Models for large populations of neurons structured by time elapsed since the last discharge

Introduction:

One of the crucial issues in neuroscience is to develop simple mathematical models that are able to capture the collective dynamics of a large number of neurons. Indeed, neurons are excitable cells connected to each other that communicate by sending stereotyped electrical discharges: action potentials. It is very difficult to understand the global dynamics of a neuronal network by observing the inextricable mess of individual connections at the microscopic level. Through mean-field limit of certain probabilistic models, it is possible to obtain a structured model, called the "time-elapsed neuron model".

The idea is to look in a neural network the time since which a randomly chosen neuron has not emitted an action potential. We therefore consider $n(s,t)$ the probability density of finding in the network at time t a neuron that has not sent an electric discharge since time s. Let X_t be a neuron taken at random in the network which at time t has not discharged since a time $s(X_t)$. Then,

$$\mathbb{P}\big(s(X_t) \in [s_1, s_2]\big) = \int_{s_1}^{s_2} n(s,t)ds.$$

The time-elapsed model writes,

$$\begin{cases} \dfrac{\partial n}{\partial t}(s,t) + \dfrac{\partial n}{\partial s}(s,t) + p\big(s, N(t)\big)n(s,t) = 0, & s,t \in (0,+\infty), \\ N(t) = \displaystyle\int_0^{+\infty} p(s, N(t))n(s,t)ds, & t \in (0,+\infty), \\ n(0,t) = N(t), \qquad n(s,0) = n_0(s), & s,t \in (0,+\infty), \end{cases} \tag{1}$$

where

$$p(s,N) = \mathbb{1}_{\{s \geqslant \sigma(N)\}} = \begin{cases} 1 & \text{if } s \geqslant \sigma(N), \\ 0 & \text{if } s < \sigma(N), \end{cases}$$

and with $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ a Lipschitz continuous function.

This model exhibits different complex behaviours such as convergence towards a stationary distribution and stable periodic oscillations of different types.

Project:

The first part of the project would be to understand the mathematical significance of each term of the equation and the proof of convergence to a stationary state in the weakly connected regime.

Then, the students will code in Python a simple numerical method to simulate the model and explore numerically the different behaviours of the equation.

Further avenues of investigation:

- Numerical study of more general two-dimensional structured neuroscience models.
- To understand the proof of sharper results, namely convergence to stationary state with more general choices of $p(s, N)$.
- Other meanings for the variable s and addition of a reset kernel $k(s, u)$:

$$\frac{\partial n}{\partial t}(s,t)+\frac{\partial n}{\partial s}(s,t)+p\big(s,N(t)\big)n(s,t) = \int_0^{+\infty} k(s,u)p(u,N(t))n(u,t)du.$$

Useful pre-requisite knowledge

The only knowledge required to start is basic real analysis and the other mathematical and computational skills can be acquired on the way. It can be helpful (but not essential) for the students to take B5.2 Applied PDEs (for better familiarity with the method of characteristics) and/or B5.5 Further mathematical biology (for a broader understanding of structured models). The computational part of the project will be in Python but it's totally fine for the students to discover the language during the project, it would even be a nice way to learn it and I will provide help for creating familiarity with the (simple) Python syntax if necessary.

Reading:

- Pakdaman, K., Perthame, B., & Salort, D. (2013). Relaxation and self-sustained oscillations in the time elapsed neuron network model. SIAM Journal on Applied Mathematics, 73(3), 1260-1279.

- Pakdaman, K., Perthame, B., & Salort, D. (2009). Dynamics of a structured neuron population. Nonlinearity, 23(1), 55.
- Perthame, B. (2006). Transport equations in biology. Springer Science & Business Media.