Mathematical Methods I MT2022

Problem Sheet 1

Eigenfunction expansions

1. Adjoint problems. Use the adjoint relation, $\langle w, Ly\rangle = \langle L^* w, y\rangle$, to determine the differential operator and boundary conditions for the adjoint problem. In each case case state if the operator and/or the full system is self-adjoint.

   (a) $Ly \equiv \dfrac{d^2y}{dx^2}, \qquad 2y(0) + y'(0) = 0, \qquad y(1) + y'(1) = 0.$

   (b) $Ly \equiv \dfrac{d^4y}{dx^4} - \dfrac{dy}{dx}, \qquad y'(0) - y''(0) = 0, \qquad y'''(0) = 0, \qquad y(1) = 0, \qquad y'(1) - y'''(1) = 0.$

2. Adjoint condition. Let

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x), \quad a < x < b$$

   such that $L^* = L$. Show that the system is fully self adjoint for any general unmixed boundary conditions (NB: the as above and αs below are distinct variables):

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{1}$$

3. An inhomogeneous problem.

   (a) Find the general homogeneous solution of the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (1+\alpha)y = 0, \tag{2}$$

   where α is a given positive constant.

   (b) Use (a) to determine the eigenvalues and eigenfunctions of the self-adjoint problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) + \lambda xy = 0 \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{3}$$

   (c) Obtain the eigenfunction expansion for the solution of the inhomogeneous problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) = x \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{4}$$

   (Give the coefficients explicitly, i.e. compute the integrals.)

4. Eigenfunction expansion gone wrong. What is wrong with the below argument?

   Let L be a second-order differential operator as usual, with boundary conditions BC. Its eigenfunctions satisfy $Ly_k = \lambda_k y_k$ (or $L^* w_k = \lambda_k w_k$) with homogeneous boundary conditions $BC = 0$ (or $BC^* = 0$) and are complete. The solution of any inhomogeneous problem $Ly = f$, $BC \neq 0$, can then be written $y(x) = \sum_k c_k y_k(x)$, and in lectures we gave a method for finding c_k in terms of f and the boundary conditions.

Now suppose I say

$$\begin{aligned}
Ly &= f \\
\Rightarrow L \sum_k c_k y_k &= f \\
\Rightarrow \sum_k c_k L y_k &= f \\
\Rightarrow \sum_k c_k \lambda_k y_k &= f \\
\Rightarrow w_j \sum_k c_k \lambda_k y_k &= w_j f \\
\Rightarrow \sum_k c_k \lambda_k \langle w_j, y_k \rangle &= \langle w_j, f \rangle \\
\Rightarrow c_j &= \frac{\langle w_j, f \rangle}{\lambda_j \langle w_j, y_j \rangle}
\end{aligned}$$

where I have (legitimately) used orthogonality to get the last line. The answer must be wrong because it contains no reference to the boundary conditions, but what is wrong? (Carefully consider the justification for going from one line to the next).

[You might try going through the steps with the system $Ly \equiv y'' = 0$, $y(0) = 0$, $y(1) = 1$. Here the problem is self-adjoint with eigenfunctions $\sin n\pi x$, and the recipe above gives $c_k = 0$, which are obviously not the Fourier sine coefficients of the solution $y = x$. The recipe in lectures *does* give the right answer, of course.]