifold. Then $g = g_{\mathbb{R}^n}|_X$ is a Riemannian metric on X. Combining this with the Whitney Embedding Theorem gives an alternative proof of Proposition 8.1.

<u>Example</u> Model the surface of the earth as a sphere S^2_R of radius $R = 6{,}371$ km about 0 in $\mathbb{R}^3$. Then the Riemannian metric $g = g_{\mathbb{R}^3}|_{S^2_R}$ determines lengths of paths on the earth.

Define spherical polar coordinates (θ, φ) on $S^2_R \setminus \{N, S\}$ by $\underline{x}(\theta,\varphi) = (R\sin\theta\cos\varphi,\ R\sin\theta\sin\varphi,\ R\cos\theta)$.

Then $g = dx_1^2 + dx_2^2 + dx_3^2 =$

$$(d(R\sin\theta\cos\varphi))^2 + (d(R\sin\theta\sin\varphi))^2 + d(R\cos\theta)^2$$

$$= R^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

8.2. Volume forms and integrating functions

Definition. Let X be an oriented n-manifold and g a Riemannian metric on X. Then there is a unique n-form $dV_g \in \Omega^n(X)$ called the volume form, such that if $v_1, \dots, v_n$ is a basis for $T_x X$ which is oriented and orthonormal w.r.t. $g|_x$, then $dV_g(v_1 \otimes \dots \otimes v_n) = 1$.

The orientation on X is $[dV_g]$.

If X is compact, the volume of X is $\mathrm{vol}(X) = \int_X dV_g$.

We can use the volume form to integrate functions on X. Suppose $f \in C^\infty(X)$ is compactly-supported. Then $f\,dV_g$ is a compactly-supported n-form on X, and $\int_X f\,dV_g \in \mathbb{R}$ as in §7.

So, on a Riemannian manifold we can integrate functions. The orientation doesn't really matter: if we change orientations we change the sign of $\int_X : \Omega^n_c(X) \to \mathbb{R}$, and the sign of dV_g, so $\int_X f\,dV_g$ doesn't change, and we can still define it if X is not oriented/orientable.

8.3. Isometries

Definition. Let (X,g) and (Y,h) be Riemannian manifolds, and $f: X \to Y$ a diffeomorphism. Then f is an immersion, so as in §8.1 we have a pullback metric $f^*(h)$. We call f an isometry if $g = f^*(h)$.

Greek iso metry, "same distance".

Definition Let (X,g) be a Riemannian manifold, and $\varphi : \mathbb{R} \times X \to X$ a 1-parameter group of diffeomorphisms, and $v \in \Gamma^\infty(TX)$ the corresponding vector field. As in §4.2, and As in §4.3, the Lie derivative is $\mathcal{L}_v g = \frac{d}{dt}\left(\varphi_t^*(g)\right)\big|_{t=0}$.

Hence if $\varphi_t : X \to X$ is an isometry for all $t \in \mathbb{R}$, then $\varphi_t^*(g)$ is independent of t, and $\mathcal{L}_v g = 0$.

Conversely, one can show that $\mathcal{L}_v g = 0$ implies φ_t is an isometry for all $t \in \mathbb{R}$. We call $v \in \Gamma^\infty(TX)$ a Killing vector field if $\mathcal{L}_v g = 0$.

Killing vector fields are infinitesimal isometries.

<u>Example</u> (a) The group of isometries of $(\mathbb{R}^n, g_{\mathbb{R}^n})$ is $Iso(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ (rotations and translations), where $O(n)$ is the orthogonal group.
(b) The group of isometries of $(S^n, g_{\mathbb{R}^{n+1}}|_{S^n})$ is $O(n+1)$.

In fact, if (X, g) is a Riemannian manifold with finitely many connected components, then $Iso(g)$ is both a group and a manifold (it is a <u>Lie group</u>), and the vector space of Killing vector fields of (X, g) is naturally identified with $T_1 Iso(X, g)$.

8.4. The geodesic flow.

<u>Definition.</u> Let X be a manifold with cotangent bundle $\pi : T^*X \to X$. Think of T^*X as a manifold in its own right, with points (x, α) for $x \in X$ and $\alpha \in T_x^*X$. We have
$T^*_{(x,\alpha)}\pi : T_x^*X \to T^*_{(x,\alpha)}(T^*X)$.

There is a <u>canonical 1-form</u> $\theta \in \Omega^1(T^*X)$ with $\theta|_{(x,\alpha)} = (T^*_{(x,\alpha)}\pi)(\alpha)$ for all $(x,\alpha) \in T^*X$.

There is a <u>canonical 2-form</u> $\omega \in \Omega^2(T^*X)$ given by $\omega = -d\theta$.

If $(x_1,\dots,x_n)$ are local coordinates on X then $(x_1,\dots,x_n,y_1,\dots,y_n)$ are local coordinates on T^*X, where $(x_1,\dots,x_n,y_1,\dots,y_n)$ represents $((x_1,\dots,x_n), y_1dx_1+\dots+y_ndx_n)$.

Then $\theta = y_1dx_1 + \dots + y_ndx_n$ and

$$\omega = dx_1\wedge dy_1 + \dots + dx_n \wedge dy_n.$$

Now let g be a Riemannian metric on X. Thinking of $g \in \Gamma^\infty(S^2T^*X)$ as a positive definite matrix, it has a matrix inverse $g^{-1} \in \Gamma^\infty(S^2TX)$, such that if

$$g = \sum_{i,j} g_{ij}\, dx_i\, dx_j \quad \text{and} \quad g^{-1} = \sum_{i,j} g^{-1}_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

in local coordinates, then $(g_{ij})^n_{i,j=1}$ and $(g^{-1}_{ij})^n_{i,j=1}$ are inverse matrices.

Define a smooth function $H : T^*X \to \mathbb{R}$ by $H(x, \alpha) = \frac{1}{2} g^{-1}|_x (\alpha, \alpha)$.

We now claim that there is a unique vector field $v \in \Gamma^\infty(T(T^*X))$ with $i_v \omega = dH$, and we call v the <u>geodesic flow</u> of g. To see this, observe that as the matrix of ω is invertible, the map $\Gamma^\infty(T(T^*X)) \to \Gamma^\infty(T^*(T^*X))$, $v \longmapsto i_v \omega$, is an isomorphism.

<u>Example</u> Let $\mathbb{R}^n$ have coordinates $(x_1, \ldots, x_n)$ and metric $g = dx_1^2 + \cdots + dx_n^2$. Then $T^*\mathbb{R}^n$ has coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. We have $H = \frac{1}{2}(y_1^2 + \cdots + y_n^2)$, so $dH = y_1 \, dy_1 + \cdots + y_n \, dy_n$, and $v = y_1 \frac{\partial}{\partial x_1} + \cdots + y_n \frac{\partial}{\partial x_n}$.

As in §4.2, we can consider <u>integral curves</u> of vector fields.

<u>Definition</u> Let (X,g) be a Riemannian manifold, and $v \in \Gamma^\infty(T(T^*X))$ be its geodesic flow. Suppose $\gamma : (a,b) \to T^*X$ is an integral curve of v. Then we call $\pi \circ \gamma : [a,b] \longrightarrow X$ a <u>geodesic</u> in (X,g).

It turns out that geodesics are <u>locally length-minimizing curves</u> in (X,g). That is, if $\gamma : (a,b) \longrightarrow X$ is a geodesic and $s,t \in [a,b]$ are sufficiently close, then $\gamma([s,t])$ is the shortest curve in (X,g) between $\gamma(s)$ and $\gamma(t)$.

In local coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$, write

$$g^{-1} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}.$$

Then

$$H = \frac{1}{2} \sum_{i,j} g^{ij} y_i y_j,$$

so

$$dH = \sum_{i,j} (g^{ij} y_i)\, dy_j + \frac{1}{2} \sum_{i,j,k} \frac{\partial g^{ij}}{\partial x_k} y_i y_j \, dx_k,$$

and $v = \sum_{i,j} (g^{ij} v_i) \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{i,j,k} \frac{\partial g^{ij}}{\partial x_k} y_i y_j \frac{\partial}{\partial y_k}$.

Thus if $\gamma(t) = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$,

the geodesic equations are

$$(*1) \quad \frac{dx_j}{dt} = \sum_{i=1}^{n} g^{ij}(x_1(t), \ldots, x_n(t))\, y_i(t)$$

$$(*2) \quad \frac{dy_k}{dt} = -\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial g^{ij}}{\partial x_k}(x_1(t), \ldots, x_n(t))\, y_i(t) y_j(t).$$

Here $(*1)$ determines $y_i(t)$ from $\frac{dx_j}{dt}$.

Then $(*2)$ is the Euler-Lagrange equation for

$$\int_a^b g\left(\frac{d(\pi \circ \gamma)}{dt}, \frac{d(\pi \circ \gamma)}{dt} \right) dt.$$

It implies the Euler-Lagrange equation for

$$\int_a^b g\left(\frac{d(\pi \circ \gamma)}{dt}, \frac{d(\pi \circ \gamma)}{dt} \right)^{1/2} dt,$$

i.e. curves of stationary length.

Also $(*2)$ implies the curves are traversed at constant speed.

Example. If $(X, g) = (\mathbb{R}^n, g_{\mathbb{R}^n})$ with the usual coordinates then $g^{ij} = \delta^{ij}$ and

$$\frac{\partial g^{ij}}{\partial x_k} = 0.$$

So $(*1)-(*n)$ become

$$\frac{dx_j}{dt} = y_j, \qquad \frac{dy_k}{dt} = 0, \quad \text{with}$$

solutions $x_j = a_j + b_j t, \quad y_j = b_j, \quad a_j, b_j \in \mathbb{R}.$

So geodesics are just straight lines in $\mathbb{R}^n$.