BO1 History of Mathematics
Lecture XIV
Linear algebra

MT 2022 Week 7

Summary

- Linear equations
- Determinants
- Eigenvalues
- Matrices
- Vector spaces

Difficulties in the historical study of linear algebra

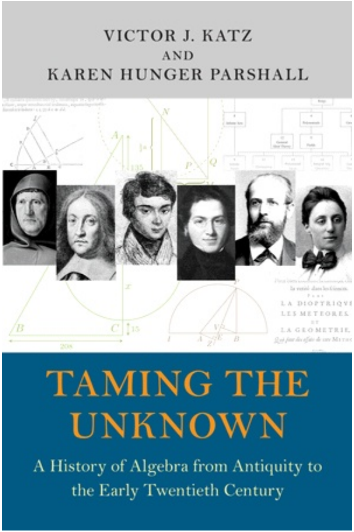
> *Linear algebra may be mathematically simple but its history is more complicated than any other topic in this book. . . . [Its development is] a very tangled tale.*

(*Mathematics Emerging*, p. 548)

- linear algebra is elementary but its manifestations are many and sophisticated
- there are hardly any obvious starting points
- theory often lagged behind practice
- practice sometimes lagged behind theory
- 19th-century reliance on theory of quadratic and bilinear forms (e.g., $ax^2 + 2bxy + cy^2$) — unfamiliar to students now

Warning: matrices (etc.) are primary in modern teaching, determinants secondary. For about 200 years until 1940 (or thereabouts) the reverse was the case: determinants came first.

On the history of linear algebra



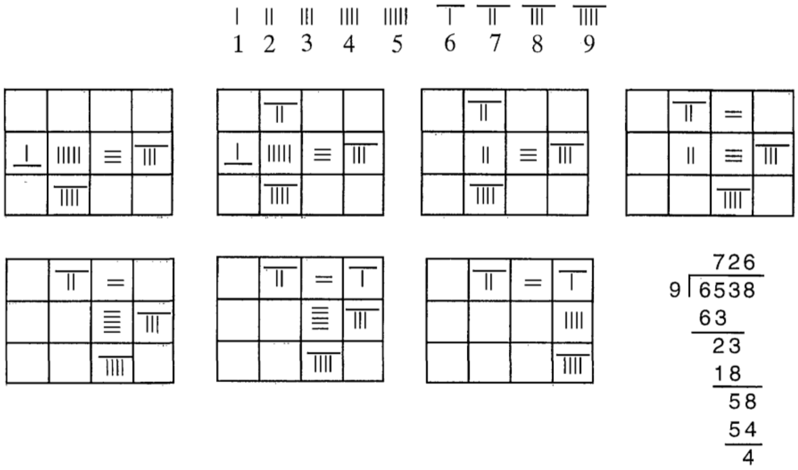
(Princeton University Press, 2014)

Jiǔzhāng Suànshù (China, c. 150 BC)

Nine chapters of the mathematical art 九章算術 (from a 16th-century edition, derived from a 3rd-century commentary by Liu Hui 劉徽)

Content: calculation of areas ($\pi \approx 3.14159$), rates of exchange, computation with fractions, proportion, extraction of square and cube roots, calculation of volumes, systems of linear equations, Pythagoras' Theorem, . . .

Chinese calculation



Base 10 system of rods on counting board: red for positive, black for negative

Early linear equations in China

Chapter 7: solution of pairs of equations in two unknowns by the method of false position

Chapter 8: solution of systems of n equations in n unknowns for $n \leq 5$

There are three types of grain

3 bundles of the first, 2 of the second, and 1 of the third contain 39 measures
2 of the first, 3 of the second, and 1 of the third contain 34
1 of the first, 2 of the second, and 3 of the third contain 26

How many measures in a bundle of each type?

Solved on a counting board by Gaussian elimination, known here as 'fāngchéng' 方程

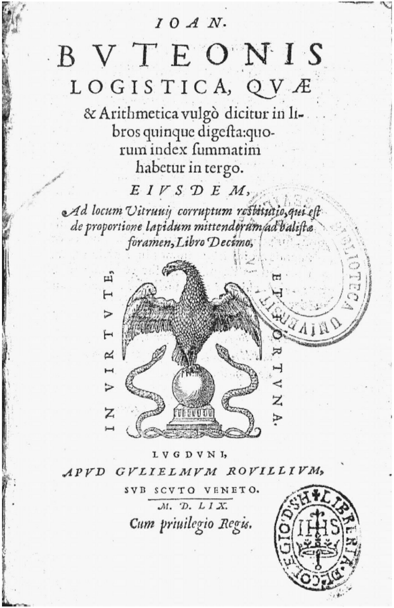
Early linear equations in China

There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.

Five equations in six unknowns, so indeterminate

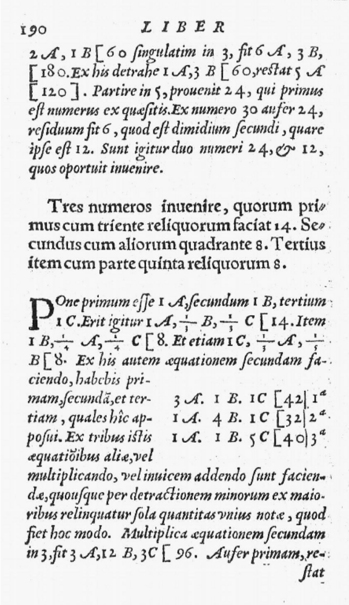
Liu Hui: we can only give a solution in terms of proportions of the lengths

Early linear equations in Europe


IOAN.
BVTEONIS
LOGISTICA, QVÆ
& Arithmetica vulgò dicitur in libros quinque digeſta: quorum index ſummatim habetur in tergo.
EIVSDEM,
Ad locum Vitruuij corruptum reſtitutio, qui eſt de proportione lapidum mittendorum ad baliſtæ foramen, Libro Decimo.
IN VIRTVTE, ET FORTVNA.
LVGDVNI,
APVD GVLIELMVM ROVILLIVM,
SVB SCVTO VENETO.
M. D. LIX.
Cum priuilegio Regis.

Jean Borrel [Ioannes Buteus] *Logistica, quæ et Arithmetica vulgo dicitur in libros quinque digesta* (*Logistic, also known as Arithmetic, digested in five books*), 1559

Linear equations in Borrel's *Logistica*



190 LIBER

2 A, 1 B [60 ſingulatim in 3, fit 6 A, 3 B, [180. Ex his detrahe 1 A, 3 B [60, reſtat 5 A [120]. Partire in 5, prouenit 24, qui primus eſt numerus ex quæſitis. Ex numero 30 aufer 24, reſiduum fit 6, quod eſt dimidium ſecundi, quare ipſe eſt 12. Sunt igitur duo numeri 24, & 12, quos oportuit inuenire.

Tres numeros inuenire, quorum primus cum triente reliquorum faciat 14. Secundus cum aliorum quadrante 8. Tertius item cum parte quinta reliquorum 8.

Pone primum eſſe 1 A, ſecundum 1 B, tertium 1 C. Erit igitur 1 A, $\frac{1}{3}$ B, $\frac{1}{3}$ C [14. Item 1 B, $\frac{1}{4}$ A, $\frac{1}{4}$ C [8. Et etiam 1 C, $\frac{1}{5}$ A, $\frac{1}{5}$ B [8. Ex his autem æquationem ſecundam faciendo, habebis primam, ſecundã, et tertiam, quales hîc appoſui. Ex tribus iſtis æquatiõibus aliæ, vel

3 A.	1 B.	1 C	[42		1ª
1 A.	4 B.	1 C	[32		2ª
1 A.	1 B.	5 C	[40		3ª

multiplicando, vel inuicem addendo ſunt faciendæ, quouſque per detractionem minorum ex maioribus relinquatur ſola quantitas vnius notæ, quod fiet hoc modo. Multiplica æquationem ſecundam in 3, fit 3 A, 12 B, 3 C [96. Aufer primam, re-

ſtat

To find three numbers, of which the first with a third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with a fifth part of the rest makes 8.

Put the first to be 1*A*, *the second* 1*B*, *the third* 1*C*. . . .

[Derives a system of equations with '.' for addition and '[' for equality.]

Multiply by 3, by 4 and by 5 respectively, etc.

(See *Mathematics emerging*, §17.1.1.)

More unknowns

GVL. GOS. DE ARTE

bunt 60 ęqualia 1 A, quare primus eſt 60, iam vero 2 B 1 C ęqualia fuerunt 100, tollamus 1 C hoc eſt 20, reſtabunt 80 æqualia 2 B, & 1 B eſt 40, ſuntque tres numeri quæſiti 60 40 20, quibus veſtigatis opus fuit.

Problema V.

Inueniamus quatuor numeros quorum primus cum ſemiſſe reliquorum faciat 17, ſecundus cum aliorum triente 12, tertius cum aliorum quadrante 13, quartus item cum aliorum ſextante 13.

Sint illi quatuor A B C D, & ſint 1 A $\frac{1}{2}$ B $\frac{1}{2}$ C $\frac{1}{2}$ D ęqualia 17, 1 B $\frac{1}{3}$ A $\frac{1}{3}$ C $\frac{1}{3}$ D ęqualia 12, 1 C $\frac{1}{4}$ A $\frac{1}{4}$ B $\frac{1}{4}$ D æqualia 13, 1 D $\frac{1}{6}$ A $\frac{1}{6}$ B $\frac{1}{6}$ C ęqualia 13, reuocentur hęc ad integros numeros, exiſtent 2 A 1 B 1 C 1 D æqualia 34, 1 A 3 B 1 C 1 D æqualia 36, 1 A 1 B 4 C 1 D æqualia 52, 1 A 1 B 1 C 6 D æqualia 78,

Guillaume Gosselin, *De arte magna seu de occulta parte numerorum quae et Algebra et Almucabala vulgo dicitur* (*On the great art or the hidden part of numbers commonly called Algebra and Almucabala*), 1577

$$1A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 17$$
$$1B + \frac{1}{3}A + \frac{1}{3}C + \frac{1}{3}D = 12$$
$$1C + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}D = 13$$
$$1D + \frac{1}{6}A + \frac{1}{6}B + \frac{1}{6}C = 13$$

A 17th-century example

After reading Gosselin ...
John Pell to Sir Charles Cavendish (1646):

Exemplum ... satis determinatis

$$3a - 4b + 5c = 2$$
$$5a + 3b - 2c = 58$$
$$7a - 5b + 4c = 14$$

(Solved via Pell's 'three-column method')

Exemplum ... non satis determinatis

$$5a + 3b - 2c = 24$$
$$-2a + 4b + 3c = 5$$

($a, b, c > 0$; found bounds for the possible values: e.g., $a < 15\frac{9}{11}$)

Linear equations — systematic practical methods

Gaussian elimination:

- *The nine chapters of the mathematical art*, China (c. 150 BC)
- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85

Maclaurin on Gaussian elimination

A

TREATISE

OF

ALGEBRA,

IN

THREE PARTS.

CONTAINING

I. *The Fundamental Rules and Operations.*

II. *The Compoſition and Reſolution of Equations of all Degrees; and the different Affections of their Roots.*

III. *The Application of* Algebra *and* Geometry *to each other.*

To which is added an

APPENDIX,

Concerning the general Properties of GEOMETRICAL LINES.

By *COLIN MACLAURIN*, M. A.

Late PROFESSOR of MATHEMATICS in the Univerſity of *Edinburgh*, and *Fellow* of the *Royal Society*.

LONDON:

Printed for A. MILLAR, and J. NOURSE, oppoſite to *Catherine-Street*, in the *Strand*.

M.DCC.XLVIII.

$$\begin{cases} x : y :: a : b \\ x^3 - y^3 = d \end{cases}$$

$$x = \frac{ay}{b} \text{ and } x^3 = \frac{a^3y^3}{b^3}$$

but $x^3 = d + y^3$

whence $d + y^3 = \frac{a^3y^3}{b^3}$

and $a^3y^3 - b^3y^3 = db^3$

$$y^3 = \frac{db^3}{a^3 - b^3}$$

$$y = \sqrt[3]{\frac{db^3}{a^3 - b^3}}$$

and $x = \sqrt[3]{\frac{da^3}{a^3 - b^3}}$.

DIRECTION V.

§ 82. "*If there are three unknown Quantities, there muſt be three Equations in order to determine them, by comparing which you may, in all Caſes, find two Equations involving only two unknown Quantities; and then, by* Direct. 3d, *from theſe two you may deduce an Equation involving only one unknown Quantity; which may be reſolved by the Rules of the laſt Chapter.*"

From 3 Equations involving any three unknown Quantities, *x*, *y*, and *z*, to deduce two Equations involving only two unknown Quantities, the following Rule will always ſerve.

RULE.

Linear equations — systematic practical methods

Gaussian elimination:

- *The nine chapters of the mathematical art*, China (c. 150 BC)
- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85
- C. F. Gauss: calculation of asteroid orbits (1810)
- from surveying, e.g., Wilhelm Jordan, *Handbuch der Vermessungskunde*, 3rd edition (1888)

Maclaurin and linear equations

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$

then $y=\frac{5\times80-3\times100}{5\times8-3\times7}=\frac{100}{19}=5\frac{5}{19}$

and $x=\frac{240}{19}=12\frac{12}{19}$.

EXAMPLE II.

$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$

$y=\frac{4\times160-3\times90}{4\times-2-3\times8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\frac{9}{16}$.

THEOREM II.

§87. Suppoſe now that there are three unknown Quantities and three Equations, then call the unknown Quantities x, y, and z. Thus,

$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$

Then ſhall $z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$.

Where the Numerator conſiſts of all the different Products that can be made of three oppoſite Coefficients taken from the Orders in which z is not found; and the Denominator conſiſts of all the Products that can be made of the three op-

Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

Three equations in three unknowns solved using a 'determinant-like' quantity

If four Equations are given, involving four unknown Quantities, their Values may be found much after the ſame Manner, by taking all the Products that can be made of four oppoſite Coefficients, and always prefixing contrary Signs to thoſe that involve the Products of two oppoſite Coefficients.

Notational difficulties — we run out of letters!

Determinants

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Determinants

Leibniz, unpublished works, 1680s/1690s.

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Leibniz and determinants

At least as early as June 1678, Leibniz devised a new notation for coefficients, writing

$$10 + 11x + 12y = 0,$$
$$20 + 21x + 22y = 0$$

for what we would write as

$$a_{10} + a_{11}x + a_{12}y = 0,$$
$$a_{20} + a_{21}x + a_{22}y = 0.$$

Leibniz used this notation to formulate general results on the solvability of systems of equations in terms of a determinant-like quantity (a sum of signed products of coefficients) — but these were not published during his lifetime

Determinants

Leibniz, unpublished works, 1680s/1690s.

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Determinants

Leibniz, unpublished works, 1680s/1690s.

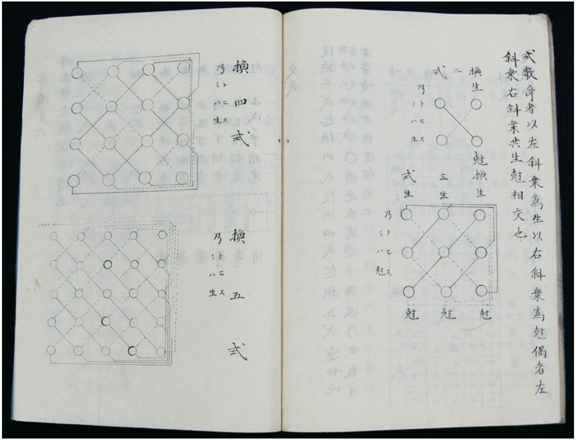
Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Seki and determinants

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法 (*Method for Solving Concealed Problems*), 1683

Arranged coefficients of systems of equations in a grid, and gave schematics for construction of determinants (dotted lines indicate positive products, and solid lines negative)



Determinants

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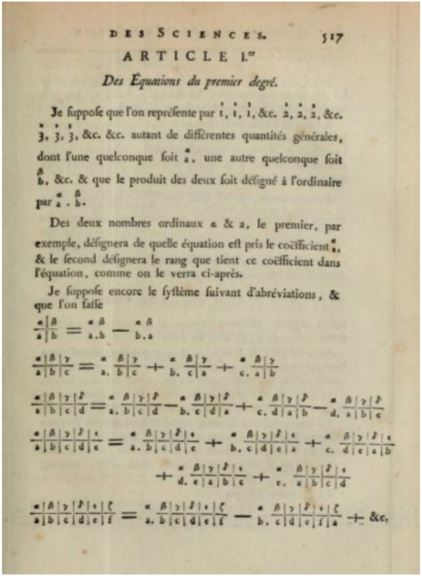
Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

Vandermonde on elimination



ARTICLE I.er

Des Équations du premier degré.

Je ſuppoſe que l'on repréſente par $\overset{1}{1}, \overset{2}{1}, \overset{3}{1}$, &c. $\overset{1}{2}, \overset{2}{2}, \overset{3}{2}$, &c. $\overset{1}{3}, \overset{2}{3}, \overset{3}{3}$, &c. &c. autant de différentes quantités générales, dont l'une quelconque ſoit $\overset{\alpha}{a}$, une autre quelconque ſoit $\overset{\beta}{b}$, &c. & que le produit des deux ſoit déſigné à l'ordinaire par $\overset{\alpha}{a} \cdot \overset{\beta}{b}$.

Des deux nombres ordinaux α & a, le premier, par exemple, déſignera de quelle équation eſt pris le coëfficient $\overset{\alpha}{a}$, & le ſecond déſignera le rang que tient ce coëfficient dans l'équation, comme on le verra ci-après.

Je ſuppoſe encore le ſyſtème ſuivant d'abréviations, & que l'on faſſe

$$\frac{\alpha|\beta}{a|b} = \overset{\alpha}{a}.\overset{\beta}{b} - \overset{\alpha}{b}.\overset{\beta}{a}$$

$$\frac{\alpha|\beta|\gamma}{a|b|c} = \overset{\alpha}{a}.\frac{\beta|\gamma}{b|c} + \overset{\alpha}{b}.\frac{\beta|\gamma}{c|a} + \overset{\alpha}{c}.\frac{\beta|\gamma}{a|b}$$

$$\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta}{b|c|d} - \overset{\alpha}{b}.\frac{\beta|\gamma|\delta}{c|d|a} + \overset{\alpha}{c}.\frac{\beta|\gamma|\delta}{d|a|b} - \overset{\alpha}{d}.\frac{\beta|\gamma|\delta}{a|b|c}$$

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon}{a|b|c|d|e} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta|\epsilon}{b|c|d|e} + \overset{\alpha}{b}.\frac{\beta|\gamma|\delta|\epsilon}{c|d|e|a} + \overset{\alpha}{c}.\frac{\beta|\gamma|\delta|\epsilon}{d|e|a|b} + \overset{\alpha}{d}.\frac{\beta|\gamma|\delta|\epsilon}{e|a|b|c} + \overset{\alpha}{e}.\frac{\beta|\gamma|\delta|\epsilon}{a|b|c|d}$$

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon|\zeta}{a|b|c|d|e|f} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta|\epsilon|\zeta}{b|c|d|e|f} - \overset{\alpha}{b}.\frac{\beta|\gamma|\delta|\epsilon|\zeta}{c|d|e|f|a} + \text{\&c.}$$

$\overset{\alpha}{a}$ denotes a single quantity, e.g., a coefficient in a linear equation

Define: $\dfrac{\alpha \,|\, \beta}{a \,|\, b} = \overset{\alpha}{a}\;\overset{\beta}{b} - \overset{\alpha}{b}\;\overset{\beta}{a}$

Anachronistically, this is the determinant of the matrix:

$$\begin{pmatrix} \overset{\alpha}{a} & \overset{\alpha}{b} \\ \overset{\beta}{a} & \overset{\beta}{b} \end{pmatrix}$$

Then continue recursively . . .

Determinants

Leibniz, unpublished works, 1680s/1690s.

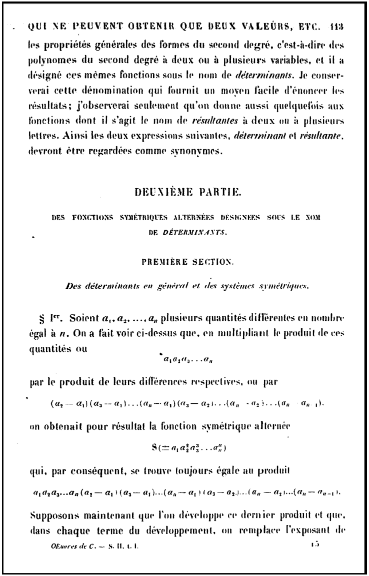
Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

Gauss in *Disquisitiones arithmeticae* (1801) gave the name 'determinant' to what is now called the 'discriminant' $B^2 - AC$ of the binary quadratic form $Ax^2 + 2Bxy + Cy^2$.

Cauchy on determinants


QUI NE PEUVENT OBTENIR QUE DEUX VALEURS, ETC. 113

les propriétés générales des formes du second degré, c'est-à-dire des polynomes du second degré à deux ou à plusieurs variables, et il a désigné ces mêmes fonctions sous le nom de *déterminants*. Je conserverai cette dénomination qui fournit un moyen facile d'énoncer les résultats; j'observerai seulement qu'on donne aussi quelquefois aux fonctions dont il s'agit le nom de *résultantes* à deux ou à plusieurs lettres. Ainsi les deux expressions suivantes, *déterminant* et *résultante*, devront être regardées comme synonymes.

DEUXIÈME PARTIE.

DES FONCTIONS SYMÉTRIQUES ALTERNÉES DÉSIGNÉES SOUS LE NOM DE *DÉTERMINANTS*.

PREMIÈRE SECTION.

Des déterminants en général et des systèmes symétriques.

§ I[er]. Soient $a_1, a_2, \ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus que, en multipliant le produit de ces quantités ou

$$a_1 a_2 a_3 \ldots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}),$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\pm a_1 a_2^2 a_3^3 \ldots a_n^n)$$

qui, par conséquent, se trouve toujours égale au produit

$$a_1 a_2 a_3 \ldots a_n (a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit et que, dans chaque terme du développement, on remplace l'exposant de

Œuvres de C. — S. II, t. I. 15

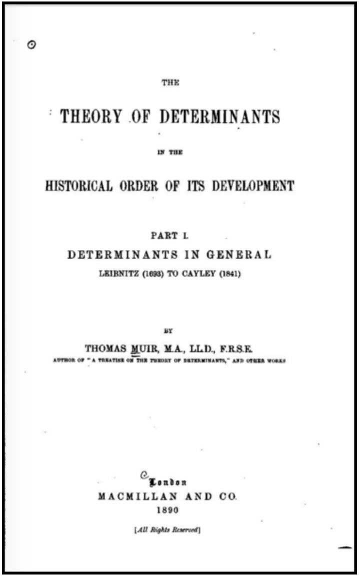
Cauchy, ‘Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu’elles renferment’, *Journal de l’École polytechnique*, 1815

Referred to Laplace, Vandermonde, Gauss, and others

Introduced the term determinant for the function of n^2 quantities (a sum of $n!$ signed products) that we now know by that name.

(See *Mathematics emerging*, §17.1.4.)

History of the theory of determinants


THE

THEORY OF DETERMINANTS

IN THE

HISTORICAL ORDER OF ITS DEVELOPMENT

PART I.

DETERMINANTS IN GENERAL

LEIBNITZ (1693) TO CAYLEY (1841)

BY

THOMAS MUIR, M.A., LL.D., F.R.S.E.

AUTHOR OF "A TREATISE ON THE THEORY OF DETERMINANTS," AND OTHER WORKS

London

MACMILLAN AND CO.

1890

[All Rights Reserved]

Determinants were studied extensively in the 19th century.

Sir Thomas Muir, *The theory of determinants in the historical order of development* (1890–1906)

- Part I: *Determinants in general: Leibnitz (1693) to Cayley (1841)*;
- Part II: *Special determinants up to 1841*

Second edition in 4 volumes, 1906–1923; supplement, 1930.

'Eigenvalue' problems

Euler (1748): change of coordinates to reduce equation of a quadric surface $\alpha z^2 + \beta yz + \gamma xz + \delta y^2 + \epsilon xy + \zeta x^2 + \eta z + \theta y + \iota x + \chi = 0$ to its simplest form $Ap^2 + Bq^2 + Cr^2 + K = 0$ (see: *Mathematics emerging*, §17.2.1.)

Laplace (1787): symmetry of coefficients in a set of linear differential equations leads to real 'eigenvalues' (see: *Mathematics emerging*, §17.2.2.)

Cauchy (1829): a symmetric matrix is diagonalisable by a real orthogonal change of variables (see: *Mathematics emerging*, §17.2.3.)

Matrices and their determinants

Gauss, *Disquisitiones arithmeticae* (1801): transformation of quadratic forms $ax^2 + 2bxy + cy^2$ by change of variables

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y'$$

followed by

$$x' = \alpha' x'' + \beta' y'', \quad y' = \gamma' x'' + \delta' y''$$

comes to the same as

$$x = (\alpha\alpha' + \beta\gamma')x'' + (\alpha\beta' + \beta\delta')y'', \quad y = (\gamma\alpha' + \delta\gamma')x'' + (\gamma\beta' + \delta\delta')y''$$

Moreover, the ‘determinants’ (our sense) multiply.

NB. All Gauss’ coefficients were integers

(See *Mathematics emerging*, §17.3.1.)

Early origins of matrices

The OED (3rd ed., March 2001) lists sense 2a of ‘matrix’ as

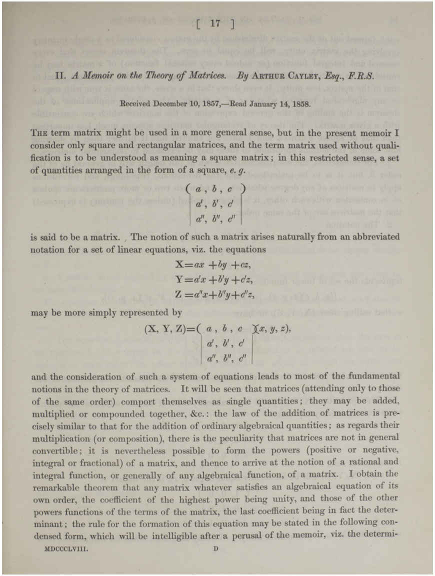
> *A place or medium in which something is originated, produced, or developed . . .*

Thus, in 1850, J. J. Sylvester applied the word to the ‘thing’ from which determinants originate:

> *For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of pth order.*

But he did not operate with matrices

The definition of matrices



II. *A Memoir on the Theory of Matrices.* By Arthur Cayley, *Esq., F.R.S.*

Received December 10, 1857,—Read January 14, 1858.

The term matrix might be used in a more general sense, but in the present memoir I consider only square and rectangular matrices, and the term matrix used without qualification is to be understood as meaning a square matrix; in this restricted sense, a set of quantities arranged in the form of a square, *e. g.*

$$\left(\begin{array}{ccc} a , & b , & c \\ a' , & b' , & c' \\ a'' , & b'' , & c'' \end{array}\right)$$

is said to be a matrix. The notion of such a matrix arises naturally from an abbreviated notation for a set of linear equations, viz. the equations

$$X = ax + by + cz,$$
$$Y = a'x + b'y + c'z,$$
$$Z = a''x + b''y + c''z,$$

may be more simply represented by

$$(X, Y, Z) = \left(\begin{array}{ccc} a , & b , & c \\ a' , & b' , & c' \\ a'' , & b'' , & c'' \end{array}\right)(x, y, z),$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices. It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities; they may be added, multiplied or compounded together, &c.: the law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix, and thence to arrive at the notion of a rational and integral function, or generally of any algebraical function, of a matrix. I obtain the remarkable theorem that any matrix whatever satisfies an algebraical equation of its own order, the coefficient of the highest power being unity, and those of the other powers functions of the terms of the matrix, the last coefficient being in fact the determinant; the rule for the formation of this equation may be stated in the following condensed form, which will be intelligible after a perusal of the memoir, viz. the determi-

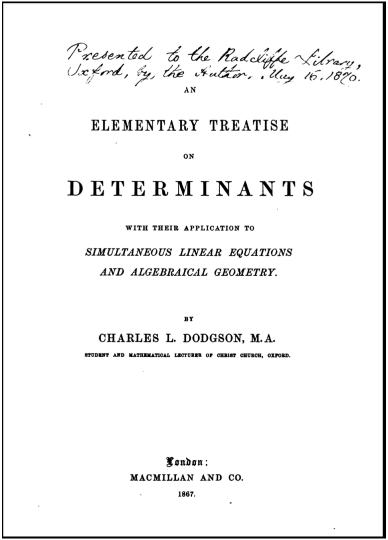
Arthur Cayley, 'A memoir on the theory of matrices', *Phil. Trans. Roy. Soc.*, 1858:

- defined matrices and their properties
- recognised connection to linear equations
- stated the Cayley–Hamilton Theorem
- investigated the matrices that commute with a given one

"It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities..."

(See *Mathematics emerging*, §17.3.2.)

Determinants persist


Presented to the Radcliffe Library, Oxford, by the Author. May 16. 1870.

AN

ELEMENTARY TREATISE

ON

DETERMINANTS

WITH THEIR APPLICATION TO

SIMULTANEOUS LINEAR EQUATIONS
AND ALGEBRAICAL GEOMETRY.

BY

CHARLES L. DODGSON, M.A.

STUDENT AND MATHEMATICAL LECTURER OF CHRIST CHURCH, OXFORD.

London:
MACMILLAN AND CO.
1867.

"I am aware that the word 'Matrix' is already in use to express the very meaning for which I use the word 'Block'; but surely the former word means rather the mould, or form, into which algebraical quantities may be introduced, than an actual assemblage of such quantities ..."

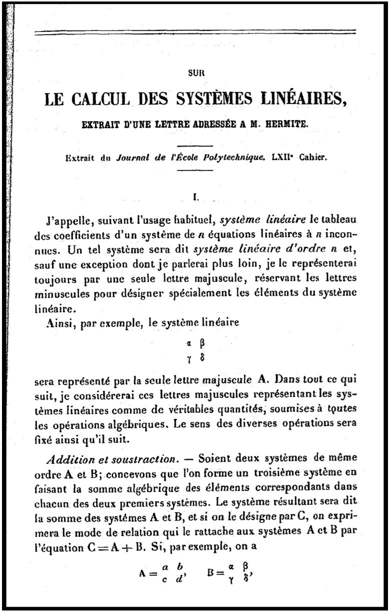
Criticised notation 'a_{ij}':
"it seems a fatal objection to this system that most of the space is occupied by a number of a's, which are wholly superfluous, while the only important part of the notation is reduced to minute subscripts, alike difficult to the writer and the reader."

Proposed $i \setminus j$ instead

Matrices elsewhere

Matrix algebra appears in Hamilton's *Lectures on Quaternions* (1853) as 'linear and vector functions' (including his version of the Cayley–Hamilton Theorem, stated and proved in terms of quaternions)

Matrices were also devised by Laguerre in his paper 'Sur le calcul des systèmes linéaires' (*J. École polytechnique*, 1867)



SUR

LE CALCUL DES SYSTÈMES LINÉAIRES,

EXTRAIT D'UNE LETTRE ADRESSÉE A M. HERMITE.

Extrait du *Journal de l'École Polytechnique*, LXII^{e} Cahier.

I.

J'appelle, suivant l'usage habituel, *système linéaire* le tableau des coefficients d'un système de n équations linéaires à n inconnues. Un tel système sera dit *système linéaire d'ordre* n et, sauf une exception dont je parlerai plus loin, je le représenterai toujours par une seule lettre majuscule, réservant les lettres minuscules pour désigner spécialement les éléments du système linéaire.

Ainsi, par exemple, le système linéaire

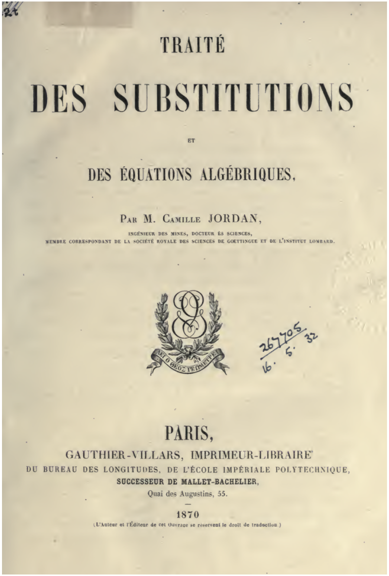
$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$$

sera représenté par la seule lettre majuscule A. Dans tout ce qui suit, je considérerai ces lettres majuscules représentant les systèmes linéaires comme de véritables quantités, soumises à toutes les opérations algébriques. Le sens des diverses opérations sera fixé ainsi qu'il suit.

Addition et soustraction. — Soient deux systèmes de même ordre A et B; concevons que l'on forme un troisième système en faisant la somme algébrique des éléments correspondants dans chacun des deux premiers systèmes. Le système résultant sera dit la somme des systèmes A et B, et si on le désigne par C, on exprimera le mode de relation qui le rattache aux systèmes A et B par l'équation $C = A + B$. Si, par exemple, on a

$$A = \begin{matrix} a & b \\ c & d \end{matrix}, \qquad B = \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix},$$

Jordan and linear substitutions



TRAITÉ

DES SUBSTITUTIONS

ET

DES ÉQUATIONS ALGÉBRIQUES,

PAR M. CAMILLE JORDAN,

INGÉNIEUR DES MINES, DOCTEUR ÈS SCIENCES,
MEMBRE CORRESPONDANT DE LA SOCIÉTÉ ROYALE DES SCIENCES DE GOETTINGUE ET DE L'INSTITUT LOMBARD.

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE,
SUCCESSEUR DE MALLET-BACHELIER,
Quai des Augustins, 55.

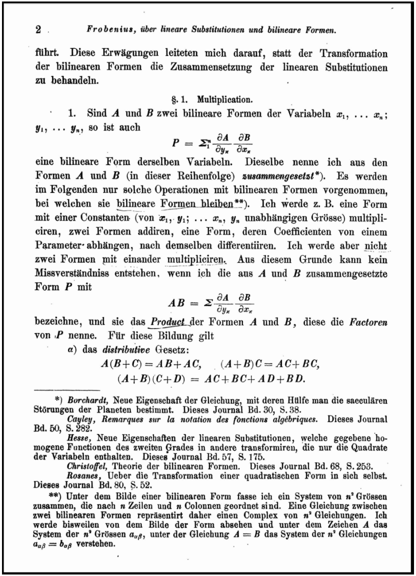
1870

(L'Auteur et l'Éditeur de cet Ouvrage se réservent le droit de traduction.)

Camille Jordan, *Traité des substitutions*, 1870:

- studied matrices over integers modulo n as part of an extensive study of linear substitutions (in connection with Galois theory); developed 'canonical forms' to study conjugacy classes in these groups
- developed his ideas to 'Jordan canonical form' for complex matrices in his studies 1872–4 of linear differential equations

German contributions



2 *Frobenius, über lineare Substitutionen und bilineare Formen.*

führt. Diese Erwägungen leiteten mich darauf, statt der Transformation der bilinearen Formen die Zusammensetzung der linearen Substitutionen zu behandeln.

§. 1. Multiplication.

1. Sind A und B zwei bilineare Formen der Variabeln $x_1, \dots x_n$; $y_1, \dots y_n$, so ist auch

$$P = \Sigma_1^n \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

eine bilineare Form derselben Variabeln. Dieselbe nenne ich aus den Formen A und B (in dieser Reihenfolge) *zusammengesetzt**). Es werden im Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben**). Ich werde z. B. eine Form mit einer Constanten (von $x_1, y_1; \dots x_n, y_n$ unabhängigen Grösse) multipliciren, zwei Formen addiren, eine Form, deren Coefficienten von einem Parameter abhängen, nach demselben differentiiren. Ich werde aber nicht zwei Formen mit einander multipliciren. Aus diesem Grunde kann kein Missverständniss entstehen, wenn ich die aus A und B zusammengesetzte Form P mit

$$AB = \Sigma \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

bezeichne, und sie das *Product* der Formen A und B, diese die *Factoren* von P nenne. Für diese Bildung gilt

α) das *distributive* Gesetz:

$$A(B+C) = AB+AC, \quad (A+B)C = AC+BC,$$
$$(A+B)(C+D) = AC+BC+AD+BD.$$

*) *Borchardt*, Neue Eigenschaft der Gleichung, mit deren Hülfe man die saeculären Störungen der Planeten bestimmt. Dieses Journal Bd. 30, S. 38.

Cayley, Remarques sur la notation des fonctions algébriques. Dieses Journal Bd. 50, S. 282.

Hesse, Neue Eigenschaften der linearen Substitutionen, welche gegebene homogene Functionen des zweiten Grades in andere transformiren, die nur die Quadrate der Variabeln enthalten. Dieses Journal Bd. 57, S. 175.

Christoffel, Theorie der bilinearen Formen. Dieses Journal Bd. 68, S. 253.

Rosanes, Ueber die Transformation einer quadratischen Form in sich selbst. Dieses Journal Bd. 80, S. 52.

**) Unter dem Bilde einer bilinearen Form fasse ich ein System von n^2 Grössen zusammen, die nach n Zeilen und n Colonnen geordnet sind. Eine Gleichung zwischen zwei bilinearen Formen repräsentirt daher einen Complex von n^2 Gleichungen. Ich werde bisweilen von dem Bilde der Form absehen und unter dem Zeichen A das System der n^2 Grössen $a_{\alpha\beta}$, unter der Gleichung $A = B$ das System der n^2 Gleichungen $a_{\alpha\beta} = b_{\alpha\beta}$ verstehen.

Georg Frobenius, in 1878, working with bilinear forms, produced more canonical forms, and gave a satisfactory proof of the Cayley–Hamilton Theorem

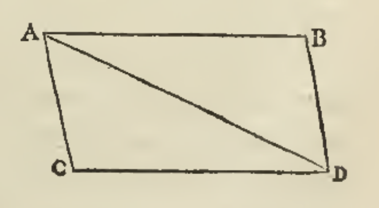
(See *Mathematics emerging*, §17.3.3.)

Other mathematicians in Germany (e.g., Kronecker, Hurwitz) contributed similarly

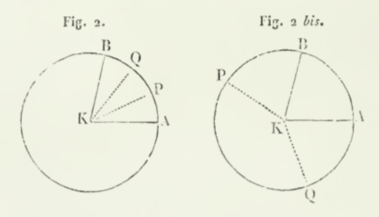
A recommended secondary source: Thomas Hawkins, 'Another look at Cayley and the theory of matrices', *Archives internationales d'histoire des sciences* **26** (1977), 82–112

Vectors

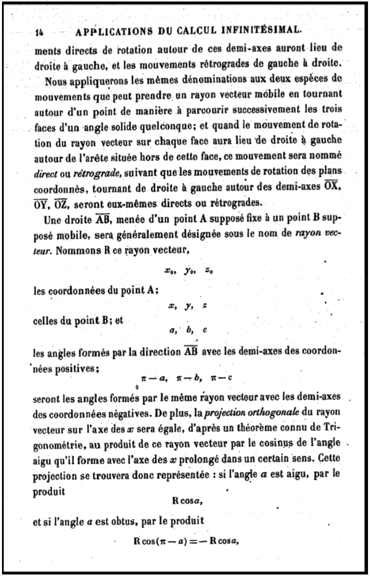
Newton (1687): parallelogram of forces



Argand (1806): complex numbers as directed quantities in the plane



Vectors


14 APPLICATIONS DU CALCUL INFINITÉSIMAL.

ments directs de rotation autour de ces demi-axes auront lieu de droite à gauche, et les mouvements rétrogrades de gauche à droite.

Nous appliquerons les mêmes dénominations aux deux espèces de mouvements que peut prendre un rayon vecteur mobile en tournant autour d'un point de manière à parcourir successivement les trois faces d'un angle solide quelconque; et quand le mouvement de rotation du rayon vecteur sur chaque face aura lieu de droite à gauche autour de l'arête située hors de cette face, ce mouvement sera nommé *direct* ou *rétrograde*, suivant que les mouvements de rotation des plans coordonnés, tournant de droite à gauche autour des demi-axes $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$, seront eux-mêmes directs ou rétrogrades.

Une droite $\overline{AB}$, menée d'un point A supposé fixe à un point B supposé mobile, sera généralement désignée sous le nom de *rayon vecteur*. Nommons R ce rayon vecteur,

$$x_0,\quad y_0,\quad z_0$$

les coordonnées du point A;

$$x,\quad y,\quad z$$

celles du point B; et

$$a,\quad b,\quad c$$

les angles formés par la direction $\overline{AB}$ avec les demi-axes des coordonnées positives;

$$\pi - a,\quad \pi - b,\quad \pi - c$$

seront les angles formés par le même rayon vecteur avec les demi-axes des coordonnées négatives. De plus, la *projection orthogonale* du rayon vecteur sur l'axe des x sera égale, d'après un théorème connu de Trigonométrie, au produit de ce rayon vecteur par le cosinus de l'angle aigu qu'il forme avec l'axe des x prolongé dans un certain sens. Cette projection se trouvera donc représentée : si l'angle a est aigu, par le produit

$$R\cos a,$$

et si l'angle a est obtus, par le produit

$$R\cos(\pi - a) = -R\cos a,$$

Word applied mostly to radius vectors

e.g., as rayon vecteur in Laplace's *Mécanique Céleste* (1799–1825)

Also in Cauchy's *Leçons sur les Applications du Calcul Infinitésimal à la Géométrie* (1826), p. 14:

A line $\overline{\mathrm{AB}}$*, taken from a point* A*, supposed to be fixed, to a moving point* B*, will in general be referred to as a radius vector.*

Hamilton and vectors

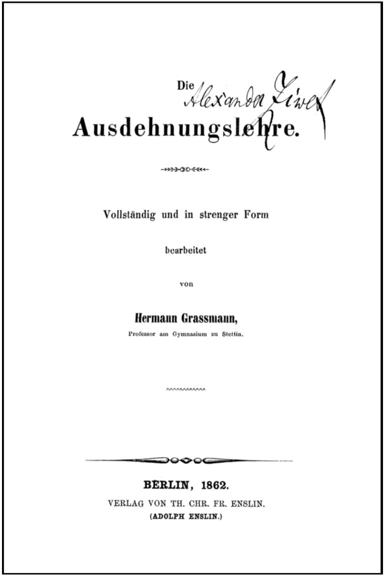
Sir William Rowan Hamilton drew a distinction between a ‘vector’ and a ‘radius vector’:

Between 1843–1866, developed quaternions — 4-dimensional quantities $a + bi + cj + dk$, where $i^2 = j^2 = k^2 = ijk = -1$, designed for use in mechanics (and geometry of 3 dimensions)

“*A* VECTOR *is thus ... a sort of* NATURAL TRIPLET *(suggested by Geometry): and accordingly we shall find that* QUATERNIONS *offer an easy mode of symbolically representing every vector by a* TRINOMIAL FORM *$(ix + jy + kz)$; which form brings the conception and expression of such a vector into the closest possible connexions with Cartesian and rectangular co-ordinates.*”

So a quaternion is a scalar + a vector

Vector spaces appear



Die

Ausdehnungslehre.

Vollständig und in strenger Form

bearbeitet

von

Hermann Grassmann,

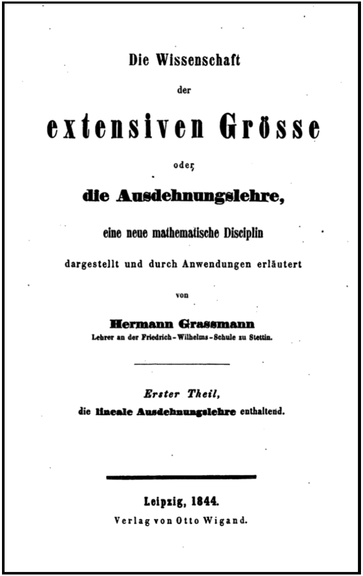
Professor am Gymnasium zu Stettin.

BERLIN, 1862.

VERLAG VON TH. CHR. FR. ENSLIN.

(ADOLPH ENSLIN.)

Grassmann's 'doctrine of extension'



Die Wissenschaft

der

extensiven Grösse

oder

die Ausdehnungslehre,

eine neue mathematische Disciplin

dargestellt und durch Anwendungen erläutert

von

Hermann Grassmann

Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

Erster Theil,

die lineale Ausdehnungslehre enthaltend.

Leipzig, 1844.

Verlag von Otto Wigand.

Die Ausdehnungslehre [*Doctrine of extension*] (1862) is a heavily reworked version of an earlier (1844) work:

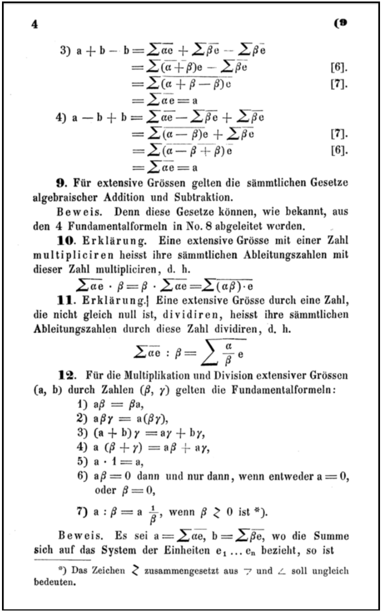
The Science of Extensive Quantities, or the Doctrine of Extension, a New Mathematical Discipline, Presented and Explained through Examples

Introduced idea of objects generated by motion — a single element generates an object of order 1, an object of order 1 generates an object of order 2, etc.

Objects of the same order can be added together or scaled by real numbers

Little impact at the time

Grassmann's 'extensive quantities'



4 (9

$$
\begin{aligned}
3)\ a + b - b &= \sum \overline{\alpha e} + \sum \overline{\beta e} - \sum \overline{\beta e} \\
&= \sum \overline{(\alpha + \beta) e} - \sum \overline{\beta e} && [6]. \\
&= \sum \overline{(\alpha + \beta - \beta) e} && [7]. \\
&= \sum \overline{\alpha e} = a \\
4)\ a - b + b &= \sum \overline{\alpha e} - \sum \overline{\beta e} + \sum \overline{\beta e} \\
&= \sum \overline{(\alpha - \beta) e} + \sum \overline{\beta e} && [7]. \\
&= \sum \overline{(\alpha - \beta + \beta) e} && [6]. \\
&= \sum \overline{\alpha e} = a
\end{aligned}
$$

9. Für extensive Grössen gelten die sämmtlichen Gesetze algebraischer Addition und Subtraktion.

B e w e i s. Denn diese Gesetze können, wie bekannt, aus den 4 Fundamentalformeln in No. 8 abgeleitet werden.

10. E r k l ä r u n g. Eine extensive Grösse mit einer Zahl m u l t i p l i c i r e n heisst ihre sämmtlichen Ableitungszahlen mit dieser Zahl multipliciren, d. h.

$$\sum \overline{\alpha e} \cdot \beta = \beta \cdot \sum \overline{\alpha e} = \sum \overline{(\alpha\beta) \cdot e}$$

11. E r k l ä r u n g.| Eine extensive Grösse durch eine Zahl, die nicht gleich null ist, d i v i d i r e n, heisst ihre sämmtlichen Ableitungszahlen durch diese Zahl dividiren, d. h.

$$\sum \overline{\alpha e} : \beta = \sum \overline{\frac{\alpha}{\beta} e}$$

12. Für die Multiplikation und Division extensiver Grössen (a, b) durch Zahlen (β, γ) gelten die Fundamentalformeln:

1) $a\beta = \beta a$,
2) $a\beta\gamma = a(\beta\gamma)$,
3) $(a + b)\gamma = a\gamma + b\gamma$,
4) $a(\beta + \gamma) = a\beta + a\gamma$,
5) $a \cdot 1 = a$,
6) $a\beta = 0$ dann und nur dann, wenn entweder $a = 0$, oder $\beta = 0$,
7) $a : \beta = a \frac{1}{\beta}$, wenn $\beta \gtrless 0$ ist *).

B e w e i s. Es sei $a = \sum \overline{\alpha e}$, $b = \sum \overline{\beta e}$, wo die Summe sich auf das System der Einheiten $e_1 \ldots e_n$ bezieht, so ist

*) Das Zeichen $\gtrless$ zusammengesetzt aus $\gt$ und $\lt$ soll ungleich bedeuten.

The 1862 text contains a theory of extensive quantities

$$a_1 e_1 + a_2 e_2 + \cdots,$$

where the e_i are 'units' and the a_i are real numbers, including

- rules for the arithmetic of such quantities
- a notion of linear independence
- dimension
- ...

But still had little impact

(See *Mathematics emerging*, §17.4.1.)

Vector spaces defined


CALCOLO

GEOMETRICO

secondo l'Ausdehnungslehre di H. Grassmann

PRECEDUTO DALLE

OPERAZIONI DELLA LOGICA DEDUTTIVA

PER

GIUSEPPE PEANO

PROFESSORE NELLA R. ACCADEMIA MILITARE
INCARICATO NELLA R. UNIVERSITÀ DI TORINO

TORINO
FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA

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Via del Corso, 216-217.
FIRENZE
Via Cerretani, 8

DEPOSITI
PALERMO
Università, 12
(N. Carosio)
NAPOLI
Piazza Plebiscito, 2
CATANIA
S. Maria al Ros.o, 23
(N. Carosio)

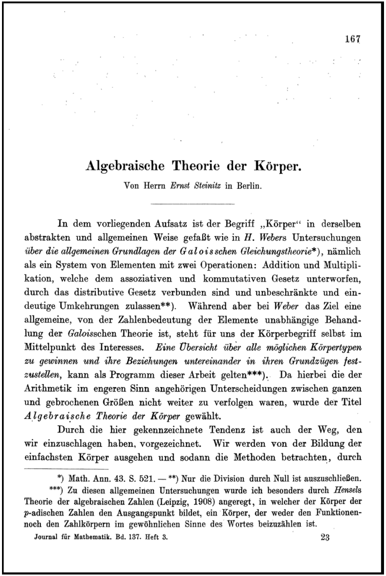
1888

On the way towards developing a 'geometric calculus', Guiseppe Peano axiomatised Grassmann's collections of extensive quantities as linear systems (sistemi lineari), and moved to a fully abstract setting

Clarified connection between dimension and linear independence — noted existence of linear systems with infinite dimension

Also no immediate impact!

Vector spaces develop



167

Algebraische Theorie der Körper.

Von Herrn *Ernst Steinitz* in Berlin.

In dem vorliegenden Aufsatz ist der Begriff „Körper" in derselben abstrakten und allgemeinen Weise gefaßt wie in *H. Webers* Untersuchungen *über die allgemeinen Grundlagen der Galoisschen Gleichungstheorie**), nämlich als ein System von Elementen mit zwei Operationen: Addition und Multiplikation, welche dem assoziativen und kommutativen Gesetz unterworfen, durch das distributive Gesetz verbunden sind und unbeschränkte und eindeutige Umkehrungen zulassen**). Während aber bei *Weber* das Ziel eine allgemeine, von der Zahlenbedeutung der Elemente unabhängige Behandlung der *Galois*schen Theorie ist, steht für uns der Körperbegriff selbst im Mittelpunkt des Interesses. *Eine Übersicht über alle möglichen Körpertypen zu gewinnen und ihre Beziehungen untereinander in ihren Grundzügen festzustellen*, kann als Programm dieser Arbeit gelten***). Da hierbei die der Arithmetik im engeren Sinn angehörigen Unterscheidungen zwischen ganzen und gebrochenen Größen nicht weiter zu verfolgen waren, wurde der Titel *Algebraische Theorie der Körper* gewählt.

Durch die hier gekennzeichnete Tendenz ist auch der Weg, den wir einzuschlagen haben, vorgezeichnet. Wir werden von der Bildung der einfachsten Körper ausgehen und sodann die Methoden betrachten, durch

*) Math. Ann. 43. S. 521. — **) Nur die Division durch Null ist auszuschließen.

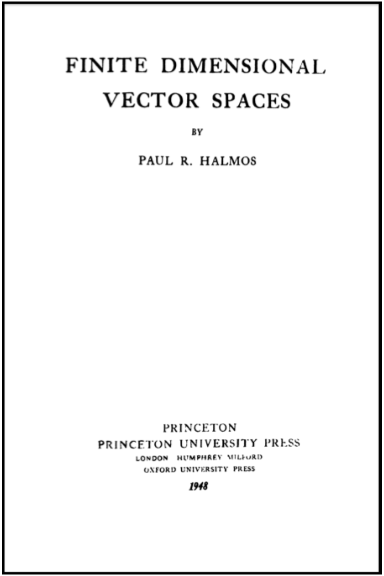
***) Zu diesen allgemeinen Untersuchungen wurde ich besonders durch *Hensels* Theorie der algebraischen Zahlen (Leipzig, 1908) angeregt, in welcher der Körper der *p*-adischen Zahlen den Ausgangspunkt bildet, ein Körper, der weder den Funktionen- noch den Zahlkörpern im gewöhnlichen Sinne des Wortes beizuzählen ist.

Journal für Mathematik. Bd. 137. Heft 3. 23

Dedekind (1879): fields and 'modules' needed for algebraic number theory in famous appendices to his third edition of Dirichlet, *Vorlesungen über Zahlentheorie* [*Lectures on number theory*]; published also separately in France, 1876–77

Ernst Steinitz (1910), 'Algebraische Theorie der Körper' ['Algebraic theory of fields'] — contains a beautifully crystallised theory of linear dependence and independence, bases, dimension, etc., in the form it is now taught

Vector spaces develop


FINITE DIMENSIONAL
VECTOR SPACES
BY
PAUL R. HALMOS

PRINCETON
PRINCETON UNIVERSITY PRESS
LONDON HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
1948

B. L. van der Waerden (1930–31), *Moderne Algebra*, incorporating material from lectures by Emil Artin and Emmy Noether (1926–1928)

Paul Halmos (1942), *Finite-dimensional vector spaces* — made the subject accessible to 1st and 2nd year undergraduates