Mathematical Methods I MT2022

Problem Sheet 4

Distributions

1. Scalar property of delta function. Show from its interpretation as an integral that $\delta(ax) = \frac{1}{|a|}\delta(x)$ for any constant a.

2. Convergence of distributions. Show that the following function sequences converge to the δ-distribution as $n \to \infty$.

   (a) $$f_n(x) = \begin{cases} n/2 & \text{for } -1/n < x < 1/n, \\ 0 & \text{else} \end{cases}$$

   (b) $$f_n(x) = \frac{e^{-nx^2/4}}{\sqrt{4\pi/n}} \quad \left(\text{using } \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \text{ without proof}\right)$$

   You have to show that $\lim_{n\to\infty} < f_n, \phi > = < \delta, \phi >$ *holds for all* $\phi \in C_0^\infty(\mathbb{R})$, *i.e. that* $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \phi(0)$

   Hints: In (a), split the integral as

   $$\int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \int_{-\infty}^{\infty} f_n(x)\ [\phi(x) - \phi(0)]\, dx\ +\ \phi(0)\int_{-\infty}^{\infty} f_n(x)dx.$$

   The second integral can be evaluated explicitly. What can you say about the first integral? A similar approach for (b) though a bit more work. In my model solution I split the integral into 4 parts, including a term

   $$\int_{-\rho}^{\rho} f_n(x)\, [\phi(x) - \phi(0)]\ dx$$

   with ρ chosen appropriately.

3. Derivatives of distributions.

   (a) For $n \geq 0$, $a \in \mathbb{R}$, let $D_n : C_0^\infty \to \mathbb{R}$ be defined by

   $$< D_n, \phi > = \left.\frac{d^n\phi}{dx^n}\right|_{x=a} = \phi^{(n)}(a) \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

   Show that D_n is a distribution, and then, by induction, that

   $$D_n = (-1)^n \delta^{(n)}(x-a),$$

   where $\delta^{(n)}$ is the n-th distributional derivative of the δ-distribution.

   (b) Let $T : C_0^\infty \to \mathbb{R}$ be a distribution (not necessarily one that is induced by a continuous function). Using the definition of distributional derivative, translation of distributions and convergence of distributions, show that

   $$\lim_{\alpha\to 0} \frac{T(x+\alpha) - T(x)}{\alpha} = T'(x)$$

   You may use (without proof) that for any $\phi \in C_0^\infty(\mathbb{R})$,

   $$\rho_\alpha(x) \equiv \frac{\phi(x) - \phi(x-\alpha)}{\alpha}$$

   converges to $\phi'(x)$ *uniformly in* x *as* $\alpha \to 0$.