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(P1)

§2.4 (supplementary notes)

Example

$$\begin{cases} y'' = f(x) & \text{for } 0 < x < 1 \\ y(0) = \alpha, & y(1) = \beta \end{cases}$$

OPTION 1 : DECOMPOSITION

Seek $u_1(x), u_2(x)$ s.t.

$$\begin{cases} u_1'' = f(x), & u_1(0) = 0 = u_1(1) \\ u_2'' = 0, & u_2(0) = \alpha, \; u_2(1) = \beta \end{cases}$$

so that $\begin{cases} u_1 \text{ solves the homog. BVP} \\ u_2 \text{ is a particular integral} \end{cases}$

it is straightforward to show that

$$u_1(x) = \sum \tilde{c}_k y_k(x) \quad \text{where} \quad \tilde{c}_k = \frac{-\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle}$$

$$= -\frac{2}{k^2\pi^2} \int_0^1 f(x) \sin(k\pi x)\, dx$$

$$u_2(x) = \alpha + (\beta - \alpha) x.$$

$\Rightarrow$ full solution: $y(x) = \sum_k \tilde{c}_k y_k(x) + \underbrace{\alpha + (\beta-\alpha)x}_{u_2(x)}$

OPTION 2 : NO DECOMPOSITION

$$y(x) = \sum c_k y_k(x)$$

where (after some algebra)

$$c_k = \underbrace{-\frac{2}{k^2\pi^2} \int_0^1 f(x)\sin(k\pi x)\, dx}_{\tilde{c}_k} + \frac{2}{k\pi}\left(\alpha - (-1)^k \beta\right)$$

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(P2)

CLAIM : $u_2(x) = \alpha + (\beta-\alpha)x = \sum_k \frac{2}{k\pi}\left(\alpha - (-1)^k\beta\right) \underbrace{\sin(k\pi x)}_{y_k(x) = \omega_k(x)}$

Proof Let :

$$\alpha + (\beta-\alpha)x = \sum A_k \sin(k\pi x)$$

$$\Rightarrow A_k \underbrace{\int_0^1 \sin^2(k\pi x)\,dx}_{=1/2} = \int_0^1 \left\{\alpha + (\beta-\alpha)x\right\}\sin(k\pi x)\,dx$$

$$= \frac{\alpha}{k\pi}\left[-\cos k\pi x\right]_0^1 + \frac{(\beta-\alpha)}{k\pi}\left[-x\cos k\pi x\right]_0^1$$

$$+ \frac{(\beta-\alpha)}{(k\pi)^2}\left[\sin k\pi x\right]_0^1$$

$$\Rightarrow \frac{A_k}{2} = \frac{\alpha}{k\pi}\left(1-(-1)^k\right) - \frac{(\beta-\alpha)}{k\pi}(-1)^k.$$

$$\Rightarrow \boxed{A_k = \frac{2}{k\pi}\left(\alpha - (-1)^k\beta\right)}$$, as claimed.

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