C4.3 Functional Analytic Methods for PDEs
Lectures 1-2

Luc Nguyen
luc.nguyen@maths

University of Oxford

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What is this course about?

- We will be concerned with linear elliptic equations of the form

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

  - ⋆ Ω: a domain in $\mathbb{R}^n$,
  - ⋆ $u : \Omega \to \mathbb{R}$ is the unknown,
  - ⋆ $f : \Omega \to \mathbb{R}$ is a given source,
  - ⋆ $a_{ij} : \Omega \to \mathbb{R}$ are given coefficients with $a_{ij} = a_{ji}$.
  - ⋆ repeated indices are summed from 1 to n, i.e.

$$\partial_i(a_{ij}\partial_j u) = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u).$$

- Linearity: L is linear in the sense that $L(\alpha u + v) = \alpha Lu + Lv$.
- Ellipticity: L is elliptic in the sense that the coefficient matrix $(a_{ij})_{i,j=1}^{n}$ is positive definite.
- Boundary condition: ignored at the moment.

What is this course about?

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- We will deal with the functional analytic aspects of $(\dagger)$:
  - ⋆ In what functional space should one look for the solutions u?
  - ⋆ In what functional space should one give the sources f?
  - ⋆ In those spaces, is $(\dagger)$ solvable?
  - ⋆ In those spaces, what other properties of solutions does one have?
- We will NOT be concerned with
  - ⋆ Solving for solutions of $(\dagger)$ in closed form.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Classical solutions:
  - ⋆ $u \in C^2(D)$: u has continuous second derivative in D.
  - ⋆ $f \in C(D)$: f is continuous in D.
  - ⋆ $\Delta : C^2(D) \to C(D)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Issue 1: Non-existence. The Poisson equation $(\star)$ has no classical solutions for some $f \in C(D)$, e.g.

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \frac{5 - 4\log(x^2 + y^2)}{(1 - \log(x^2 + y^2))^{3/2}}.$$

For this function f, all 'reasonable' solutions are of the form

$$u(x,y) = (x^2 - y^2)(1 - \log(x^2 + y^2))^{1/2} + \text{ an analytic function.}$$

These do not have continuous second derivative at $(0,0)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

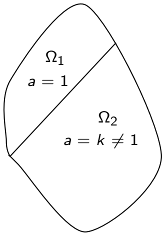
- Issue 2: In some applications, such as heat or electricity conduction on a plate, the source f is not continuous. For example, heat may be supplied only on part of the plate D. In such cases, f is at best piecewise continuous. Naturally the solutions u are no longer in C^2.

Example 2: An equation from material sciences

$$Lu := -\mathrm{div}(a\nabla u) = f \text{ in } \Omega \subset \mathbb{R}^3. \qquad (\star\star)$$

- A composite material occupies a region $\Omega = \Omega_1 \cup \Omega_2$, where each subregion models a different constituent material. The coefficient a thus assumes different values on these subregions, say

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ k \neq 1 & \text{if } x \in \Omega_2. \end{cases}$$



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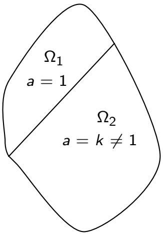

- Issue 1: As a is discontinuous, IF u is smooth, the vector $a\nabla u$ does not have to be continuous and thus the meaning of $\mathrm{div}(a\nabla u)$ is not clear.
- Issue 2: If we instead requires that $a\nabla u$ be continuous, then ∇u may be discontinuous, and so u may not be twice differentiable.

Conclusion

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- There is a need to consider (generalised/weak) solutions which are not twice differentiable.
- There is a need to consider functions whose (generalised/weak) derivatives are discontinuous.
- GOAL: Treat ($\dagger$) in Sobolev spaces $W^{1,p}$, i.e. space of functions which has first derivatives belonging to L^p.
- Agenda: L^p spaces $\rightsquigarrow$ $W^{1,p}$ spaces $\rightsquigarrow$ Treatment of ($\dagger$).

Tentative plan

- Lebesgue spaces (Chapter 1): Lectures 1-4
- Sobolev spaces (Chapter 2): Lectures 5-7.
- Embedding theorems (Chapter 3): Lecture 8-10.
- Linear elliptic equations in divergence form (Chapter 4): Lecture 11-16.

General expectation

- This course warms up rather casually with L^p theory which many of you are familiar with if you took Part A integration or the equivalence elsewhere, but the pace picks up quickly around end of W3 onwards. I'll try to be as inclusive as possible.
- Do read ahead the lecture notes.
- Though most of the materials in the lecture notes will be discussed in lectures, I may decide occasionally to go over certain topics rather briefly and use the lecture time to discuss something else which is not in the lecture notes. Those either complement what's in the lecture notes, or along the line of exam questions, etc.
- It's highly recommended to consult the various texts given in the lecture notes.

Outline for the rest of the lecture

- Definition of Lebesgue spaces $L^p(E)$.
- Hölder's and Minkowski's inequalities.
- Completeness of Lebesgue spaces – Riesz-Fischer's theorem.
- Converse to Hölder's inequality.
- Duals of Lebesgue spaces.
- L^2 as a Hilbert space.
- Density of simple functions for Lebesgue spaces.
- Separability of Lebesgue spaces.

Lebesgue spaces $L^p(E)$ with $1 \le p < \infty$

- E: a measurable subset of $\mathbb{R}^n$,
- $1 \le p < \infty$, define

$$\mathcal{L}^p(E) = \Big\{ f : E \to \mathbb{R} \Big| \; f \text{ is measurable on } E \text{ and } \int_E |f|^p \, dx < \infty \Big\}.$$

- Define $L^p(E)$ as $\mathcal{L}^p(E)/\sim$ where

$$f \sim g \text{ if } f = g \text{ a.e. in } E.$$

Lebesgue spaces $L^\infty(E)$

- E: a measurable subset of $\mathbb{R}^n$,
- For a measurable $f : E \to \mathbb{R}$, define the essential supremum of f on E by

$$\operatorname*{ess\,sup}_E f = \inf\{c > 0 : f \leq c \text{ a.e. in } E\}.$$

  When $\operatorname{ess\,sup}_E |f| < \infty$, we say f is essentially bounded on E.
- $\mathcal{L}^\infty(E)$ is defined as the set of all essentially bounded measurable functions on E.
- $L^\infty(E)$ is defined as $\mathcal{L}^\infty(E)/\sim$.

Some conventions

- Unless otherwise stated, our functions are real-valued.
- When E is clear, we will simply write L^p in place of $L^p(E)$.
- For simplicity, we will frequently refer to elements of $L^p(E)$ as functions rather than equivalent classes of functions. When there is a need to speak of a representative in an equivalent class of functions, we will make it clear.
- We will use $L^p_{loc}(E)$ to refer to the set of functions f such that, for every compact subset K of E, the restriction of f to K belongs to $L^p(K)$.

$L^p(E)$ is a normed vector space for $1 \leq p \leq \infty$

The following results were proven in Integration:

- The space $L^p(E)$ is a vector space.
- If we define

$$\|f\|_{L^p(E)} = \left\{ \int_E |f|^p \, dx \right\}^{1/p} \text{ for } 1 \leq p < \infty,$$

  and

$$\|f\|_{L^\infty(E)} = \operatorname*{ess\,sup}_E |f|,$$

  then $L^p(E)$ is a normed vector space with these norms for $1 \leq p \leq \infty$.

Recap

Recall that $(X, \|\cdot\|)$ is a normed vector space if

- $\star$ X is a vector space
- $\star$ $\|\cdot\|$ maps X into $[0, \infty)$ and satisfies
  - $\triangleright$ $\|x\| = 0$ if and only if $x = 0$.
  - $\triangleright$ $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}, x \in X$.
  - $\triangleright$ $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

$L^p(E)$ is a normed vector space for $1 \le p \le \infty$

The following results were proven in Integration:

- In particular, we have

Theorem (Minkowski's inequality)

If $1 \le p \le \infty$, *then* $\|f + g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)}$.

- The proof of the above uses the following important inequality:

Theorem (Hölder's inequality)

If $1 \le p, p' \le \infty$ *are such that* $\frac{1}{p} + \frac{1}{p'} = 1$, *then*
$\|fg\|_{L^1(E)} \le \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}$.

$L^p(E)$ is a Banach space $1 \leq p \leq \infty$

The following result was touched upon in Integration:

Theorem (Riesz-Fischer's theorem)

If $1 \leq p \leq \infty$, then $L^p(E)$ is a Banach space with norm $\| \cdot \|_{L^p(E)}$.

Recall that a normed vector space is a Banach space if it is complete with respect to its norm, i.e. all Cauchy sequences converge.

Proof of Riesz-Fischer's theorem

- Suppose that (f_k) is a Cauchy sequence in L^p. We need to show that f_k converges in L^p to some $f \in L^p$.
- Case 1: $p = \infty$.
  - ⋆ For every k, m, there exists a set $Z_{k,m}$ of zero measure such that

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z_{k,m}.$$

  - ⋆ Let $Z = \cup_{k,m} Z_{k,m}$. Then Z has zero measure and

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z \text{ for all } k \text{ and } m.$$

  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.

Proof of Riesz-Fischer's theorem

- Case 1: $p = \infty$...
  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.
  - ⋆ Now, for any k, we have

$$|f_k - f| \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z.$$

  - ⋆ As Z has measure zero, this means

$$\|f_k - f\|_{L^\infty} \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty}.$$

  - ⋆ Since $f_k \in L^\infty$, it follows from Minkowski's inequality that $f \in L^\infty$. Also, sending $k \to \infty$ in the above inequality also shows that $\|f_k - f\|_{L^\infty} \to 0$, i.e. f_k converges to f in L^∞.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \le p < \infty$.
  - ⋆ We have

$$|\{x \in E : |f_k(x) - f_m(x)| > \varepsilon\}| \le \frac{1}{\varepsilon^p} \int_E |f_k(x) - f_m(x)|^p$$
$$= \frac{1}{\varepsilon^p} \|f_k(x) - f_m(x)\|_{L^p}^p$$
$$\xrightarrow{k,m\to\infty} 0.$$

    This means that the sequence (f_k) is Cauchy in measure.
  - ⋆ A result from Integration then asserts that (f_k) converges in measure, and hence it has a subsequence, say (f_{k_j}), which converges a.e. in E to some function f. To conclude, we show that $f \in L^p$ and $f_k \to f$ in L^p.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \leq p < \infty$...
  - ⋆ Fix some $\delta > 0$, then, for large k and j,

$$\int_E |f_{k_j} - f_k|^p \, dx = \|f_{k_j} - f_k\|_{L^p}^p \leq \delta^p.$$

  - ⋆ Sending $j \to \infty$ and using Fatou's lemma, we get

$$\int_E |f - f_k|^p \, dx \leq \liminf_{j\to\infty} \int_E |f_{k_j} - f_k|^p \, dx \leq \delta^p.$$

  - ⋆ So we have $\|f - f_k\|_{L^p} \leq \delta$ for large k. By Minkowski's inequality, this implies that $f \in L^p$. As δ is arbitrary, this also gives $f_k \to f$ in L^p, as desired.

Dual space of $L^p(E)$

Recall that for a (real) normed vector space X, the dual of X, denoted as X^*, is the Banach space of bounded linear functional $T : X \to \mathbb{R}$, equipped with the dual norm

$$\|T\|_* = \sup_{\|x\|\leq 1} \|Tx\|.$$

Theorem (Riesz' representation theorem)

Let E be measurable, $1 \leq p < \infty$ and $p' = \frac{p}{p-1}$. Then there is an isometric isomorphism $\pi : (L^p(E))^* \to L^{p'}(E)$ such that

$$Tg = \int_E \pi(T) g \, dx \text{ for all } g \in L^p(E) \text{ and } T \in (L^p(E))^*.$$

Dual space of $L^p(E)$

Theorem (Riesz' representation theorem)

Let E be measurable, $1 \le p < \infty$ and $p' = \frac{p}{p-1}$. Then there is an isometric isomorphism $\pi : (L^p(E))^* \to L^{p'}(E)$ such that

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- Note the similarity of the above and Riesz' representation theorem for Hilbert spaces. In particular, observe the connection when $p = 2$.
- The theorem is false for $p = \infty$. The dual of $L^\infty(E)$ is strictly bigger than $L^1(E)$. In other words, there exists a linear functional T on $L^\infty(E)$ for which there is no $f \in L^1(E)$ satisfying

$$Tg = \int_E fg \, dx \text{ for all } g \in L^\infty(E).$$

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- Recall that for a (real) normed vector space X, the dual of X, denoted as X^*, is the Banach space of bounded linear functional $T : X \to \mathbb{R}$, equipped with the dual norm

$$\|T\|_* = \sup \|Tx\|.$$

- $(L^p(E))^* = L^{p'}(E)$ for $1 \leq p < \infty$.
- Consider $p = \infty$. Let $T_k \in (L^\infty(\mathbb{R}))^*$ given by $T_k g = \frac{1}{k} \int_0^k g\, dx$. Then, for every $g \in L^\infty(\mathbb{R})$, $(T_k g) \in \ell^\infty$.
- Let $L \in (\ell^\infty)^*$ be such that

$$L((x_k)) = \lim_{k\to\infty} x_k \text{ provided } (x_k) \text{ is convergent.}$$

  Such L exists by the Hahn-Banach theorem.
- Define $Tg = L((T_k g))$ for all $g \in L^\infty(\mathbb{R})$. It is easy to check that $T \in (L^\infty(\mathbb{R}))^*$.

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- We claim that there is no $f \in L^1(\mathbb{R})$ such that

$$Tg = \int_{\mathbb{R}} fg\,dx \text{ for all } g \in L^\infty(\mathbb{R}).$$

- Suppose by contradiction that such f exists. Fix some $m > 0$ and let $g_1(x) = sign(f(x))\chi_{(0,m)}(x)$. Then, as $|g_1| \leq \chi_{(0,m)}$, we have for $k > m$ that $|T_k g_1| \leq \frac{m}{k}$. It follows that

$$\int_0^m |f|\,dx = Tg_1 = L((T_k g_1)) = \lim_{k\to\infty} \frac{m}{k} = 0.$$

As m is arbitrary, we thus have $f = 0$ a.e. in $(0, \infty)$.

- On the other hand, with $g_2 = \chi_{(0,\infty)}$, we have $T_k g_2 = 1$ and so

$$0 = \int_0^\infty f\,dx = Tg_2 = L((T_k g_2)) = \lim_{k\to\infty} 1 = 1,$$

which is absurd.

Converse to Hölder's inequality

Proposition (Converse to Hölder's inequality)

Let E be measurable, and f be measurable on E. If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f\|_{L^p(E)} = \sup\Big\{ \int_E fg\,dx : g \in L^{p'}(E), \|g\|_{L^{p'}(E)} \leq 1 \text{ and } fg \text{ is integrable on } E \Big\}.$$

Note: We do not presume that $f \in L^p(E)$.

Proof of Converse to Hölder's inequality

- Will only present the case $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ need some justification; see notes.
- Let

$$\alpha = \sup \left\{ \int_E fg\,dx : \|g\|_{L^{p'}} \leq 1, fg \in L^1(E) \right\} \in [0, \infty].$$

  By Hölder's inequality, we have $\alpha \leq \|f\|_{L^p}$. So it suffices to show $\alpha \geq \|f\|_{L^p}$.
- If $\|f\|_{L^p} = 0$, we are done. Assume henceforth that $\|f\|_{L^p} > 0$.

Proof of Converse to Hölder's inequality

- Case 1: $0 < \|f\|_{L^p} < \infty$.
  In this case, we test the definition of α using

$$g_0(x) = \frac{sign(f(x))|f(x)|^{p-1}}{\|f\|_{L^p}^{p-1}}.$$

  - ⋆ We have, as $p' = \frac{p}{p-1}$,

$$\int_E |g_0|^{p'}\,dx = \frac{1}{\|f\|_{L^p}^p}\int_E |f|^p\,dx = 1.$$

  - ⋆ Next,

$$\int_E |f|\,|g_0|\,dx = \frac{1}{\|f\|_{L^p}^{p-1}}\int_E |f|^p\,dx < \infty.$$

  - ⋆ So by the definition of α,

$$\alpha \geq \int_E f\,g_0\,dx = \frac{1}{\|f\|_{L^p}^{p-1}}\int_E |f|^p\,dx = \|f\|_{L^p}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$.
  In this case, we need to show that $\alpha = \infty$.
  - ⋆ Consider a truncation of $|f|$ given by

$$f_k(x) = \begin{cases} \min(|f|(x), k) & \text{if } x \in E \text{ and } |x| \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

    Note that we are truncating both in the domain and in the range: $f_k(x) = \min(|f|(x), k)\chi_{E \cap \{|x| \leq k\}}(x)$.
  - ⋆ It is clear that $f_k \in L^p(E)$. Also, by Lebesgue's monotone convergence theorem,

$$\|f_k\|_{L^p}^p = \int_E |f_k|^p \, dx \to \int_E |f|^p \, dx = \infty.$$

    In addition, by Case 1,

$$\|f_k\|_{L^p} = \sup\Big\{ \int_E f_k \, g \, dx : \|g\|_{L^{p'}} \leq 1, f_k g \in L^1(E)\Big\}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$...
  - ⋆ In fact, the proof in Case 1 shows that the function $g_k = \frac{|f_k|^{p-1}}{\|f_k\|_{L^p}^{p-1}} \geq 0$ satisfies $\|g_k\|_{L^{p'}} = 1$, $f_k g_k \in L^1(E)$ and
$$\|f_k\|_{L^p} = \int_E f_k \, g_k \, dx.$$
  - ⋆ As $|f| \geq f_k \geq 0$, It follows that, as
$$\int_E |f| g_k \, dx \geq \int_E f_k \, g_k \, dx = \|f_k\|_{L^p} \to \infty.$$
  - ⋆ Letting $\tilde{g}_k(x) = sign(f(x)) g_k(x)$, we then have $\|\tilde{g}_k\|_{L^{p'}} = 1$, $f \tilde{g}_k \in L^1(E)$ and so
$$\alpha \geq \int_E f \tilde{g}_k \, dx = \int_E |f| \, g_k \, dx \to \infty.$$

  So $\alpha = \infty$, as desired.

$L^2(E)$ as a Hilbert space

Theorem

The space $L^2(E)$ is a (real) Hilbert space with inner product

$$\langle f, g\rangle = \int_E fg.$$

This means

- (Banach) $L^2(E)$ is a Banach space.
- (Inner product) The map $(f, g) \mapsto \langle f, g\rangle$ from $L^2(E) \times L^2(E)$ into $\mathbb{R}$ satisfies
    - ⋆ (Linearity) $\langle \lambda f_1 + f_2, g\rangle = \lambda\langle f_1, g\rangle + \langle f_2, g\rangle$ for all $\lambda \in \mathbb{R}, f_1, f_2, g \in L^2(E)$,
    - ⋆ (Symmetry) $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in L^2(E)$,
    - ⋆ (Positivity) $\langle f, f\rangle = \|f\|^2_{L^2(E)}$. Hence $\langle f, f\rangle \geq 0$ for all $f \in L^2(E)$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.

Density results for L^p via simple functions

We will show that the following sets are dense in L^p:

- Set of simple functions, for $1 \leq p \leq \infty$.
- Set of 'rational and dyadic' simple functions, for $1 \leq p < \infty$.

Density results for L^p via simple functions

Simple function:

$$\sum_{i=1}^{N} \alpha_i \chi_{A_i} \text{ where } \alpha_i \text{ is a constant and } A_i \text{ is measurable.}$$

Theorem

Let $1 \leq p \leq \infty$. The set of all p-integrable simple functions is dense in $L^p(E)$.

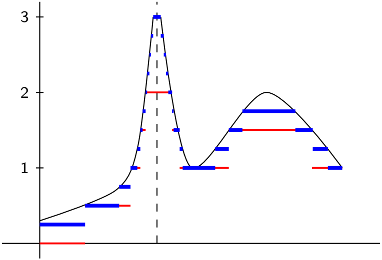
Density results for L^p via simple functions

Proof:

- Take $f \in L^p(E)$. We need to construct a sequence (f_k) of p-integrable simple function such that $\|f_k - f\|_{L^p} \to 0$.
- Using the splitting $f = f^+ - f^-$, we may assume without loss of generality that f is non-negative.
- Fact from Integration: If f is a non-negative measurable function, then there exist non-negative simple functions f_k such that $f_k \nearrow f$ a.e.
  Furthermore, if $p < \infty$, then
  - ⋆ $|f_k|^p \leq |f|^p$ and so $f_k \in L^p$;
  - ⋆ As $|f_k - f|^p \leq |f|^p \in L^1$, and so by Lebesgue dominated convergence theorem, $\int_E |f_k - f|^p \, dx \to 0$. So $f_k \to f$ in L^p.

Density results for L^p via simple functions

- When $p = \infty$, the above proof doesn't work as seen. Let us take the proof one step further by recalling how such a sequence f_k can be constructed.
  - ⋆ For each k, one partition the range $[0, \infty]$ into $2^{2k} + 1$ intervals: $J_1^{(k)} = [0, 2^{-k})$, $J_2^{(k)} = [2^{-k}, 2 \times 2^{-k})$, ..., $J_{2^{2k}}^{(k)} = [(2^{2k} - 1) \times 2^{-k}, 2^{2k} \times 2^{-k})$ and $J_{2^{2k}+1}^{(k)} = [2^k, \infty]$.
  - ⋆ f_k is then defined by $f_k(x) = (\ell - 1) \times 2^{-k}$ if $\{f(x) \in J_\ell^{(k)}\}$ for $1 \leq \ell \leq 2^{2k} + 1$.



Density results for L^p via simple functions

- When $p = \infty$...
  - ⋆ Aside from the fact that $f_k \nearrow f$, this construction has the property that, in the set $\{f(x) < 2^k\}$, i.e. outside of the set $\{f(x) \in J^{(k)}_{2^{2k}+1}\}$, it holds that

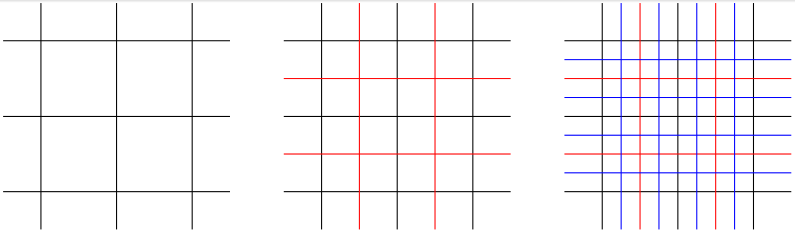
$$|f_k - f| \leq 2^{-k}.$$

  - ⋆ Now as $p = \infty$, f is essentially bounded, i.e. there is an M and a set Z of zero measure such that $f < M$ in $\mathbb{R}^n \setminus Z$. We then redefine f on Z to be zero, i.e. we work with the representative in the 'equivalent class f' which is bounded everywhere by M.
  - ⋆ After this redefinition, we see that $\{f(x) \in J^{(k)}_{2^{2k}+1}\} = \emptyset$ for large k, and so we have $|f_k - f| \leq 2^{-k}$ everywhere for all large k. This means that $f_k \to f$ in L^∞.

Density results for L^p via simple functions

Theorem

Let $1 \leq p < \infty$. The set $\mathscr{F}$ of all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes is dense in $L^p(\mathbb{R}^n)$.



$\mathscr{C}_1$ $\mathscr{C}_2$ $\mathscr{C}_3$

$$\mathscr{F} = \Big\{ g = \sum_{i=1}^{N} r_i \chi_{Q_i} \text{ where } r_i \in \mathbb{Q}, Q_i \in \bigcup_{j=1}^{\infty} \mathscr{C}_j \Big\}.$$

Density results for L^p via simple functions

Proof:

- We know that the set of p-integrable simple functions is dense in L^p. We also know that $\mathbb{Q}$ is dense in $\mathbb{R}$.
- Thus we only need to show that $\chi_E \in \overline{\mathscr{F}}$.
- By the construction of the Lebesgue measure, every open subset U of $\mathbb{R}^n$ can be written as a countable union of cubes in $\cup \mathscr{C}_i$, say $U = \cup_{i=1}^\infty Q_i$. Then

$$\sum_{i=1}^N \chi_{Q_i} \to \chi_U \text{ in } L^p, \text{ and so } \chi_U \in \overline{\mathscr{F}}.$$

- Now, for every measurable set E of finite measure, the outer regularity of the Lebesgue measure implies that there exist open U_k, $U_k \supset E$ such that $|U_k \setminus E| \to 0$. Then

$$\chi_{U_k} \to \chi_E \text{ in } L^p, \text{ and so } \chi_E \in \overline{\mathscr{F}}.$$

Application: Separability of L^p

Theorem

For $1 \leq p < \infty$, the space $L^p(E)$ is separable, i.e. it has a countable dense subset.

Proof:

- When $E = \mathbb{R}^n$, the result follows from the previous theorem, as $\mathscr{F}$ is countable.
- For general E, let $\tilde{\mathscr{F}}$ be the set of restrictions to E of functions in $\mathscr{F}$. Then $\tilde{\mathscr{F}}$ is countable. We will now show that $\tilde{\mathscr{F}}$ is dense in $L^p(E)$.
    - ⋆ Take $f \in L^p(E)$. Set $f = 0$ in $\mathbb{R}^n \setminus E$. Then $f \in L^p(\mathbb{R}^n)$ and so there exist $f_k \in \mathscr{F}$ such that $f_k \to f$ in $L^p(\mathbb{R}^n)$.
    - ⋆ Let $\tilde{f}_k = f_k|_E \in \tilde{\mathscr{F}}$. Then $\|\tilde{f}_k - f\|_{L^p(E)} \leq \|f_k - f\|_{L^p(E)} \to 0$, so we are done.