C4.3 Functional Analytic Methods for PDEs
Lectures 3-4

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In the last lecture

- Lebesgue spaces.
- Duals of Lebesgue spaces.
- L^2 as a Hilbert space.
- Density of simple functions for Lebesgue spaces.

This lecture

- Weak and weak* convergence in Lebesgue spaces.
- Continuity property of translation operators in L^p.
- Convolution. Young's inequality.
- Differentiation rule for convolution.
- Approximation of identity in Lebesgue spaces.
- Density by smooth functions.

Weak and weak* convergence in L^p

Definition

Let X be a normed vector space and X^* its dual.

(i) We say that a sequence (x_n) in X converges weakly to some $x \in X$ if $Tx_n \to Tx$ for all $T \in X^*$. We write $x_n \rightharpoonup x$.

(ii) We say that a sequence (T_n) in X^* converges weakly* to some $T \in X^*$ if $T_n x \to Tx$ for all $x \in X$. We write $T_n \overset{*}{\rightharpoonup} T$.

Weak sequential compactness

Theorem (Weak sequential compactness in reflexive Banach spaces)

Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

Corollary

Assume that $1 < p < \infty$ and (f_k) is bounded in $L^p(E)$. Then there is a subsequence f_{k_j} which converges weakly in L^p. In other words, there exists a function $f \in L^p$ such that

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^{p'}(E).$$

Weak* sequential compactness

Theorem (Helly's theorem on weak* sequential compactness in duals of separable Banach spaces)

Every bounded sequence in the dual of a separable Banach space has a weakly convergent subsequence.*

Corollary

Assume that (f_k) is bounded in $L^\infty(E)$. Then there is a subsequence f_{k_j} which converges weakly in L^∞. In other words, there exists a function $f \in L^\infty$ such that*

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^1(E).$$

A summary

	Dual	Reflexivity	Separability	Sequential compactness of $\overline{B(0,1)}$
L^p $1 < p < \infty$	$L^{p'}$	Yes	Yes	Weak and weak*
L^1	L^∞	No	Yes	Neither
L^∞	$\supsetneq L^1$	No	No	Weak*

Continuity of translation operators

Translation operators: For a $h \in \mathbb{R}^n$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define $\tau_h f$ by

$$(\tau_h f)(x) = f(x + h) \text{ for all } x \in \mathbb{R}^n.$$

Then $\tau_h : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear transformation for $1 \leq p \leq \infty$. In fact it is an isometric isomorphism.

Theorem (Continuity in L^p)

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{|h| \to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Continuity of translation operators

- In other words, for $1 \le p < \infty$, for every fixed $f \in L^p(\mathbb{R}^n)$, the map $h \mapsto \tau_h f$ is a continuous map from $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$.
- The theorem is false for $p = \infty$, e.g. with $f = \chi_Q$ with Q being the unit cube.
- The theorem does ***NOT*** assert that the maps $h \mapsto \tau_h$ is a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$. In fact,

$$\|\tau_h - Id\|_{\mathscr{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \ge 2^{1/p} \text{ when } h \neq 0.$$

  - ⋆ Let $r = |h|/4$ and let $f = c_n r^{-n/p} \chi_{B_r(0)}$ where c_n is chosen such that $\|f\|_{L^p} = 1$.
  - ⋆ Then $\tau_h f$ and f has disjoint support. So

$$\|\tau_h f - f\|_{L^p} = \left\{ \|\tau_h f\|_{L^p}^p + \|f\|_{L^p}^p \right\}^{1/p} = 2^{1/p}.$$

Continuity of translation operators

Proof:

- Let $\mathscr{A}$ denote the set of functions f in L^p such that $\|\tau_h f - f\|_{L^p} \to 0$ as $|h| \to 0$.
- It is clear that if $f, g \in \mathscr{A}$ then $f + g \in \mathscr{A}$, and $\lambda f \in \mathscr{A}$ for any $\lambda \in \mathbb{R}$. So $\mathscr{A}$ is a vector subspace of L^p.
- We claim that $\mathscr{A}$ is closed in L^p, i.e. if $(f_k) \subset \mathscr{A}$ and $f_k \to f$ in L^p, then $f \in \mathscr{A}$. Indeed, by Minkowski's inequality, we have

$$\begin{aligned}\|\tau_h f - f\|_{L^p} &\le \|\tau_h f_k - f_k\|_{L^p} + \|\tau_h f_k - \tau_h f\|_{L^p} + \|f_k - f\|_{L^p}\\ &= \|\tau_h f_k - f_k\|_{L^p} + 2\|f_k - f\|_{L^p}.\end{aligned}$$

  Now, if one is given an $\varepsilon > 0$, one can first select large k such that $\|f_k - f\|_{L^p} \le \varepsilon/3$, and then select $\delta > 0$ such that $\|\tau_h f_k - f_k\|_{L^p} \le \varepsilon/3$ for all $|h| \le \delta$, so that

$$\|\tau_h f - f\|_{L^p} \le \varepsilon \text{ for all } |h| \le \delta.$$

Continuity of translation operators

- So $\mathscr{A}$ is a closed vector subspace of L^p.
- Now, observe that if Q is a cube in $\mathbb{R}^n$, then $\|\tau_h \chi_Q - \chi_Q\|_{L^p} \to 0$ as $|h| \to 0$, by e.g. Lebesgue's dominated convergence theorem (or a direct estimate).
- So $\mathscr{A}$ contains all finite linear combinations of characteristic functions of cubes. In particular, it contains all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes. As this latter set is dense in L^p and $\mathscr{A}$ is closed, we thus have $\mathscr{A} = L^p$, as desired.

Convolution

Definition

Let f and g be measurable functions on $\mathbb{R}^n$. The convolution $f * g$ of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\,dy$$

wherever the integral converges.

Young's convolution inequality

Theorem (Young's convolution inequality)

Let p, q and r satisfy $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Young's convolution inequality

Proof: We will only deal with the case $q = 1$ and $r = p$. We are thus given $f \in L^p, g \in L^1$. We need to show that $f * g \in L^p$ and $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.

- Observe that $|f * g| \leq |f| * |g|$. We may thus assume without loss of generality in the proof that $f, g \geq 0$.
- Case 1: $p = 1$.
  - ⋆ Consider the integral

$$I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y) g(x - y)\, dx\, dy.$$

This integral is well-defined as $f, g \geq 0$ and the function $G(x, y) = g(x - y)$ is measurable as a function from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$.

Young's convolution inequality

- Case 1: $p = 1$.
  - ⋆ Consider $I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y) g(x-y)\,dx\,dy$.
  - ⋆ By Tonelli's theorem, we have

$$I = \int_{\mathbb{R}^n} \Big\{ \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy \Big\}\,dx = \int_{\mathbb{R}^n} (f*g)(x)\,dx$$
$$= \|f*g\|_{L^1}.$$
$$I = \int_{\mathbb{R}^n} f(y) \Big\{ \int_{\mathbb{R}^n} g(x-y)\,dx \Big\}\,dy = \int_{\mathbb{R}^n} f(y) \|g\|_{L^1}\,dy$$
$$= \|f\|_{L^1} \|g\|_{L^1}.$$

  - ⋆ So $\|f*g\|_{L^1} = \|f\|_{L^1}\|g\|_{L^1}$.

Young's convolution inequality

- Case 2: $p = \infty$. This case is easy, as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$
$$\leq \int_{\mathbb{R}^n} \|f\|_{L^\infty}\, g(x-y)\, dy = \|f\|_{L^\infty}\, \|g\|_{L^1}.$$

- Case 3: $1 < p < \infty$.
  - ⋆ We start by writing

$$|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y)g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\, dy$$

and applying Hölder's inequality to the above.

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - ⋆ $|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y) g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\, dy.$
  - ⋆ So

$$|(f * g)(x)| \leq \Big\{ \int_{\mathbb{R}^n} f(y)^p g(x-y)]\, dy \Big\}^{1/p} \Big\{ \int_{\mathbb{R}^n} g(x-y)\, dy \Big\}^{1/p'}$$
$$= [(f^p * g)(x)]^{1/p} \|g\|_{L^1}^{1/p'}.$$

  - ⋆ It follows that

$$\|f * g\|_{L^p} = \Big\{ \int_{\mathbb{R}^n} |(f * g)(x)|^p\, dx \Big\}^{1/p}$$
$$\leq \Big\{ \int_{\mathbb{R}^n} (f^p * g)(x)\, dx \Big\}^{1/p} \|g\|_{L^1}^{1/p'}$$
$$= \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'}$$

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - ⋆ $\|f * g\|_{L^p} \leq \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'}$.
  - ⋆ So by Case 1,

$$\begin{aligned} \|f * g\|_{L^p} &\leq \Big[\|f^p\|_{L^1}\|g\|_{L^1}\Big]^{1/p} \|g\|_{L^1}^{1/p'} \\ &= \|f\|_{L^p}\|g\|_{L^1}. \end{aligned}$$

Some notations

- If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
- If f is a function and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, we write $\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f$.
- For $k \geq 0$, $C^k(\mathbb{R}^n) = \Big\{$continuous $f : \mathbb{R}^n \to \mathbb{R}$ such that $\partial^\alpha f$ exists and is continuous whenever $|\alpha| \leq k\Big\}$.
- $C_c^k(\mathbb{R}^n) = \Big\{f \in C^k(\mathbb{R}^n)$ which has compact support$\Big\}$. Recall that, for a continuous function f,

$$Supp(f) = \text{Support of } f = \overline{\{f(x) \neq 0\}}.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in C_c^0(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.

Proof:

- Fix some $x \in \mathbb{R}^n$. We need to show that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$.
- We compute

$$
\begin{aligned}
&f * g(x+z) - f * g(x) \\
&\quad = \int_{\mathbb{R}^n} f(y) g(x+z-y)\,dy - \int_{\mathbb{R}^n} f(y) g(x-y)\,dy \\
&\quad = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\,dy.
\end{aligned}
$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- $f * g(x+z) - f * g(x) = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\, dy$.
- Since $g \in C_c^0(\mathbb{R}^n)$, $g \equiv 0$ outside of some big ball B_R centered at 0. Then, for $|z| < R$,

$$f * g(x+z) - f * g(x) = \int_{|x-y| \leq 2R} f(y)[g(x+z-y) - g(x-y)]\, dy.$$

- Note that as g is continuous, it is uniformly continuous on $\bar{B}_{3R}$. Thus, for any given $\varepsilon > 0$, there exists small $\delta \in (0, R)$ such that

$$|g(x+z-y) - g(x-y)| \leq \varepsilon$$

whenever $|z| \leq \delta$ and $|x-y| \leq 2R$.

- So when $|z| \leq \delta$, we have

$$|f * g(x+z) - f * g(x)| \leq \varepsilon \int_{|x-y| \leq 2R} |f(y)|\, dy.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- So when $|z| \leq \delta$, we have

$$\begin{aligned} |f * g(x+z) - f * g(x)| &\leq \varepsilon \|f\|_{L^1(\{|x-y|\leq 2R\})} \\ &\leq \varepsilon \|f\|_{L^p(\mathbb{R}^n)} \|1\|_{L^{p'}(\{|x-y|\leq 2R\})} \\ &= C_n R^{n/p'} \|f\|_{L^p} \varepsilon. \end{aligned}$$

- Since the right side can be made arbitrarily small, this precisely means that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$, i.e. $f * g$ is continuous.

Differentiation rule for convolution

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in C_c^k(\mathbb{R}^n)$ for some $k \ge 1$, then $f * g \in C^k(\mathbb{R}^n)$ and

$$D^\alpha(f * g)(x) = (f * D^\alpha g)(x) \text{ for all multi-index } \alpha \text{ with } |\alpha| \le k.$$

Proof

- We will only consider the case $k = 1$. The general case can be proved by applying the case $k = 1$ repeatedly.
- Suppose that $g \in C_c^1(\mathbb{R}^n)$. Fix a point x and consider $\partial_{x_1}(f * g)(x)$. We need to show that

$$\lim_{t \to 0} \underbrace{\frac{(f * g)(x + te_1) - f * g(x)}{t}}_{=: D.Q.(x,t)} = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- We have

$$D.Q.(x,t) = \int_{\mathbb{R}^n} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\, dy.$$

As $t \to 0$, the integrand converges to $f(y)\partial_{x_1} g(x-y)$. We would like to show that the above integral converges to

$$\int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x-y)\, dy = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- As before, if the support of g is contained in B_R, then, for $|t| < R$,

$$D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\,dy.$$

- When $|x-y| \leq 2R$ and $|t| < R$, we have $|x-y+te_1| \leq 3R$. Hence

$$\frac{|g(x-y+te_1) - g(x-y)|}{|t|} \leq \max_{\bar{B}_{3R}} |\partial_{x_1} g| =: M.$$

So the integrand above satisfies

$$|\text{integrand}| \leq M|f(y)|.$$

Differentiation rule for convolution

Proof

- So we have, for $|t| \leq R$,

$$D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\,dy$$

  where

  - ⋆ integrand $\to f(y)\partial_{x_1} g(x-y)$ as $t \to 0$.
  - ⋆ $|\text{integrand}| \leq M|f(y)|$, which belongs to $L^1(\{|x-y| \leq 2R\})$, as $f \in L^p(\mathbb{R}^n)$.

- By Lebesgue's dominated convergence theorem, we thus have

$$\lim_{t\to 0} D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y)\partial_{x_1} g(x-y)\,dy$$
$$= \int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x-y)\,dy = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- We conclude that $\partial_{x_1}(f * g)$ exists and is equal to $f * \partial_{x_1} g$.
- By the previous lemma, we have that $f * \partial_{x_1} g$ is continuous. So $\partial_{x_1}(f * g)$ is continuous. Applying this to all partial derivatives, we conclude that $f * g \in C^1(\mathbb{R}^n)$.

Approximation of identity

- A family of "kernels" $\{\varrho_\varepsilon : \mathbb{R}^n \to \mathbb{R}\}_{\varepsilon>0}$ is called an approximation of identity if

$$f * \varrho_\varepsilon \text{``} \to \text{''} f \text{ as } \varepsilon \to 0,$$

  where the meaning of the convergence depends on the context.
- Loosely speaking, it means that the operators T_ε defined by $T_\varepsilon f = f * \varrho_\varepsilon$ "approximates" the identity operator.

Approximation of identity in continuous settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in C(\mathbb{R}^n)$, then $f * \varrho_\varepsilon$ converges uniformly on compact subsets of $\mathbb{R}^n$ to f.

More on terminologies:

- A family (ϱ_ε) as in the statement is called a family of 'mollifiers'.
- The family $(f * \varrho_\varepsilon)$ is called a regularization of f by mollification. Note that since $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have that $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Approximation of identity in continuous settings

Proof:

- Let us first consider pointwise convergence, i.e. for every x there holds:

$$(f * \varrho_\varepsilon)(x) = \int_{\mathbb{R}^n} f(y)\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon \to 0} f(x).$$

- The idea is to convert $f(x)$ into an integral as well. For this we use the identity

$$\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} \varrho_\varepsilon(z)\,dz = \int_{\mathbb{R}^n} \varrho(w)\,dw = 1.$$

  Hence

$$f(x) = \int_{\mathbb{R}^n} f(x)\varrho_\varepsilon(x-y)\,dy.$$

Approximation of identity in continuous settings

Proof:

- So we need to show

$$\int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon\to 0} 0.$$

- By hypotheses, ϱ vanishes outside of some ball B_R centered at the origin. So $\varrho_\varepsilon(x-y) = 0$ when $|x-y| \geq \varepsilon R$. It follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy\Big| \\
&\quad \leq \sup_{\{y:|x-y|\leq \varepsilon R\}} |f(x)-f(y)| \int_{|x-y|\leq\varepsilon R} \varrho_\varepsilon(x-y)\,dy \\
&\quad = \sup_{\{y:|x-y|\leq \varepsilon R\}} |f(x)-f(y)| \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

Approximation of identity in continuous settings

Proof:

- Now we turn to prove the uniform convergence on compact sets, i.e. for every given compact set K, we need to show

$$\sup_{x\in K}\Big|(f * \varrho_\varepsilon)(x) - f(x)\Big| \xrightarrow{\varepsilon\to 0} 0.$$

  As before, this is equivalent to

$$\sup_{x\in K}\Big|\int_{\mathbb{R}^n}[f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy\Big| \xrightarrow{\varepsilon\to 0} 0,$$

  which can be turned into

$$\sup_{x\in K}\Big|\int_{\{y:|x-y|\le\varepsilon R\}}[f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy\Big| \xrightarrow{\varepsilon\to 0} 0,$$

Approximation of identity in continuous settings

Proof:

- We need to show

$$A_\varepsilon := \sup_{x \in K} \left| \int_{\{y : |x-y| \leq \varepsilon R\}} [f(x) - f(y)] \varrho_\varepsilon(x-y)\, dy \right| \xrightarrow{\varepsilon \to 0} 0,$$

- In the same way as before, we have

$$A_\varepsilon \leq \sup_{x \in K} \sup_{\{y : |x-y| \leq \varepsilon R\}} |f(x) - f(y)|.$$

- Note that if $K \subset B_{R'}$, $\varepsilon \leq 1$, $x \in K$ and $|x-y| \leq \varepsilon R$, then
  - ⋆ $|x| \leq R' \leq R + R'$,
  - ⋆ $|y| \leq |x| + |y - x| \leq R + R'$.

  So

$$A_\varepsilon \leq \sup_{\{|x|, |y| \leq R+R', |x-y| \leq \varepsilon R\}} |f(x) - f(y)| \xrightarrow{\varepsilon \to 0} 0,$$

  in view of the uniform continuity of f on $\overline{B_{R+R'}}$.

Approximation of identity in Lipschitz settings

Theorem (Approximation of identity)

Let ϱ *be a non-negative function in* $C_c^\infty(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varrho = 1$*. For* $\varepsilon > 0$*, let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in C^{0,1}(\mathbb{R}^n)$*, i.e. there exists* $L \geq 0$ *such that*

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n,$$

then, for some constant $C > 0$ *depending only on the choice of* ϱ*,*

$$\sup_{x \in \mathbb{R}^n} |f * \varrho_\varepsilon(x) - f(x)| \leq CL\varepsilon.$$

Approximation of identity in Lipschitz settings

Proof: Following the same argument as before, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \left|(f * \varrho_\varepsilon)(x) - f(x)\right| &= \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy\right| \\
&\leq \sup_{x\in\mathbb{R}^n} \sup_{\{y:|x-y|\leq\varepsilon R\}} |f(x) - f(y)| \\
&\leq \sup_{x\in\mathbb{R}^n} \sup_{\{y:|x-y|\leq\varepsilon R\}} L|x-y| \\
&\leq L\varepsilon R.
\end{aligned}$$

Approximation of identity in L^p settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

$f * \varrho_\varepsilon \not\to f$ in L^∞

Remark

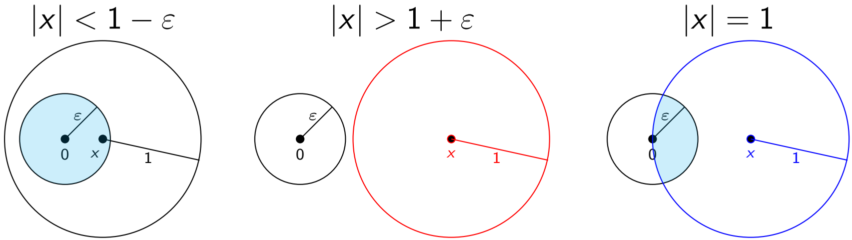
There exist $f \in L^\infty(\mathbb{R}^n)$ and $\varrho \in C_c^\infty(B_1(0))$ such that $f * \varrho_\varepsilon$ does not converge to f in L^∞.

- Take $f = \chi_{B_1(0)}$.
- Then

$$f * \varrho_\varepsilon(x) = \int_{B_1(0)} \varrho_\varepsilon(x-y)\,dy$$
$$= \int_{B_1(x)} \varrho_\varepsilon(z)\,dz$$
$$= \int_{B_1(x)\cap B_\varepsilon(0)} \varrho_\varepsilon(z)\,dz.$$

$f * \varrho_\varepsilon \not\to f$ in L^∞

- $f * \varrho_\varepsilon(x) = \int_{B_1(x)\cap B_\varepsilon(0)} \varrho_\varepsilon(z)\,dz$.



$f * \varrho_\varepsilon(x) = 1$

$f * \varrho_\varepsilon(x) = 0$

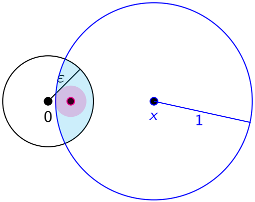
$f * \varrho_\varepsilon(x) \in [0, 1]$
$\to \frac{1}{2}$ in symmetry,
i.e. $\varrho = \varrho(|x|)$

$f * \varrho_\varepsilon \not\to f$ in L^∞

- We now take some ϱ of the form $\varrho(x) = \varrho(|x|)$ such that, in addition to the condition $\|\varrho\|_{L^1} = 1$, we have

$$\int_{B_{1/4}(p)} \varrho(z)\,dz = c_0 \in (0,1) \text{ for all } |p| = 1/2.$$

- Consider $1 < |x| < 1 + \varepsilon/4$.



  - $\star$ $B_1(x) \cap B_\varepsilon(0)$ contains a ball $B_{\varepsilon/4}(p_\varepsilon)$ with $|p_\varepsilon| = \varepsilon/2$.
  - $\star$ So $f * \varrho_\varepsilon(x) \geq \int_{B_{\varepsilon/4}(p_\varepsilon)} \varrho_\varepsilon(z)\,dz = c_0 \in (0,1)$.
  - $\star$ As $f(x) = 0$ here, we thus have

$$\|f * \varrho_\varepsilon - f\|_{L^\infty} \geq c_0 \not\to 0.$$

Approximation of identity in L^p settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

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Proof

- Let $f_\varepsilon(x) := f * \varrho_\varepsilon(x)$. Then

$$f_\varepsilon(x) := f * \varrho_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\varrho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)\varrho_\varepsilon(y)\,dy.$$

- Recall that, as $\int_{\mathbb{R}^n} \varrho_\varepsilon = 1$, we have

$$f(x) = \int_{\mathbb{R}^n} f(x)\varrho_\varepsilon(y)\,dy.$$

- Hence

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|dy\\ &= \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|^{\frac{1}{p}}|\varrho_\varepsilon(y)|^{\frac{1}{p'}}\,dy.\end{aligned}$$

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Proof

- $|f_\varepsilon(x) - f(x)| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|^{\frac{1}{p}}|\varrho_\varepsilon(y)|^{\frac{1}{p'}}\,dy.$
- Applying Hölder's inequality, the above is less than or equal to

$$\leq \Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p}}\Big\{\int_{\mathbb{R}^n} |\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p'}}$$
$$= \Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p}}.$$

- Integrating and using Tonelli's theorem,

$$\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\,dx$$
$$= \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)|\Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p\,dx\Big\}dy.$$

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Proof

- $\|f_\varepsilon - f\|_{L^p}^p \le \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \Big\{ \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \, dx \Big\} dy.$
- In other words,

$$\|f_\varepsilon - f\|_{L^p}^p \le \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy.$$

- If we had that $Supp(\varrho) \subset B_R$, then $Supp(\varrho_\varepsilon) \subset B_{\varepsilon R}$, and so

$$\begin{aligned} \|f_\varepsilon - f\|_{L^p}^p &\le \sup_{|y| \le \varepsilon R} \|\tau_{-y} f - f\|_{L^p}^p \int_{B_{\varepsilon R}} |\varrho_\varepsilon(y)| dy \\ &= \sup_{|y| \le \varepsilon R} \|\tau_{-y} f - f\|_{L^p}^p \xrightarrow{\varepsilon \to 0} 0, \end{aligned}$$

in view of the theorem on the continuity of the translation operator in L^p.

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Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- In the general case where ϱ may or may not have compact support, we argue as follows: For every fixed $\hat{R} > 0$,

$$\begin{aligned}
&\int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \\
&\quad \leq \sup_{|y| \leq \varepsilon \hat{R}} \|\tau_{-y} f - f\|_{L^p}^p \int_{B_{\varepsilon \hat{R}}} |\varrho_\varepsilon(y)| dy \\
&\quad \leq \sup_{|y| \leq \varepsilon \hat{R}} \|\tau_{-y} f - f\|_{L^p}^p \xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

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Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- $\forall\ \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- On the other hand,

$$\begin{aligned}
&\int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \\
&\quad \leq \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| (\|\tau_{-y} f\|_{L^p} + \|f\|_{L^p})^p dy \\
&\quad = 2^p \|f\|_{L^p}^p \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| dy = 2^p \|f\|_{L^p}^p \int_{|z| \geq \hat{R}} |\varrho(z)| dz.
\end{aligned}$$

- As $\varrho \in L^1(\mathbb{R}^n)$, we have by Lebesgue's dominated convergence theorem that

$$\lim_{\hat{R} \to \infty} \int_{|z| \geq \hat{R}} |\varrho(z)| dz = 0.$$

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Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- $\forall\ \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- $\forall\ \varepsilon$, $\lim_{\hat{R} \to \infty} \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- We are ready to wrap up the proof: Fix some $\eta > 0$ and select some large $\hat{R}$ so that

$$\int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta/2.$$

- Then we select small ε_0 such that, for all $\varepsilon < \varepsilon_0$,

$$\int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta/2.$$

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Proof

- For $\varepsilon < \varepsilon_0$,

$$\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta.$$

- As η is arbitrary, we conclude that $\|f_\varepsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$.

$\overline{C_c^\infty(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$

Theorem

For $1 \leq p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Proof

- Fix $f \in L^p(\mathbb{R}^n)$. We need to produce $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in L^p.
- If f has compact support, say $Supp(f) \subset B_R$, this follows from the previous theorem:
  - ⋆ Take a smooth non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and define the mollifiers $\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho(x/\varepsilon)$.
  - ⋆ Then $f_k := f * \varrho_{1/k} \in C^\infty$ and $f_k \to f$ in L^p.
  - ⋆ Recall that $f_k(x) = \int_{\mathbb{R}^n} f(y)\varrho_{1/k}(x-y)\,dy$, and observe that for $|x| > R + 1/k$, then, by triangle inequality, $|y| > R$ or $|x - y| > 1/k$. So $f(y)\varrho_{1/k}(x-y) \equiv 0$ for those x. So $Supp(f_k) \subset B_{R+1/k}$, and $f_k \in C_c^\infty(\mathbb{R}^n)$.

$\overline{C_c^\infty(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$

Proof

- In general, we produce $f_k \in C_c^\infty(\mathbb{R}^n)$ with $\|f_k - f\|_{L^p} \leq 1/k$ as follows:
  - ⋆ Note that, as $R \to \infty$, $f\chi_{B_R} \to f$ in L^p by Lebesgue's dominated convergence theorem.
  - ⋆ So we can pick large R_k such that $\|f\chi_{B_{R_k}} - f\|_{L^p} \leq \frac{1}{2k}$.
  - ⋆ Now $f\chi_{B_{R_k}}$ has compact support, so by the previous consideration, there exists a function $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $\|f\chi_{B_{R_k}} - f_k\|_{L^p} \leq \frac{1}{2k}$.
  - ⋆ Then $\|f_k - f\|_{L^p} \leq 1/k$ and so $f_k \to f$ in L^p as wanted.

$\overline{C^\infty(E) \cap L^p(E)} = L^p(E)$

Let $C^\infty(E) = \left\{ f|_E : f \in C^\infty(\mathbb{R}^n) \right\}$.

Theorem

For $1 \le p < \infty$, the space $C^\infty(E) \cap L^p(E)$ is dense in $L^p(E)$.

Proof

- Fix $f \in L^p(E)$. We will produce functions $f_k \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $f_k|_E \to f$ in $L^p(E)$.
- Extend f to $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $\tilde{f} = 0$ in $\mathbb{R}^n \setminus E$. Then $\tilde{f} \in L^p(\mathbb{R}^n)$. By the previous theorem, there exist $f_k \in C_c^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $f_k \to \tilde{f}$ in $L^p(\mathbb{R}^n)$.
- Now,
$$\int_E |f_k - f|^p \, dx \le \int_{\mathbb{R}^n} |f_k - \tilde{f}|^p \, dx \to 0,$$
and so $f_k|_E \to f$ in $L^p(E)$.