C4.3 Functional Analytic Methods for PDEs
Lectures 5-6

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In the last 4 lectures

- L^p spaces and their properties.

This lecture

- Pre-compactness criterion in $L^p(\Omega)$.
- Divergence theorem and Integration by parts formula.
- Weak derivatives.
- Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ as Banach spaces.
- Differentiation rule for convolution of Sobolev functions.
- Dual of $W^{1,p}$.
- Sobolev spaces $W_0^{k,p}(\Omega)$.
- Differentiation rule for convolution of Sobolev functions.

Pre-compactness criterion in $C(K)$

Theorem (Ascoli-Arzelà's theorem)

Let K be a compact subset of $\mathbb{R}^n$. Suppose that (f_i) is a sequence of functions of $C(K)$ such that

1. *(Boundedness)* $\sup_i \|f_i\|_{C(K)} < \infty$,
2. *(Equi-continuity) For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|f_i(x) - f_i(y)| < \varepsilon$ *for all* i *and all* $x, y \in K$ *with* $|x - y| < \delta$.

Then there exists a subsequence (f_{i_j}) *which converges uniformly on* K.

In other words, the set $\{f_i\}$ is pre-compact.

Pre-compactness criterion in $C(K)$

Proof

- We would like to show that (f_i) has a subsequence (f_{i_j}) which is Cauchy in $C(K)$, i.e. for every given $\varepsilon > 0$,

$$\|f_{i_j} - f_{i_k}\|_{C(K)} \leq \varepsilon \text{ for all large } j, k. \qquad (*)$$

- We claim that a slightly softer statement holds: For every given ε, there is a subsequence $(f_{i_j}^{\varepsilon})$ of (f_i) such that

$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{C(K)} \leq 3\varepsilon \text{ for large } j, k. \qquad (**)$$

- Suppose that (**) holds for the moment, we will now show how (*) can be obtained.

Pre-compactness criterion in $C(K)$

Proof

- (**) $\Rightarrow$ (*): We will use a diagonal procedure.
  - ⋆ Using (**), take a subsequence $(f_{i_j}^1)$ of (f_i) such that $\|f_{i_j}^1 - f_{i_k}^1\|_{C(K)} \leq 1$ eventually.
  - ⋆ Now the sequence $(f_{i_j}^1)$ satisfies the condition of theorem. Since we are assuming (**), we can thus take a subsequence $(f_{i_j}^2)$ of $(f_{i_j}^1)$ such that $\|f_{i_j}^2 - f_{i_k}^2\|_{C(K)} \leq 1/2$ eventually.
  - ⋆ Proceeding inductively, we have a nest sequence of subsequences $(f_i) \supset (f_{i_j}^1) \supset (f_{i_j}^2) \supset \ldots$ such that, for each $m \geq 1$,

$$\|f_{i_j}^m - f_{i_k}^m\|_{C(K)} \leq 1/m \text{ eventually.}$$

  - ⋆ Now let $f_{i_j} = f_{i_j}^j$. Then, for every fixed m, the sequence (f_{i_j}) is eventually a subsequence of $(f_{i_j}^m)$ and so $\|f_{i_j} - f_{i_k}\|_{C(K)} \leq 1/m$ eventually. So (f_{i_j}) satisfies (*).

Pre-compactness criterion in $C(K)$

Proof

- We now prove (**), i.e. for every given ε, there is a subsequence $(f_{i_j}^{\varepsilon})$ of (f_i) such that

$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{C(K)} \leq 3\varepsilon \text{ for large } j, k.$$

  - ⋆ By equi-continuity, there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for all i and all $x, y \in K$ with $|x - y| < \delta$.
  - ⋆ As K is compact, we can cover K by finitely many open balls $B(x_1, \delta), \ldots, B(x_N, \delta)$ with x_ℓ's in K.
  - ⋆ By uniform boundedness, for each ℓ, the sequence $(f_i(x_\ell))$ is bounded in $\mathbb{R}$. By Bolzano-Weierstrass' theorem, we can select a subsequence $(f_{i_j}^{\varepsilon})$ such that $(f_{i_j}^{\varepsilon}(x_\ell))$ is convergent for all ℓ. So

$$|f_{i_j}^{\varepsilon}(x_\ell) - f_{i_k}^{\varepsilon}(x_\ell)| \leq \varepsilon \text{ for all } \ell \text{ and for all large } j, k.$$

Proof

- We now prove (**).
  - ⋆ ...$|f_i(x) - f_i(y)| \leq \varepsilon$ for all i and all $x, y \in K$ with $|x - y| < \delta$.
  - ⋆ $B(x_1, \delta), \ldots, B(x_N, \delta)$ covers K.
  - ⋆ ...$|f_{i_j}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x_\ell)| \leq \varepsilon$ for all ℓ and for all large j and k.
  - ⋆ Now if $x \in K$, then $x \in B(x_\ell, \delta)$ for some ℓ. Then, for large j, k,

$$\begin{aligned} |f_{i_j}^\varepsilon(x) - f_{i_k}^\varepsilon(x)| &\leq |f_{i_j}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x_\ell)| + |f_{i_j}^\varepsilon(x_\ell) - f_{i_j}^\varepsilon(x)| \\ &\qquad + |f_{i_k}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x)| \\ &\leq 3\varepsilon. \end{aligned}$$

  - ⋆ So $\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{C(K)} \leq 3\varepsilon$, which proves (**).

Pre-compactness criterion in $L^p(\Omega)$

Theorem (Kolmogorov-Riesz-Fréchet's theorem)

Let $1 \leq p < \infty$ and Ω be an open bounded subset of $\mathbb{R}^n$. Suppose that a sequence (f_i) of $L^p(\Omega)$ satisfies

1. *(Boundedness)* $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
2. *(Equi-continuity in L^p) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where $\tilde{f}_i$ is the extension by zero of f_i to the whole of $\mathbb{R}^n$.*

Then, there exists a subsequence (f_{i_j}) which converges in $L^p(\Omega)$.

By definition $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{f}_i = f_i$ in Ω and $\tilde{f}_i = 0$ in $\mathbb{R}^n \setminus \Omega$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- As in the proof of Ascoli-Arzelà's theorem, it suffices to show that, for every given $\varepsilon > 0$, there exists a subsequence $(f_{i_j}^\varepsilon)$ of (f_i) such that

$$\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{L^p(\Omega)} \leq 3\varepsilon \text{ for large } j, k. \qquad (***)$$

- Claim: For every fixed $\varphi \in C_c^\infty(\mathbb{R}^n)$, the sequence $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
  - ⋆ First, by Hölder's inequality, we have

$$\|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} = \|f_i\|_{L^1(\Omega)} \leq \|f_i\|_{L^p(\Omega)} |\Omega|^{1/p'}.$$

    Thus, by the boundedness of (f_i) in $L^p(\Omega)$, we have that $(\tilde{f}_i)$ is bounded in $L^1(\mathbb{R}^n)$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- Claim: $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
  - ⋆ $\sup_i \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} < \infty$.
  - ⋆ By Young's convolution inequality

$$\|\tilde{f}_i * \varphi\|_{L^\infty(\mathbb{R}^n)} \leq \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^\infty(\mathbb{R}^n)}.$$

    So $\sup_i \|\tilde{f}_i * \varphi\|_{C(\bar{\Omega})} < \infty$.
  - ⋆ Next,

$$\begin{aligned} |\tilde{f}_i * \varphi(x) - \tilde{f}_i * \varphi(y)| &\leq \int_{\mathbb{R}^n} \Big|\varphi(x-z) - \varphi(y-z)\Big| |\tilde{f}(z)|\, dz \\ &\leq \|\varphi\|_{Lip(\mathbb{R}^n)} |x-y| \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

    So by squeezing $|x-y|$, we can make $\sup_i |\tilde{f}_i * \varphi(x) - \tilde{f}_i * \varphi(y)|$ as small as we want.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- Now, take a non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and, for $\eta > 0$, let $\varrho_\eta(x) = \frac{1}{\eta^n}\varrho(x/\eta)$ be the standard mollifiers. Recall that we have the estimate

$$\begin{aligned}\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p}^p &\leq \int_{\mathbb{R}^n} |\varrho_\eta(y)| \|\tau_{-y}\tilde{f}_i - \tilde{f}_i\|_{L^p}^p \, dy \\ &\leq \sup_{|y|\leq\eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}^p \int_{\mathbb{R}^n} |\varrho_\eta(y)| \, dy \\ &= \sup_{|y|\leq\eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}^p.\end{aligned}$$

Pre-compactness criterion in $L^p(\Omega)$

Proof

- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- $\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p} \leq \sup_{|y| \leq \eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}$.
- We are now ready to prove (***):
  - ⋆ By the equi-continuity, there exists a small $\eta > 0$ such that $\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p} \leq \varepsilon$ for all i.
  - ⋆ Using Ascoli-Arzelà's theorem, select a subsequence $(f_{i_j}^\varepsilon)$ of (f_i) such that $(\tilde{f}_{i_j}^\varepsilon * \varrho_\eta|_{\bar{\Omega}})$ is convergent in $C(\bar{\Omega})$.
  - ⋆ It follows that $\|\tilde{f}_{i_j}^\varepsilon * \varrho_\eta - \tilde{f}_{i_k}^\varepsilon * \varrho_\eta\|_{L^p(\Omega)} \leq \varepsilon$ for large j, k.
  - ⋆ Consequently, by triangle inequality,

$$\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{L^p(\Omega)} \leq \|f_{i_j}^\varepsilon * \varrho_\eta - f_{i_k}^\varepsilon * \varrho_\eta\|_{L^p(\Omega)} + \|f_{i_j}^\varepsilon * \varrho_\eta - f_{i_j}^\varepsilon\|_{L^p(\Omega)} + \|f_{i_k}^\varepsilon * \varrho_\eta - f_{i_k}^\varepsilon\|_{L^p(\Omega)} \leq 3\varepsilon \text{ for large } j, k,$$

    which is (***).

Frequently used terminologies/notations

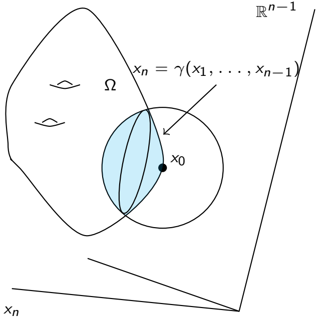
- Ω denotes a domain in $\mathbb{R}^n$.
- $C^k(\Omega)$ denotes the space of functions which are k-times continuously differentiable in Ω.
- $C^k(\bar{\Omega})$ denotes the subspace of $C^k(\Omega)$ consisting of functions which can be extended to a k-times continuously differentiable functions on some open set containing $\bar{\Omega}$.
- $C^k_c(\Omega)$ denotes the subspace of $C^k(\Omega)$ consisting of functions f such that $Supp(f) = \overline{\{f \neq 0\}}$ is a bounded closed subset of Ω.

Frequently used terminologies/notations

- Ω is said to be a Lipschitz (resp. C^k) domain, or equivalently, $\partial\Omega$ is said to be Lipschitz (resp. C^k), if for every $x_0 \in \partial\Omega$ there exists a radius $r_0 > 0$ such that, after a relabeling of coordinate axes if necessary,

$$\Omega \cap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

for some Lipschitz (resp. C^k) function γ.



Divergence theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fact: $\partial\Omega$ admits an 'outward pointing' unit normal n.

Theorem (Divergence theorem)

Let $F \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Then

$$\int_\Omega div\, F\, dx = \int_{\partial\Omega} F \cdot n\, dS.$$

In particular, if $F \in C^1_c(\Omega; \mathbb{R}^n)$, then

$$\int_\Omega div\, F\, dx = 0.$$

IBP formula

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.

Theorem (Integration by parts formula)

Let $f, g \in C^1(\bar{\Omega})$. Then

$$\int_\Omega f\,\partial_i g\,dx = \int_{\partial\Omega} f g n_i\,dS - \int_\Omega \partial_i f\,g\,dx.$$

In particular, if f or g has compact support in Ω, then

$$\int_\Omega f\,\partial_i g\,dx = -\int_\Omega \partial_i f\,g\,dx.$$

Weak derivatives

Let Ω be a domain in $\mathbb{R}^n$.

Definition

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. A function $g \in L^1_{loc}(\Omega)$ is said to be a weak α-derivative of f if

$$\int_\Omega f\, \partial^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega g \varphi\, dx \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{1}$$

We write $g = \partial^\alpha f$ in the weak sense.

The function φ is called a *test function*.

Example of weak derivatives

- If $f \in C^1(\bar{\Omega})$ and Ω is a bounded Lipschitz domain, then its classical derivatives are also its weak derivatives.
- Suppose $\Omega = (-1, 1)$ and $f(x) = |x|$. Then, if $\varphi \in C_c^\infty(-1, 1)$, we have by IBP that

$$\begin{aligned}
\int_{-1}^{1} f(x)\,\varphi'(x)\,dx &= \int_{-1}^{0} (-x)\varphi'(x)\,dx + \int_0^1 x\varphi'(x)\,dx \\
&= -x\varphi(x)\Big|_{-1}^{0} - \int_{-1}^{0} (-1)\varphi(x)\,dx \\
&\qquad +x\varphi(x)\Big|_0^1 - \int_0^1 (1)\varphi(x)\,dx \\
&= -\int_{-1}^{1} \mathrm{sign}(x)\varphi(x)\,dx.
\end{aligned}$$

So $f'(x) = \mathrm{sign}(x)$ in the weak sense.

Uniqueness of weak derivatives

Lemma

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. The weak α-derivative of f, if exists, is uniquely defined up to a set of measure zero.

This follows from the definition of weak derivative and the following:

Lemma (Fundamental lemma of the Calculus of Variations)

Let $g \in L^1_{loc}(\Omega)$. If $\int_\Omega g\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $g = 0$ a.e. in Ω.

Uniqueness of weak derivatives

Proof

- We will only consider the case Ω is a bounded domain and $g \in L^1(\Omega)$. The general case is left as an exercise.
- In Sheet 1, you showed that $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$. Thus, for any $\varepsilon > 0$, we can select $h \in C_c^\infty(\Omega)$ such that $\|g - h\|_{L^1} \leq \varepsilon$. Furthermore, by triangle inequality $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$.
- For $\delta > 0$, let $h_\delta = \frac{h}{\sqrt{\delta^2 + h^2}}$ so that $h_\delta \in C_c^\infty(\Omega)$ and $|h_\delta| \leq 1$.
- By hypotheses, $\int_\Omega g h_\delta \, dx = 0$.
- By construction, $\left| \int_\Omega (g - h) h_\delta \, dx \right| \leq \|g - h\|_{L^1} \|h_\delta\|_{L^\infty} \leq \varepsilon$.
- It follows that

$$\varepsilon \geq \int_\Omega g h_\delta \, dx - \int_\Omega (g - h) h_\delta \, dx = \int_\Omega h h_\delta \, dx.$$

Uniqueness of weak derivatives

Proof

- Recalling the expression of h_δ, we have

$$\varepsilon \geq \int_\Omega \frac{h^2}{\sqrt{\delta^2 + h^2}}\,dx.$$

- The integrand on the right hand side converges monotonically increasingly to $|h|$. Thus, by Lebesgue's monotone convergence theorem,

$$\varepsilon \geq \int_\Omega |h|\,dx = \|h\|_{L^1}.$$

- Recall that $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$, we obtain that $2\varepsilon \geq \|g\|_{L^1}$. Sending $\varepsilon \to 0$, we obtain $\|g\|_{L^1} = 0$, i.e. $g = 0$ a.e. in Ω.

A relation between classical and weak derivatives

Remark

Suppose that

1. $f \in L^1(\Omega)$ *is weakly differentiable with weak derivatives* $\partial_1^w f$, $\ldots$, $\partial_n^w f$,
2. *and, for some subdomain* $\omega \subset \Omega$, f *is classically differentiable in* ω *with classical derivatives* $\partial_1^c f$, $\ldots$, $\partial_n^c f$.

Then

$$\partial_i^w f = \partial_i^c f \text{ a.e. in } \omega \text{ for all } i = 1, \ldots, n.$$

A relation between classical and weak derivatives

Sketch of proof

- Using the definition of weak derivatives, $f|_\omega$ is weakly differentiable with weak derivatives $\partial_1^w f|_\omega, \ldots, \partial_n^w f|_\omega$.
- As f is classically differentiable in ω, its classical derivatives are also weak derivatives of $f|_\omega$.
- By the uniqueness of weak derivatives, the conclusion follows.

Example of non-existence of weak derivatives

If $\Omega = (-1,1)$ and $u(x) = \mathrm{sign}(x)$, then u has no weak derivative.

Proof

- Suppose otherwise that $u' = g \in L^1_{loc}(-1,1)$. Then, for $\varphi \in C_c^\infty(-1,1)$,

$$\int_{-1}^{1} g(x)\varphi(x)\,dx = \int_{-1}^{0} \varphi'(x)\,dx - \int_{0}^{1} \varphi'(x)\,dx$$
$$= [\varphi(0) - \varphi(-1)] - [\varphi(1) - \varphi(0)]$$
$$= 2\varphi(0).$$

- In particular, if we take $\underline{\varphi \in C_c^\infty(-1,0)}$, we have $\int_{-1}^{0} g(x)\varphi(x)\,dx = 0$. So $g = 0$ a.e. in $(-1,0)$. Likewise, $g = 0$ a.e. in $(0,1)$. So $g = 0$ a.e. in $(-1,1)$.
- We thus have $0 = \int_{-1}^{1} g(x)\varphi(x)\,dx = 2\varphi(0)$ for all $\varphi \in C_c^\infty(-1,1)$, which is impossible.

The Sobolev spaces $W^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p \leq \infty$, define

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \Big| \forall |\alpha| \leq k, \text{ the weak derivative } \partial^\alpha f \text{ exists and belongs to } L^p(\Omega) \Big\}.$$

  We equip $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big[\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \Big]^{\frac{1}{p}}$$

  so that $W^{k,p}(\Omega)$ is a normed vector space (check this!).
- For $p = 2$, we also write $H^k(\Omega)$ for $W^{k,2}(\Omega)$. These are inner product spaces (check this!) with inner product

$$\langle u, v \rangle_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

Examples of Sobolev functions

Let $\Omega = (-1, 1)$ and $f(x) = |x|$.

- We have that $f'(x) = \mathrm{sign}(x)$ and so $f \in W^{1,p}(-1, 1)$ for every $p \in [1, \infty]$.
- The function $f'(x) = \mathrm{sign}(x)$ has no weak derivatives, and so $f \notin W^{2,p}(-1, 1)$ for any $p \in [1, \infty]$.

Completeness of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is a Banach space. When $p = 2$, $W^{k,2}(\Omega)$ is a Hilbert space.

Proof

- We have seen that $W^{k,p}$ is a normed vector space and $W^{k,2}$ is an inner product space. It remains to show that $W^{k,p}$ is complete.
- Suppose that (u_m) is a Cauchy sequence in $W^{k,p}$. We need to show that there exists $u \in W^{k,p}$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ is Cauchy in L^p, as

  $$\|\partial^\alpha u_m - \partial^\alpha u_j\|_{L^p} \leq \|u_m - u_j\|_{W^{k,p}}.$$

  By Riesz-Fischer's theorem, we have that $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.

Completeness of $W^{k,p}(\Omega)$

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- To conclude, we show that $u := v_{(0,\dots,0)}$ belongs to $W^{k,p}$ and $u_m \to u$ in $W^{k,p}$.
  - ⋆ By definition of weak derivatives, we have for $|\alpha| \leq k$ that

$$\int_\Omega u_m \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \, \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega),$$

  - ⋆ Now we would like to pass $m \to \infty$. By Hölder's inequality

$$\Big| \int_\Omega (u_m - u) \partial^\alpha \varphi \, dx \Big| \leq \|u_m - u\|_{L^p} \|\partial^\alpha \varphi\|_{L^{p'}} \to 0.$$

    So $\int_\Omega u_m \partial^\alpha \varphi \, dx \to \int_\Omega u \partial^\alpha \varphi \, dx$.
  - ⋆ Similarly, $\int_\Omega \partial^\alpha u_m \, \varphi \, dx \to \int_\Omega v_\alpha \varphi \, dx$.

Completeness of $W^{k,p}(\Omega)$

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- ⋆ We thus have

$$\int_\Omega u \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  So v_α is the weak α-derivative of u. So $u \in W^{k,p}$.

  ⋆ Now

$$\begin{aligned} \|u_m - u\|_{W^{k,p}}^p &= \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - \partial^\alpha u\|_{L^p}^p \\ &= \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - v_\alpha\|_{L^p}^p \stackrel{m\to\infty}{\longrightarrow} 0. \end{aligned}$$

  So $u_m \to u$ in $W^{k,p}$.
- We conclude that $W^{k,p}$ is complete.

Reflexivity of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 < p < \infty$, $W^{k,p}(\Omega)$ is reflexive.

Proof

- We will only consider the case $k = 1$. The general case requires some minor changes.
- By Eberlein's theorem, we only need to show that every bounded sequence in $W^{1,p}$ has a weakly convergent subsequence.
- Suppose $(u_m) \subset W^{1,p}$ is bounded. Then, (u_m) and $(\partial_i u_m)$ are bounded in L^p.
- By the weak sequential compactness property of L^p for $1 < p < \infty$, there exists a subsequence (u_{m_j}) such that (u_{m_j}) and $(\partial_i u_{m_j})$ are weakly convergent in L^p. Let u be the L^p weak limit of (u_{m_j}) and v_i be the L^p weak limit of $(\partial_i u_{m_j})$.

Reflexivity of $W^{k,p}(\Omega)$

- To conclude, we show that u belongs to $W^{1,p}$ and $u_{m_j} \rightharpoonup u$ in $W^{1,p}$.
- The proof that $u \in W^{1,p}$ is similar to the one we did moment ago, but also has some subtle difference: By definition of weak derivatives, we have

$$\int_\Omega u_{m_j} \partial_i \varphi = - \int_\Omega \partial_i u_{m_j} \, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega),$$

Sending $j \to \infty$ by using the definition weak convergence, we obtain

$$\int_\Omega u \partial_i \varphi = - \int_\Omega v_i \, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

So $v_i = \partial_i u$ in the weak sense. So $u \in W^{1,p}$.

Reflexivity of $W^{k,p}(\Omega)$

- It remains to show that, if $A \in (W^{1,p})^*$, then $Au_{m_j} \to Au$.
  - ⋆ Define $E : W^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ by $Ef = (f, \partial_1 f, \ldots, \partial_n f)$. Then E is an isometry.
  - ⋆ Let $X := E(W^{1,p}(\Omega))$ and $Y := (L^p(\Omega))^{n+1}$. Define $\tilde{A} : X \to \mathbb{R}$ by $\tilde{A}p = AE^{-1}p$ for $p \in X$. Then $\tilde{A} \in X^*$. By Hahn-Banach's theorem, it has an extension $\hat{A} \in Y^*$.
  - ⋆ It follows that

$$\begin{aligned} Au_{m_j} = \tilde{A}Eu_{m_j} &= \hat{A}Eu_{m_j} \\ &= \hat{A}(u_{m_j}, 0, \ldots, 0) + \sum_i \hat{A}(0, 0, \ldots, 0, \partial_i u_{m_j}, 0, \ldots, 0) \\ &=: B(u_{m_j}) + \sum_i B_i(\partial_i u_{m_j}) \\ &\to B(u) + \sum_i B_i(\partial_i u) = Au. \end{aligned}$$

This concludes the proof.

The Sobolev spaces $W_0^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p < \infty$, define

$$W_0^{k,p}(\Omega) = \text{the closure of } C_c^\infty(\Omega) \text{ in } W^{k,p}(\Omega).$$

  When $p = 2$, we also write $H_0^k(\Omega)$ for $W_0^{k,2}(\Omega)$.
- In other words, $u \in W_0^{k,p}(\Omega)$ if there exist $u_m \in C_c^\infty(\Omega)$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- When $k = 0$, $1 \leq p < \infty$, and Ω is a bounded domain, we have seen in Sheet 1 that $W_0^{0,p}(\Omega) = W^{0,p}(\Omega) = L^p(\Omega)$.
  In general, this is not true for $k \geq 1$. Roughly speaking, $W_0^{k,p}(\Omega)$ consists of functions f in $W^{k,p}(\Omega)$ such that

$$\text{`}\partial^\alpha f = 0 \text{ on } \partial\Omega\text{' for all } |\alpha| \leq k - 1.$$

IBP formula for Sobolev functions

Proposition (Integration by parts)

Let $u \in W^{k,p}(\Omega)$ *and* $v \in W_0^{k,p'}(\Omega)$ *with* $k \geq 0$, $1 < p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\int_\Omega \partial^\alpha u v\,dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v\,dx \text{ for all } |\alpha| \leq k.$$

Proof

- By definition of $W_0^{k,p'}$, there exists $v_m \in C_c^\infty(\Omega)$ such that $v_m \to v$ in $W^{k,p'}$. In particular, $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- By the definition of weak derivatives,

$$\int_\Omega \partial^\alpha u v_m\,dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m\,dx \text{ for all } |\alpha| \leq k.$$

IBP formula for Sobolev functions

Proof

- $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- $\int_\Omega \partial^\alpha u v_m \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m \, dx$ for all $|\alpha| \leq k$.
- We can now pass $m \to \infty$ as in the proof of the completeness of Sobolev spaces.
  - ⋆ By Hölder's inequality
  $$\left| \int_\Omega \partial^\alpha u (v_m - v) \, dx \right| \leq \|\partial^\alpha u\|_{L^p} \|v_m - v\|_{L^{p'}} \to 0.$$
  So $\int_\Omega \partial^\alpha u v_m \, dx \to \int_\Omega \partial^\alpha u v \, dx$.
  - ⋆ Similarly, $\int_\Omega u \partial^\alpha v_m \, dx \to \int_\Omega u \partial^\alpha v \, dx$.
  - ⋆ We conclude that
  $$\int_\Omega \partial^\alpha u v \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v \, dx.$$

Differentiation rule for convolution of Sobolev functions

- Suppose $k \geq 0$ and $1 \leq p < \infty$.
- Let $f \in L^p(\mathbb{R}^n)$ and $g \in C_c^k(\mathbb{R}^n)$. We knew that $f * g \in C^k(\mathbb{R}^n)$ and

$$\partial^\alpha(f * g) = f * (\partial^\alpha g) \text{ for all } |\alpha| \leq k.$$

Lemma

Assume $f \in W^{k,p}(\mathbb{R}^n)$ and $g \in C_c^k(\mathbb{R}^n)$ for some $k \geq 0$ and $1 \leq p < \infty$, then

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g \textit{ for all } |\alpha| \leq k.$$

Differentiation rule for convolution of Sobolev functions

Proof

- We will only consider the case $k = 1$. We aim to prove that

$$\partial_{x_1}(f * g) = (\partial_{x_1} f) * g$$

- We compute

$$\begin{aligned}\partial_{x_1}(f * g)(x) = f * (\partial_{x_1} g)(x) &= \int_{\mathbb{R}^n} f(y)\, \partial_{x_1} g(x-y)\, dy \\ &= -\int_{\mathbb{R}^n} f(y)\, \partial_{y_1} g(x-y)\, dy \\ &= \int_{\mathbb{R}^n} \partial_{y_1} f(y)\, g(x-y)\, dy = ((\partial_{x_1} f) * g)(x).\end{aligned}$$

So we are done.