Mathematical Institute University of Oxford

B1.2 Set Theory MT22

Problem Set 1

For this sheet assume the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme. In question 1 **only** assume also the Powerset Axiom: if X is a set there is a set $\mathcal{P}(X)$ whose elements are precisely the subsets of X (this set is called the powerset of X).

1. For each statement give either a proof or a counterexample.

(i) $\mathcal{P}\big(\bigcup X\big) = X$
(ii) $\bigcup \mathcal{P}(X) = X$
(iii) If $\mathcal{P}(a) \subseteq \mathcal{P}(b)$ then $a \subseteq b$.

2. (a) Prove that the unordered pair $\{x, y\}$ of x and y is the unique set whose elements are precisely x and y.

(b) Let $\phi(z, w_1, \ldots, w_k)$ be a formula of $\mathcal{L}$ and $w_1, \ldots, w_k, x$ sets. Prove that the subset y of x afforded by the Comprehension Scheme is unique with the stated property.

3. Let a be a set. Prove that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

4. (a) Show that if we define an ordered triple (a, b, c) of sets to be $\langle\langle a, b\rangle, c\rangle$ then this definition "works": i.e. if $(a, b, c) = (a', b', c')$ then $a = a', b = b', c = c'$. You may use the fact (from lectures) that $\langle a, b\rangle$ "works".

(b) for each of the following alternative possible definitions of an ordered triple, prove that the definition "works" or give a counterexample.

(i) $(a, b, c)_1 = \{\{a\}, \{a, b\}, \{a, b, c\}\}$
(ii) $(a, b, c)_2 = \{\langle 0, a\rangle, \langle 1, b\rangle, \langle 2, c\rangle\}$ (where $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}$)
(iii) $(a, b, c)_3 = (\{0, a\}, \{1, b\}, \{2, c\})$ where $(., ., .)$ is as in part (a)
(iv) $(a, b, c)_4 = \{\{0, a\}, \{1, b\}, \{2, c\}\}$.

5. A set a is called *transitive* if $\bigcup a \subseteq a$, i.e. if, for all sets x, if $x \in a$ then $x \subseteq a$. Prove that

(i) $\emptyset$ is transitive
(ii) if a is transitive then so is $a \cup \{a\}$ (this set is denoted a^+)
(iii) a is transitive iff $\bigcup \big(a \cup \{a\}\big) = a$
(iv) a is transitive iff, for all sets x, y, if $x \in y \in a$ then $x \in a$
(v) the intersection of any (non-empty) set of transitive sets is transitive
(vi) the union of any set of transitive sets is transitive
(vii) write a formula in $\mathcal{L}$ with a free variable x expressing "x is transitive".

PTO

6. Prove the following.

(i) If x is a set, there is no set whose elements are all the sets y with $y \notin x$.

(ii) There is no set of all one-element sets.

(iii) There is no set of all two-element sets.

7. (a) Prove that

(i) if a, b, c are sets then $\{a, b, c\}$ is a set.

(ii) if $x_1, \ldots, x_n$ are sets then $\{x_1, \ldots, x_n\}$ is a set (here $n \in \mathbb{N}$).

(iii) if X is a finite set then $\mathcal{P}(X)$ is a set (do not assume the Powerset Axiom!).

(iv) if X is a finite set then the collection of all two-element subsets of X is a set.

(b) Suppose X is a set all of whose elements are finite sets. Prove that there is a set Y consisting of all the elements of X that have an *even* number of elements. (Note it is not sufficient that Y "is" a subset of X.)

Hint: You must not use the power set axiom. However, from part (a) (ii) you know that a finite number of sets can always be assembled into a set with just them as elements. (So if X is a finite set then there is a set which is its powerset, but you do not necessarily need the power set...)

8. Prove that there exist infinitely many sets.