§4 Matched asymptotic expansions

4.1 Singular perturbations

small parameter ε.

Consider a differential equation of the form $D_\varepsilon y = 0$

↳ naturally we would look at $D_0 y = 0$ as an approximation for the solution.

However - problem if ε multiplies the highest derivative eg $d^k y/dx^k$ since then taking $\varepsilon = 0$ reduces the order of the problem.

↳ an issue since $D_\varepsilon y = 0$ is a kth order eqn with k boundary conditions but $D_0 y = 0$ is a $(k-1)$th order eqn with k boundary conditions - they cannot, in general, all be satisfied.

- called a singular perturbation problem.

Example $\quad \varepsilon y'' + y' + y = 0$ for $x \in (0,1)$ with $y(0) = a$ and $y(1) = b$.

$\varepsilon = 0 \quad y' + y = 0 \Rightarrow y = Ae^{-x}$ which cannot satisfy $y(0) = a$, $y(1) = b$

Interpretation and procedure - the method of matched asymptotic expansions

One possible explanation: If y satisfies $D_\varepsilon y = 0$ then

- over most of the range, $\varepsilon\, d^k y/dx^k$ is small, and y approximately satisfies $D_0 y = 0$.
- in certain regions (often at the ends of the range), $\varepsilon\, d^k y/dx^k$ is not small and y adjusts itself to the boundary conditions (ie it varies rapidly).

↑ regions often known as boundary layers

Procedure

1. determine the scaling of the boundary layers (eg $x \propto \varepsilon / \varepsilon^{1/2}$ etc)
2. rescale the independent variable in the boundary layer
3. find the asymptotic expansions in, and outside of, the boundary layers.
4. fix the arbitrary constants
   - obey problem boundary conditions
   - match - inner and outer solutions.

called inner and outer solutions.

Back to the example: $\varepsilon y'' + y' + y = 0$ with $y(0) = a$, $y(1) = b$.
(NB can be solved exactly, but will pretend o/w, for now...)

Scaling

Near $x = 0$: let $X_L = \dfrac{x}{\varepsilon^\alpha}$

local variable for inspecting BL on LHS.

$$\frac{d}{dx} = \frac{d}{dX_L}\frac{dX_L}{dx} = \varepsilon^{-\alpha}\frac{d}{dX_L}$$

$$\frac{d^2}{dx^2} = \varepsilon^{-2\alpha}\frac{d^2}{dX_L^2}$$

$$\Rightarrow \underbrace{\varepsilon^{1-2\alpha}\frac{d^2y_L}{dX_L^2}}_{} + \varepsilon^{-\alpha}\frac{dy_L}{dX_L} + y_L = 0$$

significant in the BL $\Rightarrow$ increase α until this term balances the largest of the others

ie $1 - 2\alpha = \min(-\alpha, 0)$

$\Rightarrow \alpha = 1$

Hence $X_L = \dfrac{x}{\varepsilon}$.

(NB choosing α larger, to balance 1st and 3rd terms gives second term of $O(\varepsilon^{-\frac{1}{2}})$ which is bigger than the other two...)

Similarly, $X_R = \dfrac{(x-1)}{\varepsilon}$ (or, $x = 1 + \varepsilon X_R$) so that $X_R < 0$.

Expand

LH: $y(x) = y_L(X_L) = y_{L0}(X_L) + \varepsilon y_{L1}(X_L) + \dots$ $\quad \left(X_L = \dfrac{x}{\varepsilon}\right)$

middle: $y(x) = y_m(x) = y_{m0}(x) + \varepsilon y_{m1}(x) + \dots$

RH: $y(x) = y_R(X_R) = y_{R0}(X_R) + \varepsilon y_{R1}(X_R) + \dots$ $\quad \left(X_R = \dfrac{(x-1)}{\varepsilon}\right)$

Solution on the left (inner left)

$$\frac{d^2y_L}{dX_L^2} + \frac{dy_L}{dX} + \varepsilon y_L = 0 \Rightarrow O(1): \frac{d^2y_{L0}}{dX_L^2} + \frac{dy_{L0}}{dX_L} = 0$$

$$O(\varepsilon): \frac{d^2y_{L1}}{dX_L^2} + \frac{dy_{L1}}{dX_L} + \underbrace{y_{L0}}_{} = 0$$

forcing term for y_{L1}

$\therefore\ y_{L0}(X_L) = A_{L0} + B_{L0}e^{-X_L}$

$y_{L1}(X_L) = A_{L1} + B_{L1}e^{-X_L} + (B_{L0}X_Le^{-X_L} - A_{L0}X_L)$

with $y_{L0}(0) = a = \underline{A_{L0} + B_{L0}}$

Solution in the middle (outer)

$$\varepsilon \frac{d^2 y_m}{dx^2} + \frac{dy_m}{dx} + y_m = 0 \quad \Rightarrow \quad O(1): \quad \frac{dy_{m0}}{dx} + y_{m0} = 0$$

$$O(\varepsilon): \quad \underbrace{\frac{d^2 y_{m0}}{dx^2}}_{\text{inhomogeneous part}} + \frac{dy_{m1}}{dx} + y_{m1} = 0$$

$$\therefore \quad y_{m0}(x) = A_{m0} e^{-x}$$

$$y_{m1}(x) = A_{m1} e^{-x} - A_{m0}\, x e^{-x}$$

we will match solutions to determine A_{m0}, A_{m1}

Solution on the right (inner right)

$$\frac{d^2 y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \quad \Rightarrow \quad O(1): \quad \frac{d^2 y_{R0}}{dx_R^2} + \frac{dy_{R0}}{dx_R} = 0$$

$(x_R < 0)$

$$O(\varepsilon): \quad \frac{d^2 y_{R1}}{dx_R^2} + \frac{dy_{R1}}{dx_R} + \underbrace{y_{R0}}_{\text{inhom. part.}} = 0$$

$$\therefore \quad y_{R0}(x_R) = A_{R0} + B_{R0} e^{-x_R}$$

$$y_{R1}(x_R) = A_{R1} + B_{R1} e^{-x_R} + (B_{R0}\, x_R e^{-x_R} - A_{R0}\, x_R)$$

Boundary condition @ $x = 1 \Rightarrow y_{R1}(0) = b = \underline{A_{R0} + B_{R0}}$

Matching to establish lowest order coefficients

- Have five constants: A_{L0}, B_{L0}, A_{m0}, A_{R0}, B_{R0}

  and two equations: $A_{L0} + B_{L0} = a$ and $A_{R0} + B_{R0} = b$

- We obtain three more conditions by matching.

  ↳ Idea: ∃ overlap region where both expansions should hold, and hence be equal.

  i.e. $y_L(x_L) \sim y_m(x)$ as $x_L = \frac{x}{\varepsilon} \to \infty$ and $x \to 0$.

One approach – introduce a scaling – should be 'intermediate'

ie $\hat{x} = \dfrac{x}{\varepsilon^{\alpha}}$ where $0<\alpha<1$.

Then, with $\varepsilon \to 0^+$ and $\hat{x}$ fixed, $x = \varepsilon^{\alpha}\hat{x} \to 0$

$x_L = \varepsilon^{\alpha-1}\hat{x} \to \infty$

{ as $\varepsilon \to 0$ then $x \to 0$ and outer soln → that of the BL, and $x_L \to \infty$ so inner soln tends to the value in the interior

Matching at the LH end: we want $y_L(\varepsilon^{\alpha-1}\hat{x}) \sim y_m(\varepsilon^{\alpha}\hat{x})$ as $\varepsilon \to 0^+$

ie they generate the same expansion

with $\hat{x}>0$, $\hat{x} \sim \mathrm{ord}(1)$.

$$y_L = A_{L0} + B_{L0}e^{-\varepsilon^{\alpha-1}\hat{x}} + o(\varepsilon)$$
$$= A_{L0} + o(\varepsilon)$$

since $\alpha \in (0,1)$ then this term is exp. small as $\varepsilon \to 0^+$

$$y_m = A_{m0}e^{-\varepsilon^{\alpha}\hat{x}} + o(\varepsilon)$$
$$= A_{m0}(1 - \varepsilon^{\alpha}\hat{x} + \dots) + o(\varepsilon)$$

Hence, at leading order, $\underline{A_{L0} = A_{m0}}$.
(as $\varepsilon \to 0^+$)

} ie y values need to match – the outer limit of the inner problem matches the inner limit of the outer problem.

Matching at the RH end: this time use $x = 1 + \varepsilon^{\alpha}\tilde{x}$ ie $\tilde{x} = \dfrac{(x-1)}{\varepsilon^{\alpha}} \leq 0$

so that $x_R = \dfrac{x-1}{\varepsilon} = \varepsilon^{\alpha-1}\tilde{x}$

We want $y_R(\varepsilon^{\alpha-1}\tilde{x}) = y_m(\varepsilon^{\alpha}\tilde{x})$

$$y_R = A_{R0} + B_{R0}e^{-\varepsilon^{\alpha-1}\tilde{x}} + o(\varepsilon)$$

this term blows up as $\varepsilon \to 0^+ \Rightarrow$ we need $\underline{B_{R0} = 0}$.

$$y_m = A_{m0}e^{-1-\varepsilon^{\alpha}\tilde{x}} + o(\varepsilon)$$
$$= \frac{A_{m0}}{e}(1 - \varepsilon^{\alpha}\hat{x} + \dots) + o(\varepsilon)$$

Hence, at leading order, $\underline{A_{R0} = \dfrac{A_{m0}}{e}}$.
(as $\varepsilon \to 0^+$)

(Again – y values must match . .)

→ Now have five equations and five unknowns :)

$$A_{L0} + B_{L0} = a,\quad A_{R0} + B_{R0} = b,\quad A_{L0} = A_{m0},\quad B_{R0} = 0,\quad A_{R0} = A_{m0}e^{-1}$$

$$\Rightarrow\quad A_{L0} = eb,\quad B_{L0} = a - eb,\quad A_{R0} = b,\quad B_{R0} = 0,\quad A_{m0} = eb.$$

Putting this all together: $y_{L0} = eb + (a-eb)e^{-X_L}$

$$y_{M0} = ebe^{-x}$$

$$y_{R0} = b.$$

← No rapid variation in the RH BL – we don't really need it!

NB Exact solution is $y(x) = A_+ e^{\lambda_+ x} + A_- e^{\lambda_- x}$ with $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

Expanding eg. $\lambda_+ \sim -1 + o(\varepsilon)$, $\lambda_- \sim -\frac{1}{\varepsilon} + 1 + o(\varepsilon)$ as $\varepsilon \to 0^+$ etc.

one can show that (as $\varepsilon \to 0^+$)

$$\left.\begin{aligned} y(\varepsilon X_L) &= y_{L0}(X_L) + o(\varepsilon) \\ y(x) &= y_{M0}(x) + o(\varepsilon) \\ y(\varepsilon X_R) &= y_{R0}(X_R) + o(\varepsilon) \end{aligned}\right\}$$

$X_L > 0$, $X_L \sim \mathrm{ord}(1)$

$X_R < 0$, $X_R \sim \mathrm{ord}(1)$

Matching to establish coefficients at the next order

we have

$$y_{L1} = -ebX_L + (a-eb)X_Le^{-X_L} + \underline{A_{L1}} + \underline{B_{L1}}e^{-X_L}$$

$$y_{M1} = -ebxe^{-x} + \underline{A_{M1}}e^{-x}$$

$$y_{R1} = -bX_R + \underline{A_{R1}} + \underline{B_{R1}}e^{-X_R}$$

again, are constants.

The boundary conditions supply two eqns: $A_{L1} + B_{L1} = 0$ $\{ y_{L1}(0) = 0$

$A_{R1} + B_{R1} = 0$ $\{ y_{R1}(0) = 0$

Matching at the LH end: as before, we write $x = \varepsilon^\alpha \hat{x} \Rightarrow X_L = \varepsilon^{\alpha-1}\hat{x}$

with $\alpha \in (0,1)$, $\hat{x} \sim \mathrm{ord}(1)$.

we have, on the left,

$$y_L = y_{L0}(\varepsilon^{\alpha-1}\hat{x}) + \varepsilon y_{L1}(\varepsilon^{\alpha-1}\hat{x}) + o(\varepsilon^2)$$

$$= eb + (a-eb)\underline{e^{-\varepsilon^{\alpha-1}\hat{x}}}$$

$$+ \varepsilon\left(-eb\varepsilon^{\alpha-1}\hat{x} + (a-eb)\varepsilon^{\alpha-1}\hat{x}\underline{e^{-\varepsilon^{\alpha-1}\hat{x}}} + A_{L1} + B_{L1}\underline{e^{-\varepsilon^{\alpha-1}\hat{x}}}\right) + o(\varepsilon^2)$$

all $\to 0$ as $\varepsilon \to 0^+$ for $\alpha \in (0,1)$ and $\hat{x} > 0$

$$= eb - eb\varepsilon^\alpha\hat{x} + A_{L1}\varepsilon + o(\varepsilon^2)$$

and, in the outer,

$$y_m = y_{m0}(\varepsilon^\alpha \hat{x}) + \varepsilon y_{m1}(\varepsilon^\alpha \hat{x}) + o(\varepsilon^2)$$

$$= eb e^{-\varepsilon^\alpha \hat{x}} + \varepsilon\left(-eb\varepsilon^\alpha \hat{x} e^{-\varepsilon^\alpha \hat{x}} + A_{m1} \underbrace{e^{-\varepsilon^\alpha \hat{x}}}_{\text{expand as } 1-\varepsilon^\alpha \hat{x}+\cdots}\right) + o(\varepsilon^2)$$

$$= eb\left(1 - \varepsilon^\alpha \hat{x} + \frac{\varepsilon^{2\alpha}\hat{x}^2}{2!} + \cdots\right)$$

$$-eb\varepsilon^{\alpha+1}\hat{x}\left(1 - \varepsilon^\alpha \hat{x} + \cdots\right) + A_{m1}\varepsilon\left(1 - \varepsilon^\alpha \hat{x} + \cdots\right) + o(\varepsilon^2)$$

$$= \underline{eb - eb\hat{x}\varepsilon^\alpha} + \underline{\frac{\varepsilon^{2\alpha}\hat{x}^2}{2} eb} + \cdots - eb\varepsilon^{\alpha+1}\hat{x} + \cdots$$

the $O(1)$ term, matched in y_L ✓✓

$$+\varepsilon A_{m1} - A_{m1}\varepsilon^{\alpha+1}\hat{x} + \cdots + o(\varepsilon^2)$$

in order to be able to neglect these terms we need $2\alpha > 1$ ie $\alpha > \frac{1}{2}$

$\Rightarrow \alpha \in (\frac{1}{2}, 1)$.

Comparing terms that are $O(\varepsilon)$ gives $\underline{A_{L1} = A_{m1}}$.

NB some terms jump order: $-\varepsilon^\alpha eb\hat{x}$ comes from the inner expansion of the first outer term, but from the outer expansion of the second inner term!

Matching at the RH end: as before, we write $x = 1 + \varepsilon^\alpha \tilde{x}$, $\tilde{x} < 0$

$$\Rightarrow \tilde{x} = \frac{x-1}{\varepsilon^\alpha}, \quad X_R = \frac{x-1}{\varepsilon} = \varepsilon^{\alpha-1}\tilde{x}$$

we have, on the right,

$$y_R = b + \varepsilon\left(-b\varepsilon^{\alpha-1}\tilde{x} + A_{R1} + B_{R1}e^{-\varepsilon^{\alpha-1}\tilde{x}}\right) + o(\varepsilon^2)$$

$$= b - b\varepsilon^\alpha \tilde{x} + \varepsilon A_{R1} + B_{R1}\underbrace{\varepsilon e^{-\varepsilon^{\alpha-1}\tilde{x}}}_{\text{blows up as } \varepsilon \to 0^+} + o(\varepsilon^2)$$

$\Rightarrow$ $\underline{B_{R1} = 0}$

and, in the outer,

expand $e^{\varepsilon^\alpha \hat{x}}$

$$y_m = ebe^{-1-\varepsilon^\alpha \hat{x}} + \varepsilon\left(-eb(1+\varepsilon^\alpha \hat{x})e^{-1-\varepsilon^\alpha\hat{x}} + A_{m1}e^{-1-\varepsilon^\alpha \hat{x}}\right) + O(\varepsilon^2)$$

$$= \frac{\cancel{e}b}{\cancel{e}}\left(1-\varepsilon^\alpha \hat{x} + \frac{\varepsilon^{2\alpha}\hat{x}^2}{2} + \dots\right) - \frac{b\cancel{e}}{\cancel{e}}(\varepsilon + \varepsilon^{\alpha+1}\hat{x})(1-\varepsilon^\alpha\hat{x}+\dots)$$

$$+ \varepsilon \frac{A_{m1}}{e}\left(1-\varepsilon^\alpha \hat{x} + \dots\right) + O(\varepsilon^2)$$

$$= \underline{b} + (A_{m1}e^{-1} - b)\varepsilon + \dots$$

matches the $O(1)$ contribution ✓✓

Hence, collecting terms at $O(\varepsilon)$ gives $\underline{A_{m1}e^{-1} - b = A_{R1}}$

Again, we now have five equations and five unknowns :)

$$A_{L1} + B_{L1} = 0, \quad A_{R1} + B_{R1} = 0, \quad A_{L1} = A_{m1}, \quad B_{R1} = 0, \quad A_{m1}e^{-1} - b = A_{R1}$$

$$\Rightarrow A_{R1} = 0, \quad B_{R1} = 0, \quad A_{m1} = be, \quad A_{L1} = be, \quad B_{L1} = -be$$

Putting it all together:

$$y_{L1} = -ebx_L + (a-eb)x_L e^{-x_L} + eb - ebx^{-L}$$

$$y_{m1} = -ebxe^{-x} + ebe^{-x}$$

$$y_{R1} = -bx_R$$

Note that $\lim_{x\to 1} y_m = ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2) = b + O(\varepsilon^2)$

which satisfies the BC @ $x=1$. However $\lim_{x\to 0} y_m = eb$ which does not satisfy the BC. Hence don't actually need the RH BL, but we do need the LH one!

was indicated by the blow up in the inner solution...

4.1.4 Van Dyke's matching rule

- using the intermediate rule is tiresome! (even for that simple example it was bad..)
- Van Dyke's rule usually works, and it's simple / convenient.

$$\underbrace{(m \text{ term inner})(n \text{ term outer})}_{\text{in the outer term expand to } n \text{ terms, then switch to the inner variables and re-expand to } m \text{ terms}} = \underbrace{(n \text{ term outer})(m \text{ term inner})}_{\text{in the inner expand to } m \text{ terms, then switch to the outer variables and re-expand to } n \text{ terms.}}$$

Example

$y_{L0} = A_{L0} + B_{L0} e^{-X_L}$	$y_{m0} = A_{m0} e^{-x}$	$y_{R0} = A_{R0} + B_{R0} e^{-X_R}$
$y_{L1} = A_{L1} + B_{L1} e^{-X_L} + (B_{L0} X_L e^{-X_L} - A_{L0} X_L)$	$y_{m1} = A_{m1} e^{-x} - A_{m0} x e^{-x}$	$y_{R1} = A_{R1} + B_{R1} e^{-X_R} + (B_{R0} X_R e^{-X_R} - A_{R0} X_R)$

with constraints $A_{L0} + B_{L0} = a$, $A_{R0} + B_{R0} = b$, $A_{L1} + B_{L1} = 0$, $A_{R1} + B_{R1} = 0$.

and $x = \varepsilon X_L = 1 + \varepsilon X_R \in [0,1]$ $(X_L > 0, X_R < 0)$.

First - consider what happens at the RH boundary: $X_R < 0$ so $e^{-X_R} \to \infty$ as $X_R \to \infty$ i.e. as we go from in the RH BL $\to$ outer soln.

$\Rightarrow$ $B_{R0} = 0$, $B_{R1} = 0$.

Again, demonstrates that assuming fast variation in the RH inner region (BL) ↙ incorrectly! gives $y_{R0} = $ constant. Then

$$\varepsilon y_{R1} = \varepsilon A_{R0} X_R = -\varepsilon A_{R0} \frac{(x-1)}{\varepsilon} = -A_{R0}(x-1)$$

i.e. the variation is not quick relative to x so there is no BL at the RH end and we can just consider the outer solution, y_m, all the way to the boundary.

Applying VD's matching rule for $\underline{m=n=1}$:

$$(1to) = A_{m0}e^{-x} \underset{\text{switch to inner variables}}{=} A_{m0}e^{-\varepsilon x_L} \underset{\text{expand}}{=} A_{m0}\left(1-\varepsilon x_L - \frac{\varepsilon^2 x_L^2}{2} + \dots\right)$$

$$\therefore \quad (1ti)(1to) = A_{m0}$$
$$(2ti)(1to) = A_{m0} - \varepsilon A_{m0} x_L \quad \text{etc.}$$

Then
$$(1ti) = A_{L0} + B_{L0}e^{-x_L} \underset{\text{switch to outer variables}}{=} A_{L0} + B_{L0}e^{-x/\varepsilon} = A_{L0} + \underbrace{\exp.}_{\text{small terms}}$$

$$\therefore \quad (1to)(1ti) = A_{L0}$$

Hence
$$(1ti)(1to) = (1to)(1ti) \Rightarrow A_{m0} = A_{L0} = eb$$

comes from evaluating: $y_{m0}(1) = b$ ie BC @ $x=1$ since no RHBL

$$\therefore \quad y_{m0} = ebe^{-x}$$
$$y_{L0} = A_{L0} + B_{L0}e^{-x_L} = eb + (a-eb)e^{-x_L}$$

since $A_{L0} + B_{L0} = a$

This automatically satisfies $\lim_{x\to 0} y_{m0}(x) = \lim_{x_L\to\infty} y_L(x_L)$ as we previously observed. This will generally be the case.

Now, apply van Dyke's matching rule for $\underline{m=n=2}$:

2 term outer:
$$y_m(x) = \underbrace{\overbrace{eb}^{A_{m0}} e^{-x}}_{y_{m0}(x)} + \underbrace{\varepsilon(A_{m1}e^{-x} - \overbrace{eb}^{A_{m0}}xe^{-x})}_{y_{m1}(x)}$$

change to inner variable

$$= ebe^{-\varepsilon x_L} + \varepsilon(A_{m1}e^{-\varepsilon x_L} - eb\varepsilon x_L e^{-\varepsilon x_L})$$

expand

$$= eb(1-\varepsilon x_L + \dots) + \varepsilon(A_{m1}(1-\varepsilon x_L + \dots) - eb\varepsilon x_L(1-\varepsilon x_L + \dots))$$

keep 2 terms

$$= eb - ebx_L\varepsilon + \varepsilon A_{m1} + O(\varepsilon^2)$$

2 term inner:

$$y_L = \underbrace{\overbrace{\underbrace{\epsilon b}_{A_{L0}} + \underbrace{(a-\epsilon b)}_{B_{L0}} e^{-x_L}}^{y_{L0}}} + \varepsilon\overbrace{(A_{L1} + B_{L1}e^{-x_L} + (a-\epsilon b)x_L e^{-x_L} - \epsilon b x_L)}^{y_{L1}} + \dots$$

change to outer variable

$$= \epsilon b + (a-\epsilon b)e^{-x/\varepsilon} + \varepsilon(A_{L1} + B_{L1}e^{-x/\varepsilon} + (a-\epsilon b)\tfrac{x}{\varepsilon}e^{-x/\varepsilon} - \epsilon b\tfrac{x}{\varepsilon}) + \dots$$

← exponentially small →

expand + keep 2 terms

$$= \epsilon b + \varepsilon A_{L1} - \epsilon b x + \text{exp. small terms}$$

Recall $y_m = \epsilon b - \epsilon b \underbrace{x_L\varepsilon}_{=x} + \varepsilon A_{m1} + O(\varepsilon^2)$

$\therefore$ $\underline{A_{L1} = A_{m1} = \epsilon b}$ (by the RH boundary condition – since we know that there is no boundary layer...)

Also, $A_{L1} + B_{L1} = 0 \Rightarrow \underline{B_{L1} = -\epsilon b}$.

(Have all the same information and we get it much faster!)

Summary

$$y_m(x) = \epsilon b e^{-x} + \varepsilon \epsilon b(1-x)e^{-x} + \dots$$

$$y_L(x) = \epsilon b + (a-\epsilon b)e^{-x_L} + \varepsilon\left(\epsilon b(1-e^{-x_L}) - \epsilon b x_L + (a-\epsilon b)x_L e^{-x_L}\right) + \dots$$

NB – can check what happens for $(1t_i)(2t_o) = (2t_o)(1t_i)$
↳ no new information!

Composite expansion

Neither of y_m or y_L is valid across the whole domain
↳ but when eg. plotting it can be really useful!

- Aim – combine the inner and outer expansions to obtain a uniformly valid expansion.

$$y_{composite} = (p \text{ terms outer}) + (p \text{ terms inner}) - \underbrace{(p \text{ terms inner})(p \text{ terms outer})}$$

(by VD's MR) $= (p \text{ terms outer})(p \text{ terms inner})$

need to subtract as it has been counted twice in the overlap region...

Example: $p=1$

$$y_{composite} = y_{mo}(x) + y_{Lo}\left(\tfrac{x}{\varepsilon}\right) - (\text{1 term inner})(\text{1 term outer})$$

$$= \underbrace{ebe^{-x}}_{y_{mo}} + \underbrace{eb + (a-eb)e^{-x/\varepsilon}}_{y_{Lo}} - eb$$

1 term outer $= ebe^{-x} = ebe^{-\varepsilon x_L}$

$\Rightarrow$ (1 term inner)(1 term outer) $= eb$

$$= ebe^{-x} + (a-eb)e^{-x/\varepsilon} + O(\varepsilon).$$

rapid change at LH boundary (as $x \sim O(\varepsilon)$) ensures the BC is satisfied.

Example: $p=2$

$$y_{composite} = y_{mo}(x) + \varepsilon y_{m1}(x) + y_{Lo}(x/\varepsilon) + \varepsilon y_{L1}(x/\varepsilon) - (2ti)(2to)$$

$$= ebe^{-x} + \varepsilon eb(1-x)e^{-x} + eb + (a-eb)e^{-x/\varepsilon}$$

$$+ \varepsilon\left(eb(1-e^{-x/\varepsilon}) - eb\tfrac{x}{\varepsilon} + (a-eb)\tfrac{x}{\varepsilon}e^{-x/\varepsilon}\right)$$

$$- \underbrace{eb + ebx - \varepsilon eb}_{(2ti)(2to)}$$

$$= ebe^{-x} + (a-eb)(1+x)e^{-x/\varepsilon} - \varepsilon eb(1-x)e^{-x} - \varepsilon eb\, e^{-x/\varepsilon} + O(\varepsilon^2)$$

Choice of scaling re-visited

Near $x=0$ let $x_L = x/\varepsilon^\alpha$, $y(x) = y_L(x_L)$

$$\Rightarrow \quad \varepsilon^{1-2\alpha}\frac{d^2y_L}{dx_L^2} + \varepsilon^{-\alpha}\frac{dy_L}{dx_L} + y_L = 0$$

$\alpha=0$		balance	balance
$0<\alpha<1$		dominant	
$\alpha=1$	balance	balance	
$\alpha>1$	dominant		

OUTER SOLUTION

can match since they share a common term which is dominant in overlap region

INNER SOLUTION

interesting scalings - balance 2+ terms, called distinguished limits.

Next - we will think about how to determine where the BL is...

4.2 Where is the boundary layer?

- To have a non-trivial boundary layer possible – need a solution in the inner region that decays as we move towards the outer.
  (saw this in the previous example with the LH BL.)
- In the problem that we considered, though, the solution in the RH BL grew exponentially as we moved towards the outer ⇒ cannot have a RH BL!
- Note – BLs don't need to be at boundaries! We can have small regions of high gradient in the interior (⇒ interior layer).

Example: consider the general problem

$$\varepsilon y'' + p(x)y' + q(x)y = 0 \quad \text{for } 0<x<1 \text{ with } p(x)>0,\ y(0)=A,\ y(1)=b.$$

and p, q smooth, and $\varepsilon \ll 1$.

RH boundary layer

Rescale: $x = 1+\delta\hat{x}$, $y(x) = y_R(\hat{x})$ with $\hat{x}<0$

('= derivative wrt argument)

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2} y_R'' + \frac{p(1+\delta\hat{x})}{\delta} y_R' + q(1+\delta\hat{x}) y_R = 0$$

We want this term to be included since seeking a solution s.t. y'' large

↳ here, the $\frac{p(1+\delta\hat{x})}{\delta} y_R'$ will dominate the $q(1+\delta\hat{x}) y_R$ term

Hence, the dominant balance is $\frac{\varepsilon}{\delta^2} = \frac{1}{\delta} \Rightarrow \varepsilon = \delta$.

$$\therefore \quad y_R'' + p(1+\varepsilon\hat{x})y_R' + \varepsilon q(1+\varepsilon\hat{x}) y_R = 0$$

$$\Rightarrow \quad y_R'' + [p(1) + \varepsilon\hat{x}p'(1) + \dots]y_R' + \varepsilon[q(1) + \varepsilon\hat{x}q'(1) + \dots]y_R = 0$$

Let $y_R(\hat{x}) = y_{R0}(\hat{x}) + \varepsilon y_{R1}(\hat{x}) + \dots$, substitute and collect terms of the same order:

$O(1)$: $y_{R0}'' + p(1)y_{R0}' = 0 \Rightarrow y_{R0}(\hat{x}) = J + Ke^{-p(1)\hat{x}}$ with $\hat{x} < 0$.

We want to match the outer solution when $\hat{x}$ large and negative.
BUT, then $Ke^{-p(1)\hat{x}} \to \infty$ and so we have to take $k = 0$, and
$y_{R0} = J$ (constant).

ie no fast variation near the RH boundary $\Rightarrow$ NO BOUNDARY LAYER,
(we can match the outer at $x = 1$ to $y(1) = b$).

(*) Blow up as the inner solution is extended towards the outer solution $\Rightarrow$ NO BOUNDARY LAYER.

LH Boundary layer

Let $x = \varepsilon\hat{x}$ with $\hat{x} > 0$ and $y(x) = y_L(\hat{x})$.

Similarly,

$$y_L''(\hat{x}) + [p(0) + \varepsilon\hat{x}p'(0) + \dots]y_L'(\hat{x}) + \varepsilon[q(0) + \varepsilon\hat{x}q'(0) + \dots]y_L(\hat{x}) = 0.$$

Expanding: $y_L(\hat{x}) = y_{L0}(\hat{x}) + \varepsilon y_{L1}(\hat{x}) + \dots$ gives $y_{L0}(\hat{x}) = M + Ne^{-p(0)\hat{x}}$

Then, moving from the inner to the outer ($\hat{x} \to \infty$) gives $Ne^{-p(0)\hat{x}} \to 0$ ✓

$\therefore y_{L0}(\hat{x}) = M + Ne^{-p(0)x/\varepsilon}$

ie have a very rapid change in the solution near the boundary, and can match outer.

NB If $p(x) < 0$ then the situation is reversed and we expect to find a boundary layer at the RH e $x = 1$. If $p(x_0) = 0$ for some $x_0 \in (0,1)$ then there may be an interior layer.

(↑ next example!)

Example

$$\varepsilon^2 y'' + 2y(1-y^2) = 0 \quad \text{for } -1<x<1 \text{ with } y(-1)=-1 \text{ and } y(1)=1.$$

Then for $\varepsilon = 0$ we can have outer solutions with $y = 0, \pm 1$.

To satisfy the BCs, we take $\left.\begin{array}{l} y_{0L} = -1 \\ y_{0R} = +1 \end{array}\right\}$ $\Rightarrow$ Must be a transition between them in the interior.

outer solns on LH/RH sides

By inspection, we see that we need to rescale near $x = x_0$ $(x_0 \in (0,1))$ by setting $x = x_0 + \varepsilon X$ and $y(x) = Y(X)$

$$\Rightarrow \quad Y''(X) + 2Y(1-Y^2) = 0 \quad \text{for } -\infty < X < \infty \quad \begin{array}{l} Y \to -1 \text{ as } X \to -\infty \\ Y \to +1 \text{ as } X \to +\infty \end{array}$$

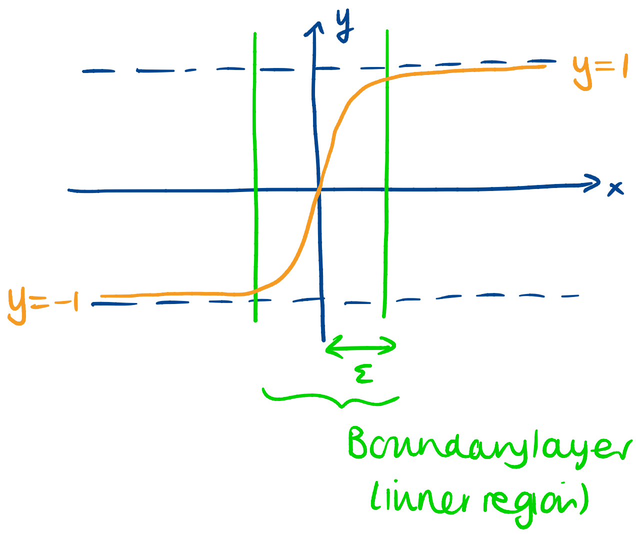
If scaling not obvious then let $x = x_0 + \delta(\varepsilon) X$ and establish dominant balance.

Solution is $Y(X) = \tanh(X - X_*)$

Recall $X = \frac{x - x_0}{\varepsilon}$ and let $x_* = \varepsilon X_*$ to write $y(x) = \tanh\left(\frac{x - x_0 - x_*}{\varepsilon}\right)$

Note that if $y(x)$ is a solution then $-y(-x)$ is also a solution, and by Picard, the solution is unique. In particular $y(0) = -y(0) = 0$ and so $x_0 + x_* = 0$ and we have $y(x) \sim \tanh\left(\frac{x}{\varepsilon}\right)$.

NB! The position of the BL is exponentially sensitive to the boundary data. Finding the location for other data is nontrivial.

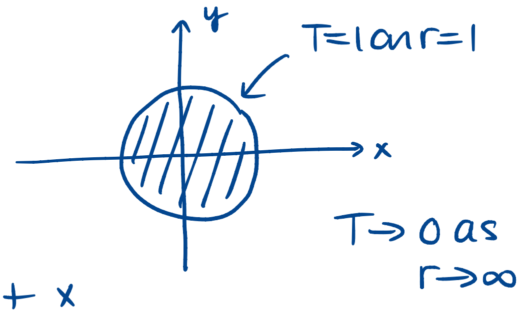


4.3 Boundary layers in PDEs

Heat transfer from a cylinder in potential flow with small diffusion (high Peclet number).

$(0<\varepsilon \ll 1)$

$$\underbrace{\underline{u}\cdot\nabla T}_{\text{advection}} = \underbrace{\varepsilon\nabla^2 T}_{\text{diffusion}} \qquad r \geq 1$$



$$\underline{u} = \nabla\varphi, \quad \varphi = \left(r + \frac{1}{r}\right)\cos\theta = x + \frac{x}{x^2+y^2}$$

velocity   gradient flow vector field

NB $\nabla^2\varphi = 0 \Rightarrow \underline{u}$ irrotational and incompressible, and has zero normal component on $r=1$. $\Rightarrow$ flow is around the cylinder @ $r=1$, and this advects thermal energy.

Physical problem: steady state temperature profile with diffusion and advection of internal energy represented through temperature T.

Outer solution

Expand $T \sim T_0 + \varepsilon T_1 + \ldots$ as $\varepsilon \to 0$ and substitute to get

$O(1)$: $\underline{u}\cdot\nabla T_0 = 0$, with the BC $T \to 0$ as $r \to \infty$

(will need the inner solution to match the BC at $r=1$.)

Consider any curve with $\frac{d\underline{r}}{ds} = \underline{u} \qquad (\underline{r} = (x,y))$

Then $\frac{dT_0}{ds} = \nabla T_0 \cdot \frac{d\underline{r}}{ds} = \nabla T_0 \cdot \underline{u} = 0$

For $r>1$, $\frac{dx}{ds} = u_1 = \varphi_x = 1 + \frac{1}{x^2+y^2} - \frac{2xy}{x^2+y^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{1-\cos 2\theta}{r^2} > 0$

$\therefore$ all curves $\underline{u} = \nabla\varphi$ end up at infinity where $T_0 = 0$.

Hence $T_0(s)$ is constant along such curves and, using the BC, this constant must be zero, ie $T_0 = 0$.

Proceeding further, we have $T_n = 0 \;\forall n$. $\Rightarrow \exists$ thermal boundary layer near the cylinder.

Inner solution

In cylindrical coordinates, ← to accommodate the BC

$$\underbrace{\left(1-\frac{1}{r^2}\right)\cos\theta\frac{\partial T}{\partial r}-\left(1+\frac{1}{r^2}\right)\frac{\sin\theta}{r}\frac{\partial T}{\partial\theta}}_{\underline{u}\cdot\nabla T}=\varepsilon\underbrace{\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right)}_{\nabla^2 T}$$

- we need to scale r close to $r=1$ so that the diffusive term balances:
  let $r=1+\delta(\varepsilon)\rho$ with $0<\delta(\varepsilon)\ll 1$ and let $T(r,\theta)=T_i(1+\delta(\varepsilon)\rho,\theta)$.

Then,

$$\left(1-\frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\frac{\partial T_i}{\partial\rho}-\left(1+\frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{1+\delta\rho}\frac{\partial T_i}{\partial\theta}$$

$$=\varepsilon\left(\frac{1}{\delta^2}\frac{\partial^2 T_i}{\partial\rho^2}+\frac{1}{\delta(1+\delta\rho)}\frac{\partial T_i}{\partial\rho}+\frac{1}{(1+\delta\rho)^2}\frac{\partial^2 T_i}{\partial\theta^2}\right)$$

Expand to give

$$(2\delta\rho+O(\delta^2))\frac{\cos\theta}{\delta}\frac{\partial T_i}{\partial\rho}-(2+O(\delta))\sin\theta\frac{\partial T_i}{\partial\theta}$$

$$=\varepsilon\left(\frac{1}{\delta^2}\frac{\partial^2 T_i}{\partial\rho^2}+\underbrace{\frac{1}{\delta}(1+O(\delta))\frac{\partial T_i}{\partial\rho}+(1+O(\delta))\frac{\partial^2 T_i}{\partial\theta^2}}_{\text{both subleading compared to the } \partial^2 T_i/\partial\rho^2 \text{ term.}}\right)$$

Hence, we need $\delta=\sqrt{\varepsilon}$.

Expand: let $T_i(\rho,\theta)=T_{i0}(\rho,\theta)+\varepsilon^{\frac{1}{2}}T_{i1}(\rho,\theta)+\ldots$

$O(1)$: $\quad 2\rho\cos\theta\dfrac{\partial T_{i0}}{\partial\rho}-2\sin\theta\dfrac{\partial T_{i0}}{\partial\theta}=\dfrac{\partial^2 T_{i0}}{\partial\rho^2}$ still a non-trivial PDE to solve – but we can make analytic progress.

with $T_{i0}(0,\theta)=1$ and $\underbrace{T_{i0}\to 0 \text{ as } \rho\to\infty}_{\text{match to outer solution}}$

$\rho=0$ – ie on the cylinder.

Seek a similarity solution for T_{i0} of the form

$$T_{i0}(\rho,\theta)=f(\eta)\quad\text{with}\quad \eta=\rho g(\theta)$$

$$\frac{\partial T_{i0}}{\partial\rho}=g(\theta)f'(\eta),\quad \frac{\partial^2 T_{i0}}{\partial\rho^2}=g^2(\theta)f''(\eta),\quad \frac{\partial T_{i0}}{\partial\theta}=f'(\eta)\rho g'(\theta)$$

Substitute into the eqn for T_{10}:

$$2\rho\cos\theta\, g(\theta) f'(\eta) - 2\sin\theta\, f'(\eta)\,\rho\, g'(\theta) = g^2(\theta) f''(\eta)$$

$$\Rightarrow \quad \underbrace{\rho g(\theta)}_{\eta}\left[\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta}{g^3(\theta)} g'(\theta)\right] f'(\eta) = f''(\eta) \quad \circledast$$

Note that if $f(\eta)$ is a similarity soln then $\circledast$ should be a function of η only $\Rightarrow$

$$\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta}{g^3(\theta)} g'(\theta) = \text{constant, } c$$

- If $c>0$ then the solution will blow up at infinity
  $\Rightarrow$ must have $c<0$.
- Note that c can be re-scaled without changing g
  $\Rightarrow$ WLOG we can take $c=-1$

$\therefore$ Can find a similarity solution as long as we can find a solution to

$$\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta}{g^3(\theta)} g'(\theta) = -1$$

Let $g = \frac{1}{\sqrt{p}}$ $\rightarrow$ then we can solve to get $g(\theta) = \frac{|\sin\theta|}{(J+\cos\theta)^{\frac{1}{2}}}$.

- If $J>1$ then the solution will blow up.
- If $J<1$ then we have further problems

as we send $r\to\infty$

at $\theta=\pi$: $T(r,\pi) \sim T_{10}(\rho,\pi) = f(\rho \underbrace{g(\pi)}_{=0}) = f(0) = 0$

but $\qquad T(0,\frac{\pi}{2}) \sim T_{10}(0,\frac{\pi}{2}) = f(0.g(\frac{\pi}{2})) = f(0) = 1$

by the BCs.

$\therefore$ Must have $J=1$ so that $g(\theta) = \frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}$.

Can now solve for f: $f(\eta) = A\int_\eta^\infty e^{-\frac{1}{2}u^2}\,du + B$

$f \to 0$ as $\eta \to \infty \Rightarrow B = 0$

$f = 1$ for $\eta = 0 \Rightarrow A = \sqrt{\frac{2}{\pi}}$

$\rho = \frac{r-1}{\sqrt{\varepsilon}}$

$\therefore \quad T_{i0} = \sqrt{\frac{2}{\pi}}\int_\eta^\infty e^{-\frac{1}{2}u^2}\,du$ with $\eta = \underbrace{\frac{\rho|\sin\theta|}{(1+\cos\theta)^{1/2}}}_{\rho g(\theta)}$

NB as $\rho \to \infty$, T_{i0} decays exponentially, to match with outer solution (solution is exponentially small in the outer region).

NB1 As the outer solution is equal to zero, the composite solution is given by T_{i0}.

NB2 Solution fails for $\theta = 0$ and r large

↳ since r large, expect (from a physical perspective) to have $T = 0$, but the lower limit on the integral is zero $\Rightarrow T(\eta, \theta) = 1$.

$\Rightarrow$ we need another distinguished limit for this region!! similar argument applies for $\theta = \pi$.

$\theta = 0, \pi$ – stagnation points.

Also BL in the wake – streamline here comes from cylinder, not infinity.

Heat loss: $\frac{\partial T}{\partial r} \sim O\left(\frac{1}{\varepsilon^{1/2}}\right)$ (reason for wind chill factor).

Example – boundary layer at infinity

(also an asymptotic power series that isn't in terms of powers of ε).

$$(x^2y')' + \varepsilon x^2 y y' = 0 \quad \text{with } x>1, \; y(1)=0, \; y(\infty)=1, \; 0<\varepsilon \ll 1$$

We will try to find a solution of the form $y \sim y_0 + \varepsilon y_2 + \dots$

will find that this doesn't work on its own and that we need another term – of the form $\varepsilon \log(\frac{1}{\varepsilon}) y_1(x)$

To see that this is the case – substitute, and collect terms:

$O(\varepsilon^0)$: $(x^2 y_0')' = 0 \Rightarrow y_0 = 1 - \frac{1}{x}$ (using BCs)

$O(\varepsilon^1)$: $(x^2 y_2')' = -x^2 y_0 y_0' = -1 + \frac{1}{x} \Rightarrow y_2(x) = A\left(1 - \frac{1}{x}\right) - \ln x - \frac{\ln x}{x}$

integrate and solve with $y_2(1) = 0$

cannot satisfy $y_2(\infty) = 0$ for any A since $\ln x \to \infty$ as $x \to \infty$

$\Rightarrow$ BL at infinity!

$\therefore$ we need to expand in an inner region where x is large, and match to the outer solution where $x = \text{ord}(1)$.

$\to$ this is given by $y_0 + \varepsilon y_2$

Inner solution – use a new variable $x = \frac{X}{\delta_1(\varepsilon)}$ where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, so that $X \sim \text{ord}(1)$ as $x \to \infty$ and $\varepsilon \to 0^+$, and let

$y = 1 + \delta_2(\varepsilon) Y$ with $\delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

satisfies the BC $y(\infty) = 1$.

Substituting:

$$\underbrace{\delta_2 \frac{d}{dX}\left(X^2 \frac{dY}{dX}\right)}_{1} + \underbrace{\frac{\varepsilon\delta_2}{\delta_1} X^2 \frac{dY}{dX}}_{2} + \underbrace{\frac{\varepsilon\delta_2^2}{\delta_1} X^2 Y \frac{dY}{dX}}_{3} = 0$$

Want a dominant balance.

Note that $\frac{\varepsilon\delta_2}{\delta_1} \gg \frac{\varepsilon\delta_2^2}{\delta_1} \Rightarrow$ 3 will never contribute.

Hence the dominant balance comes from matching ① and ② :

$$\frac{\varepsilon\delta_2}{\delta_1} = \delta_2 \Rightarrow \delta_1 = \varepsilon \text{ and } \delta_2 \text{ (as yet) undetermined.}$$

Let $Y = Y_0(X) + o(1) \Rightarrow \frac{d}{dX}\left(X^2\frac{dY_0}{dX}\right) + X^2\frac{dY_0}{dX} = 0$

$$\Rightarrow X^2\frac{dY_0}{dX} = e^{-X}$$

$$Y_0(X) = B\int_X^\infty \frac{e^{-s}}{s^2}\,ds$$

using the BC $Y_0 \to 0$ as $X \to \infty$ Since $y = 1+\delta_2 Y$.

inner solution satisfying inner BC

Single constant B ⇒ anticipate we can match to the outer solution.

Evaluate as $X \to 0^+$ by splitting the integral:

$$\int_X^\infty \frac{1}{s^2}e^{-s}\,ds = \int_X^1 \frac{1}{s^2}e^{-s}\,ds + \underbrace{\int_1^\infty \frac{1}{s^2}e^{-s}\,ds}$$

$$< \int_1^\infty e^{-s}\,ds \sim \operatorname{ord}(1)$$

$$= \int_X^1 \frac{1-s}{s^2}\,ds + \int_X^1 \frac{e^{-s}-1+s}{s^2}\,ds$$

$$= \left\{\frac{1}{X} + \ln X + \operatorname{ord}(1)\right\}$$

$\sim \frac{1}{s^2}\left(1 - s + \frac{1}{2}s^2 + \dots - 1 + s\right)$

⇒ will generate a power series with first term $\frac{1}{2}s^2$ — which is ord(1).

$$\therefore\ Y_0(X) = B\left[\frac{1}{X} + \ln X + \operatorname{ord}(1)\right].$$

To match — consider an intermediate variable: $\hat{x} = \varepsilon^\alpha x = \varepsilon^{\alpha-1}X$, $0<\alpha<1$.

Expand both the outer and inner solutions in the intermediate variable:

then $\hat{x} = \operatorname{ord}(1)$ as $\varepsilon \to 0^+ \Rightarrow \underbrace{x \to \infty}_{\text{towards BL}}$ and $\underbrace{X \to 0}_{\text{towards outer}}$

INNER: $Y_0 \sim B\left(\frac{1}{X} + \ln X + \text{ord}(1)\right)$

$\Rightarrow y = 1 + \delta_2 Y \sim 1 + \delta_2 B \frac{\varepsilon^{\alpha-1}}{\hat{x}} + \delta_2 B \ln(\varepsilon^{\alpha-1}\hat{x})$

OUTER: $y_0 = 1 - \frac{1}{x}$

$y_2 = A\left(1 - \frac{1}{x}\right) - \ln x - \frac{\ln x}{x}$ ⊛

$\Rightarrow y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \text{ord}(\varepsilon)$

Need to match these

$\Rightarrow -\frac{\varepsilon^\alpha}{\hat{x}} = \frac{\delta_2 B \varepsilon^{\alpha-1}}{\hat{x}} \Rightarrow \delta_2 = \varepsilon$

$B = -1$

$\therefore\ y = 1 - \frac{\varepsilon}{\hat{x}} + (1-\alpha)\varepsilon \ln\left(\frac{1}{\varepsilon}\right) - \underbrace{\varepsilon \ln \hat{x}}_{o(\varepsilon)}$

However, the ord(ε) term in the outer solution will never be matched by the $(1-\alpha)\varepsilon\ln\left(\frac{1}{\varepsilon}\right)$ term in the inner solution.

↳ The y_2 term in ⊛ generates terms of the form $\varepsilon\ln\left(\frac{1}{\varepsilon}\right)$ but we should really have the $\varepsilon\ln\left(\frac{1}{\varepsilon}\right)$ in the asymptotic sequence as the missing y_1 in the outer solution!

i.e. we should have originally taken $y(x) = y_0(x) + \varepsilon\ln\left(\frac{1}{\varepsilon}\right)y_1(x) + \varepsilon y_2(x) + \dots$

Then, $(x^2 y_1')' = 0 \Rightarrow y_1 = c\left(1 - \frac{1}{x}\right)$ ← doesn't need to satisfy $y(\infty) = 1$ – will match with the inner solution.

The outer solution in the intermediate variable is

$$y(x) \sim \left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) + c\varepsilon\ln\left(\frac{1}{\varepsilon}\right)\left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) + A\left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) \underbrace{- \ln(\varepsilon^{-\alpha}\hat{x}) - \frac{\ln(\varepsilon^{-\alpha}\hat{x})}{\varepsilon^{-\alpha}\hat{x}}}_{\text{expand}}$$

$$\sim \left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) + \varepsilon\ln\left(\frac{1}{\varepsilon}\right)(c - \alpha) + o(\varepsilon)$$

↑ additional term that includes the parameter c – which we determine by matching to the inner solution ($\delta = \varepsilon$, $B = -1$).

$(1-\alpha)\varepsilon\ln\left(\frac{1}{\varepsilon}\right) = \varepsilon\ln\left(\frac{1}{\varepsilon}\right)(1-c) \Rightarrow c = \alpha$

$\Rightarrow$ α's cancel ✓✓ (since we need this to hold for a range of α).

NB Have not determined A at this order – would need to go to higher order!

SUMMARY

Inner: $y \sim -\left[\frac{1}{X} + \ln X + \mathrm{ord}(1)\right]$

Outer: $y \sim 1 - \frac{1}{x} + \varepsilon \ln\left(\frac{1}{\varepsilon}\right)\left(1 - \frac{1}{x}\right) + O(\varepsilon)$

To go to higher orders – we need to use the following expansion sequence:

$1,\ \varepsilon \ln\left(\frac{1}{\varepsilon}\right),\ \varepsilon,\ \varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right),\ \varepsilon^2\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^2,\ \varepsilon^2, \ldots$

NB we can only use van Dyke's matching rule if we let $\ln\left(\frac{1}{\varepsilon}\right) \sim \mathrm{ord}(1)$ ⁂

directly contradicts our assumption – we treated $\ln\left(\frac{1}{\varepsilon}\right) \gg 1$!