Mathematical Methods I MT2022

Problem Sheet 2

Sturm-Liouville, Point source

1. Sturm-Liouville form. Consider the eigenvalue problem $Ly = -\lambda y$ for the general second order linear equation

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = -\lambda y, \qquad a \le x \le b \tag{1}$$

where $A(x), B(x), C(x)$ are given functions with $A(x) \neq 0$ for $x \in [a, b]$.

(a) Show that (1) can always be put into Sturm-Liouville form,

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = -\lambda r(x)y. \tag{2}$$

Namely, determine $p(x), q(x), r(x)$ in terms of $A(x), B(x), C(x)$.

What orthogonality condition will the eigenfunctions satisfy?

(b) Show that the substitution $y(x) = \psi(x)\exp\left(-\frac{1}{2}\int \frac{B(x)}{A(x)}\,dx\right)$ yields another self-adjoint form, called a Schroedinger equation, for $\psi(x)$

$$\frac{d^2\psi}{dx^2} + U(x)\psi = -\lambda V(x)\psi. \tag{3}$$

Find $U(x), V(x)$ in terms of A, B, C.

2. Eigenvalue expansion. Consider the eigenvalue problem on $0 \le x \le 1$,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (1+\lambda)y = 0,$$

$$y'(0) + y(0) = 0, \qquad y'(1) + y(1) = 0.$$

(a) Assuming λ to be a non-negative constant, find the general solution of the homogeneous ODE. Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(b) Obtain the adjoint eigenfunctions.

(c) Consider the problem

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = f(x),$$

$$y'(0) + y(0) = 0, \qquad y'(1) + y(1) = 0.$$

i. Obtain the coefficients in an eigenfunction expansion

$$y(x) = \sum_k^\infty c_k y_k(x)$$

ii. Show that the coefficients in an eigenfunction expansion for the equivalent Sturm-Liouville problem match those you get in part (i).

Note: You may notice that the expansion procedure only works if $f(x)$ satisfies a particular condition. More background on this will follow in “Further Mathematical Methods” in HT. For now, you may assume f satisfies what is needed for a solution to exist, and focus only on the coefficients c_k for $k > 0$.

3. Kick stop. Consider a harmonic oscillator, i.e. a mass on a spring. The displacement of the spring satisfies

$$m\ddot{x} + kx = 0, \tag{4}$$

where $x(t)$ is the displacement from rest at time t, m is the mass, and $k > 0$ is a spring constant. Suppose the mass has initial displacement $x(0) = 1$, zero initial velocity, and at time $t = \tau$ is acted upon by a point force of strength f.

Obtain the motion of the mass for any time $t > 0$, and find conditions on f and τ such that the point force completely stops the motion. Explain the result physically.