I describe here in some more detail (and in a hopefully clearer way) an example we discussed in class (at the end of lecture 6), but that is not in the lecture notes.

Definition 1 *A function* $f : \mathbf{C} \to \mathbf{C}$ *is said to be* holomorphic *at* $z = z_0 \in \mathbf{C}$ *if the following limit exists:*

$$\lim \frac{f(z) - f(z_0)}{z - z_0}$$

as z *tends to* z_0 *in the complex plane* $\mathbf{C}$.

This limit is called the complex derivative *of* f *at* $z = z_0$ *and denoted by* $f'(z_0)$.

For example if $f(z) = z^n$ then $f'(z) = nz^{n-1}$.

How does this notion relate to that of partial derivatives?

To make sense of this question observe first that the function f can be seen as a complex valued function of 2 real variables by setting:

$$g(x, y) := f(x + iy)$$

Conversely, given a complex valued function $g(x, y)$ of 2 real variables x, y we can set $f(z) = g(\Re(z), \Im(z))$.

So the natural question is to understand what are the conditions on a complex valued function $g(x, y)$ for the associated function $f(z) = g(\Re(z), \Im(z))$ to be holomorphic.

The answer is as follows:

Proposition 1 *The function* $f(z)$ *is holomorphic at* $z = z_0$ *if and only if the partial derivatives of* g *satisfy the equation:*

$$\frac{\partial g}{\partial x} = -i \frac{\partial g}{\partial y} \tag{1}$$

at $(x, y) = (x_0, y_0)$, *where* $z_0 = x_0 + iy_0$.

Proof: (i) first suppose that f is holomorphic at $z = z_0$. Let's compute the partial derivatives of g:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \lim_{x \to x_0} \frac{g(x, y_0) - g(x_0, y_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x + iy_0 - (x_0 + iy_0)} = f'(z_0) \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial g}{\partial y} &= \lim_{y \to y_0} \frac{g(x_0, y) - g(x_0, y_0)}{y - y_0} \\ &= \lim_{y \to y_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{y - y_0} = \lim_{y \to y_0} i \frac{f(x + iy_0) - f(x_0 + iy_0)}{x_0 + iy - (x_0 + iy_0)} = if'(z_0) \end{aligned}$$

Hence (1) holds.

(ii) Conversely, suppose that (1) holds. Pick a curve $z(t)$ through z_0 and assume that $z(0) = z_0$, say, and $z(t) = x(t) + iy(t)$. Consider $f(z(t))$. We need to show that

$$\lim_{t \to 0} \frac{f(z(t)) - f(z_0)}{z(t) - z_0} \tag{2}$$

exists and is independent of the choice of the curve $z(t)$ through z_0. It this is true, then

$$\frac{f(z) - f(z_0)}{z - z_0}$$

will have a well-defined limit as $z \to z_0$ in the complex plane, and thus f will be holomorphic at $z = z_0$.

To check (2), we note that

$$\lim_{t \to 0} \frac{f(z(t)) - f(z_0)}{z(t) - z_0} = \lim_{t \to 0} \frac{t}{z(t) - z(0)} \frac{f(z(t)) - f(z_0)}{t} = \frac{1}{z'(0)} \frac{d}{dt}\Big|_{t=0} f(z(t))$$

But by the Chain Rule:

$$\frac{d}{dt}\Big|_{t=0} f(z(t)) = \frac{d}{dt}\Big|_{t=0} g(x(t), y(t)) = \frac{\partial g}{\partial x}(x_0, y_0)x'(0) + \frac{\partial g}{\partial y}(x_0, y_0)y'(0)$$

hence by (1):

$$\frac{d}{dt}\Big|_{t=0} f(z(t)) = \frac{\partial g}{\partial x}(x_0, y_0)(x'(0) + iy'(0)) = \frac{\partial g}{\partial x}(x_0, y_0)z'(0)$$

So

$$\lim_{t \to 0} \frac{f(z(t)) - f(z_0)}{z(t) - z_0} = \lim_{t \to 0} \frac{t}{z(t) - z(0)} \frac{f(z(t)) - f(z_0)}{t} = \frac{\partial g}{\partial x}(x_0, y_0)$$

This ends the proof.