C4.3 Functional Analytic Methods for PDEs Lectures 7-8

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MT 2022

In the last lectures

- Definition of Sobolev spaces
- Differentiation rule for convolution of Sobolev functions.

This lecture

- Density results for Sobolev spaces.
- Extension theorems for Sobolev functions.
- Trace (boundary value) of Sobolev functions.
- Gagliardo-Nirenberg-Sobolev's inequality

Approximation of identity in Sobolev spaces

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in W^{k,p}(\mathbb{R}^n)$ for some $k \geq 0$ and $1 \leq p < \infty$, then $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{W^{k,p}(\mathbb{R}^n)} = 0.$$

In particular $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Approximation of identity in Sobolev spaces

Proof

- Let $f_\varepsilon = f * \varrho_\varepsilon$.
  - ⋆ As $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.
  - ⋆ As $f \in L^p(\mathbb{R}^n)$ and $\varrho_\varepsilon \in L^1(\mathbb{R}^n)$, Young's inequality gives that $f_\varepsilon \in L^p(\mathbb{R}^n)$.
  - ⋆ The approximation of identity theorem in L^p gives that $\|f_\varepsilon - f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- By the differentiation rule for convolution of Sobolev functions, we have $\partial^\alpha f_\varepsilon = (\partial^\alpha f) * \varrho_\varepsilon$ for $|\alpha| \leq k$. Repeat the argument as above, we have $\partial^\alpha f_\varepsilon \in L^p(\mathbb{R}^n)$ and $\|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- We deduce that $f_\varepsilon \in W^{k,p}(\mathbb{R}^n)$ and

$$\|f_\varepsilon - f\|_{W^{k,p}} = \Big[\sum_{|\alpha|\leq k} \|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p}^p\Big]^{1/p} \xrightarrow{\varepsilon\to 0} 0.$$

Meyers-Serrin's theorem

Theorem (Meyers-Serrin)

Suppose Ω is a domain in $\mathbb{R}^n$, $k \geq 0$ and $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. Namely, for every $u \in W^{k,p}(\Omega)$ there exists a sequence $(u_m) \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that u_m converges to u in $W^{k,p}(\Omega)$.

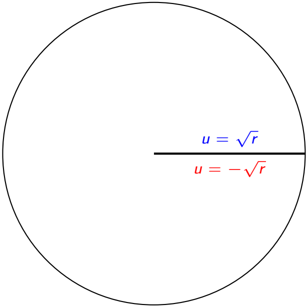
Remark: No regularity on Ω is assumed.

A question and an obstruction

Question

Is $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ dense in $W^{k,p}(\Omega)$?

Answer: Not always.



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$

$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

Consider $u(x,y) = \sqrt{r}\cos\frac{\theta}{2}$ where $(x,y) = (r\cos\theta, r\sin\theta)$.

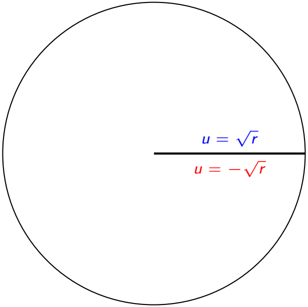
$u \in C^\infty(\Omega)$.

u is discontinuous in $\bar{\Omega}$.

One computes

$$\|u\|_{L^2}^2 = \int_\Omega u^2\,dx\,dy = \int_0^1 \int_0^{2\pi} r\cos^2\frac{\theta}{2}\,r\,dr\,d\theta = \frac{\pi}{3},$$

A question and an obstruction



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$
$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$
$D = \{x^2 + y^2 < 1\}$

Consider $u(x,y) = \sqrt{r} \cos \frac{\theta}{2}$.
$u \in C^\infty(\Omega)$ and $u \notin C(\bar{\Omega})$.
One computes $\|u\|_{L^2}^2 = \frac{\pi}{3}$,

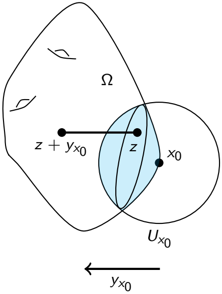
$$|\nabla u|^2 = (\partial_r u)^2 + \frac{1}{r^2}(\partial_\theta u)^2 = \frac{1}{4r},$$

$$\|\nabla u\|_{L^2}^2 = \int_\Omega |\nabla u|^2 \, dx \, dy$$
$$= \int_0^1 \int_0^{2\pi} \frac{1}{4r} \, r \, dr \, d\theta = \frac{\pi}{2},$$

So $u \in W^{1,2}(\Omega)$.
The jump discontinuity across $\theta = 0$ is an obstruction to approximate u by functions in $C^\infty(\bar{\Omega})$. It is in fact not possible, as $u \notin W^{1,2}(D)$.

The segment condition

- Ω: a domain in $\mathbb{R}^n$.
- Ω is said to satisfy <u>the segment condition</u> if every $x_0 \in \partial\Omega$ has a neighborhood U_{x_0} and a non-zero vector y_{x_0} such that if $z \in \bar{\Omega} \cap U_{x_0}$, then $z + ty_{x_0} \in \Omega$ for all $t \in (0, 1)$.



- Note that if Ω is Lipschitz, then it satisfies the segment condition.

Approximation by functions in $C^\infty(\bar{\Omega})$

Theorem (Global approximation by functions smooth up to the boundary)

Suppose $k \geq 1$ and $1 \leq p < \infty$. If Ω satisfies the segment condition, then the set of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\Omega)$. In particular $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

- An important consequence of the theorem is the statement that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ when $1 \leq p < \infty$. In order words $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$.
- You will do the special when Ω is star-shaped in Sheet 2.

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Lemma

Assume that $k \geq 0$ and $1 \leq p < \infty$. If $u \in W_0^{k,p}(\Omega)$, then its extension by zero $\bar{u}$ to $\mathbb{R}^n$ belongs to $W_0^{k,p}(\mathbb{R}^n)$.

Proof

- Suppose $u \in W_0^{k,p}(\Omega)$ and let $\bar{u}$ be its extension by zero to $\mathbb{R}^n$. It is tempted to say that, as $\bar{u} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$,

$$\partial^\alpha \bar{u} = \begin{cases} \partial^\alpha u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{*}$$

  which belongs to $L^p(\mathbb{R}^n)$, and call it the end of the proof. For this to work, we need to show first that $\bar{u}$ is weakly differentiable!

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Proof

- Let $(u_m) \subset C_c^\infty(\Omega)$ be such that $u_m \to u$ in $W^{k,p}(\Omega)$. Let $\bar{u}_m$ be the extension by zero of u_m to $\mathbb{R}^n$. Then $\bar{u}_m \in C_c^\infty(\mathbb{R}^n)$ and

$$\|\bar{u}_m - \bar{u}_j\|_{W^{k,p}(\mathbb{R}^n)} = \|u_m - u_j\|_{W^{k,p}(\Omega)} \xrightarrow{m,j\to\infty} 0.$$

- We thus have that $(\bar{u}_m)$ is Cauchy in $W^{k,p}(\mathbb{R}^n)$ and thus converges in $W^{k,p}$ to some $\bar{u}_* \in W^{k,p}(\mathbb{R}^n)$.
- To conclude, we show that $\bar{u}_* = \bar{u}$ a.e. in $\mathbb{R}^n$.
  - ⋆ As $\bar{u}_m$ converges to $\bar{u}_*$ in $L^p(\mathbb{R}^n)$, there is a subsequence $\bar{u}_{m_j}$ which converges a.e. to $\bar{u}_*$ in $\mathbb{R}^n$. This implies that $\bar{u}_* = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and u_{m_j} converges a.e. to $\bar{u}_*$ in Ω.
  - ⋆ Likewise, as u_{m_j} converges to u in $L^p(\Omega)$, we can extract yet another subsequence $u_{m_{j_l}}$ which converges a.e. to u in Ω. It follows that $\bar{u}_* = u$ a.e. in Ω.
  - ⋆ So $\bar{u} = \bar{u}_*$ a.e. in $\mathbb{R}^n$.

More on extension

Theorem (Stein's extension theorem)

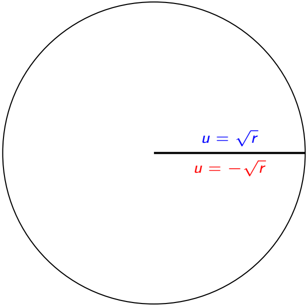
Assume that Ω is a bounded Lipschitz domain. Then there exists a linear operator E sending functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ such that for every $k \geq 0$, $1 \leq p < \infty$ and $u \in W^{k,p}(\Omega)$ it hold that $Eu = u$ a.e. in Ω and

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,p,\Omega}\|u\|_{W^{k,p}(\Omega)}$$

The operator E is called a total extension for Ω.
You will have the opportunity to see how to construct such extension in a very specific case in Sheet 2.

More on extension

- There exists domain Ω for which there is no bounded linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ such that $Eu = u$ a.e. in Ω.



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$

$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

$D = \{x^2 + y^2 < 1\}$

We knew that the function $u(x,y) = \sqrt{r} \cos \frac{\theta}{2}$ satisfies

- ⋆ $u \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$.
- ⋆ $u \notin W^{1,2}(D)$.

So no extension of u belongs to $W^{1,2}(\mathbb{R}^2)$.

Values of Sobolev functions on the boundary

- As prompted at the beginning of the course, in our later applications in the analysis of PDEs, solutions will live in a Sobolev space.
- When discussing PDEs on a domain, one needs to specify boundary conditions.
- A complication arises:
  - ⋆ On one hand, Sobolev 'functions' are equivalent classes of functions which are equal almost everywhere. Thus one can redefine the value of a Sobolev function on set of measure zero at will without changing the equivalent class it represents.
  - ⋆ On the other hand, the boundary of a domain usually has measure zero. So the boundary value of a Sobolev function cannot simply be defined by restricting as is the case for continuous functions.

Values of Sobolev functions on the boundary

Remark

Suppose $1 \le p < \infty$, Ω is a bounded smooth domain and let $(X, \|\cdot\|)$ be a normed vector space which contains $C(\partial\Omega)$. There is NO <u>*bounded*</u> *linear operator $T : L^p(\Omega) \to X$ such that $Tu = u|_{\partial\Omega}$ for all $u \in C(\bar{\Omega})$.*

Proof

- Suppose by contradiction that such T exists. Consider $f_m \in C(\bar{\Omega})$ defined by

$$f_m(x) = \begin{cases} m\, dist(x, \partial\Omega) & \text{if } dist(x, \partial\Omega) < 1/m, \\ 1 & \text{if } dist(x, \partial\Omega) \ge 1/m. \end{cases}$$

- Then $\|f_m - 1\|^p_{L^p(\Omega)} \le |\{dist(x, \partial\Omega) < 1/m\}| \le \frac{C}{m}$ and so $f_m \to 1$ in $L^p(\Omega)$.
- Now as $Tf_m = 0 \not\to 1 = T1$ in X, T cannot be bounded.

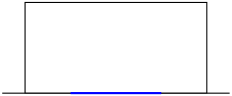
Values of Sobolev functions on the boundary

Theorem

Suppose $1 \leq p < \infty$, *and that* Ω *is a bounded Lipschitz domain. Then there exists a* *__bounded__* *linear operator* $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, *called the trace operator, such that* $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.

We will only proof a weaker statement in a simpler situation:

$\Omega = \{x = (x', x_n) : |x'| < 2, \ 0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$

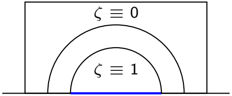
$\Gamma = \{x = (x', 0) : |x'| < 1\}$

We would like to define the trace operator relative to Γ: There exists a bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ such that

$$T_\Gamma u = u|_\Gamma \text{ for all } u \in C^1(\bar{\Omega}).$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2,$
$0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$
$\Gamma = \{x = (x', 0) : |x'| < 1\}$

$0 \leq \zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1

- We first prove the key estimate

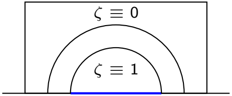
$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

  - ⋆ We have

$$\int_\Gamma |u|^p \, dx' \leq \int_{\hat{\Gamma}} \zeta |u|^p \, dx' = -\int_{\hat{\Gamma}} \Big[\int_0^2 \partial_{x_n}(\zeta |u|^p) \, dx_n \Big] \, dx'$$
$$= -\int_\Omega \partial_{x_n}(\zeta |u|^p) \, dx \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx.$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2, 0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$
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$\zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1.

- We first prove the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

⋆ We have $\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx$.

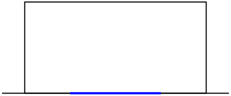
⋆ Using the inequality $|a||b|^{p-1} \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^p$, we obtain

$$\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du|^p] \, dx$$

This proves (*).

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2,$
$0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$
$\Gamma = \{x = (x', 0) : |x'| < 1\}$

- We have proved the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

- It follows that the map $u \mapsto u|_\Gamma =: Au$ is a bounded linear operator from $(C^1(\bar{\Omega}), \|\cdot\|_{W^{1,p}})$ into $L^p(\Gamma)$.
- As Ω is Lipschitz, $C^\infty(\bar{\Omega})$ and hence $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists a unique bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ which extends A, i.e. $T_\Gamma u = u|_\Gamma$ for all $u \in C^1(\bar{\Omega})$.

IBP formula revisited

Proposition (Integration by parts)

Suppose that $1 \le p < \infty$, Ω is a bounded Lipschitz domain, n be the outward unit normal to $\partial\Omega$, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then

$$\int_\Omega \partial_i u \, v \, dx = \int_{\partial\Omega} Tu \, v \, n_i \, dS - \int_\Omega u \, \partial_i v \, dx \text{ for all } v \in C^1(\bar{\Omega}).$$

Proof

- We knew that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{1,p}$.
- Fix some $v \in C^1(\bar{\Omega})$. We have

$$\int_\Omega \partial_i u_m \, v \, dx = \int_{\partial\Omega} u_m \, v \, n_i \, dS - \int_\Omega u_m \, \partial_i v \, dx.$$

IBP formula revisited

Proof

- $\int_\Omega \partial_i u_m \, v \, dx = \int_{\partial\Omega} u_m \, v \, n_i \, dS - \int_\Omega u_m \, \partial_i v \, dx$.
- Note that $\partial_i u_m \to \partial_i u$, $u_m \to u$ in $L^p(\Omega)$ and $u_m|_{\partial\Omega} = Tu_m \to Tu$ in $L^p(\partial\Omega)$. We can thus argue using Hölder's inequality to send $m \to \infty$ to obtain

$$\int_\Omega \partial_i u \, v \, dx = \int_{\partial\Omega} Tu \, v \, n_i \, dS - \int_\Omega u \, \partial_i v \, dx$$

as wanted.

Functions of zero trace

Theorem (Trace-zero functions in $W^{1,p}$)

Suppose that $1 \leq p < \infty$, Ω is a bounded Lipschitz domain, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$.

Proof

- $(\Rightarrow)$ Suppose $u \in W_0^{1,p}(\Omega)$. By definition, there exists $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{1,p}$. Clearly $Tu_m = 0$ and so by continuity, $Tu = 0$.
- $(\Leftarrow)$ We will only consider the case Ω is the unit ball B. This proof can be generalised fairly quickly to star-shaped domains. The proof for Lipschitz domains is more challenging.

Functions of zero trace

Proof

- ($\Leftarrow$) Suppose that $u \in W^{1,p}(B)$ and $Tu = 0$. We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Let $\bar{u}$ be the extension by zero of u to $\mathbb{R}^n$.
  - ⋆ As $Tu = 0$, we have by the IBP formula that

$$\int_B \partial_i u \, v \, dx = -\int_B u \, \partial_i v \, dx \text{ for all } v \in C^1(\bar{B}).$$

It follows that

$$\int_B \partial_i u \, v \, dx = -\int_B \bar{u} \, \partial_i v \, dx \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

By definition of weak derivatives, this means

$$\partial_i \bar{u} = \begin{cases} \partial_i u & \text{in } B \\ 0 & \text{elsewhere} \end{cases} \quad \text{in the weak sense.}$$

So $\bar{u} \in W^{1,p}(\mathbb{R}^n)$.

Functions of zero trace

Proof

- ($\Leftarrow$) We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}(B)$.
  - ⋆ Let $\bar{u}_\lambda(x) = \bar{u}(\lambda x)$. Observe that $Supp(\bar{u}_\lambda) \subset B_{1/\lambda}$.
  - ⋆ In Sheet 1, you showed that $\bar{u}_\lambda \to \bar{u}$ in L^p as $\lambda \to 1$.
    Noting also that $\partial_i \bar{u}_\lambda(x) = \lambda \partial_i u(\lambda x)$, we also have that $\partial_i \bar{u}_\lambda \to \partial_i \bar{u}$ in L^p as $\lambda \to 1$.
    Hence $\bar{u}_\lambda \to \bar{u}$ in $W^{1,p}$ as $\lambda \to 1$.
  - ⋆ Fix $\lambda_m > 1$ such that $\|\bar{u}_{\lambda_m} - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - ⋆ Let (ϱ_ε) be a family of mollifiers: $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$ with $\varrho \in C_c^\infty(B)$, $\int_{\mathbb{R}^n} \varrho = 1$. Then $\bar{u}_{\lambda_m} * \varrho_\varepsilon \to \bar{u}_{\lambda_m}$ in $W^{1,p}$ as $\varepsilon \to 0$. Also, $Supp(\bar{u}_{\lambda_m} * \varrho_\varepsilon) \subset B_{\lambda_m^{-1}+\varepsilon}$. Thus, we can select ε_m sufficiently small such that $u_m := \bar{u}_{\lambda_m} * \varrho_{\varepsilon_m} \in C_c^\infty(B)$ and $\|u_m - \bar{u}_{\lambda_m}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - ⋆ Now $\|u_m - u\|_{W^{1,p}(B)} \leq 2/m$ and so we are done.

Embeddings

Let X_1 and X_2 be two Banach spaces.

- We say X_1 is embedded in X_2 if $X_1 \subset X_2$.
- We say X_1 is continuously embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a bounded linear operator, i.e. there exists a constant C such that $\|x\|_{X_2} \leq C\|x\|_{X_1}$. We write $X_1 \hookrightarrow X_2$.
- We say X_1 is compactly embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a compact bounded linear operator. This means that I is continuous and every bounded sequence $(x_n) \subset X_1$ has a subsequence which is convergent with respect to the norm on X_2.

Our interest: The possibility of embedding $W^{k,p}$ in L^q or C^0.

Gagliardo-Nirenberg-Sobolev's inequality

Theorem (Gagliardo-Nirenberg-Sobolev's inequality)

Assume $1 \leq p < n$ and let $p^* = \frac{np}{n-p}$. Then there exists a constant $C_{n,p}$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p}\|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in W^{1,p}(\mathbb{R}^n).$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

The number $p^* = \frac{np}{n-p}$ is called the Sobolev conjugate of p. It satisfies $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.
The case $p = 1$ is referred to as Gagliardo-Nirenberg's inequality.

GNS's inequality – Why $p < n$ and why p^*?

Question

For what p and q does it hold

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n)? \qquad (*)$$

This will be answered by a scaling argument:

- Fix a non-zero function $u \in C_c^\infty(\mathbb{R}^n)$. Define $u_\lambda(x) = u(\lambda x)$. Then $u_\lambda \in C_c^\infty(\mathbb{R}^n)$ and so

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute

$$\|u_\lambda\|_{L^q}^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \, dy = \lambda^{-n} \|u\|_{L^q}^q.$$

GNS's inequality – Why $p < n$ and why p^*?

- $u_\lambda(x) = u(\lambda x)$ and

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q}\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute $\|u_\lambda\|_{L^q} = \lambda^{-n/q}\|u\|_{L^q}$.
- Next,

$$\|\nabla u_\lambda\|_{L^p}^p = \int_{\mathbb{R}^n} |\lambda \nabla u(\lambda x)|^p \, dx$$
$$= \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy = \lambda^{p-n}\|\nabla u\|_{L^p}^p.$$

That is $\|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p}\|\nabla u\|_{L^p}$.

GNS's inequality – Why $p < n$ and why p^*?

- Putting in (**), we get

$$\lambda^{-n/q}\|u\|_{L^q} \leq C_{n,p,q}\lambda^{1-n/p}\|\nabla u\|_{L^p}.$$

Rearranging, we have

$$\lambda^{-1+\frac{n}{p}-\frac{n}{q}} \leq \frac{C_{n,p,q}\|\nabla u\|_{L^p}}{\|u\|_{L^q}}.$$

- Since the last inequality is valid for all λ, we must have that $-1 + \frac{n}{p} - \frac{n}{q} = 0$, i.e. $q = \frac{np}{n-p} = p^*$. As $q > 0$, we must also have $p \leq n$.
- We conclude that a <u>necessary</u> condition in order for the inequality (*) to hold is that $p \leq n$ and $q = p^*$.

Proof of GNS's inequality

- Recall that we would like to show, for $1 \leq p < n$ and $p^* = \frac{np}{n-p}$ that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in W^{1,p}(\mathbb{R}^n). \qquad (\#)$$

- Claim 1: If $(\#)$ holds for functions in $C_c^\infty(\mathbb{R}^n)$, then it holds for functions in $W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Take an arbitrary $u \in W^{1,p}(\mathbb{R}^n)$. As $p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Hence, we can select $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ If $(\#)$ holds for functions in $C_c^\infty(\mathbb{R}^n)$, then $\|u_m\|_{L^{p^*}} \leq C_{n,p} \|\nabla u_m\|_{L^p}$.
  - ⋆ As $u_m \to u$ in $W^{1,p}$, we have $\partial_i u_m \to \partial_i u$ in L^p and so $\|\nabla u_m\|_{L^p} \to \|\nabla u\|_{L^p}$.
  - ⋆ Warning: It is tempted to try to show $\|u_m\|_{L^{p^*}} \to \|u\|_{L^{p^*}}$. However, this is false in general.

Proof of GNS's inequality

- Proof of Claim 1:
  - ⋆ $\|u_m\|_{L^{p^*}} \leq C_{n,p} \|\nabla u_m\|_{L^p}$.
  - ⋆ $\|\nabla u_m\|_{L^p} \to \|\nabla u\|_{L^p}$.
  - ⋆ As $u_m \to u$ in $W^{1,p}$, we have $u_m \to u$ in L^p, and so, we can extract a subsequence (u_{m_j}) which converges a.e. in $\mathbb{R}^n$ to u. By Fatou's lemma, we have

$$\int_{\mathbb{R}^n} |u|^{p^*}\,dx \leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} |u_{m_j}|^{p^*}\,dx.$$

  - ⋆ So

$$\|u\|_{L^{p^*}} \leq \liminf_{j\to\infty} \|u_{m_j}\|_{L^{p^*}} \leq C_{n,p} \liminf_{j\to\infty} \|\nabla u_{m_j}\|_{L^p} = C_{n,p} \|\nabla u\|_{L^p}.$$

    So (#) holds.

Proof of GNS's inequality

- Claim 2: If (#) holds for $p = 1$, then it holds for all $1 < p < n$.
  - ⋆ Take an arbitrary non-trivial $u \in C_c^\infty(\mathbb{R}^n)$ and consider the function $v = |u|^\gamma$ with $\gamma > 1$ to be fixed. Clearly $v \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
  - ⋆ In Sheet 3, you will show that $|u|$ is weakly differentiable and

$$\nabla|u| = \begin{cases} \nabla u & \text{in } \{x : u(x) > 0\}, \\ -\nabla u & \text{in } \{x : u(x) < 0\}, \\ 0 & \text{in } \{x : u(x) = 0\}. \end{cases}$$

  - ⋆ It follows that $\nabla v = \gamma |u|^{\gamma-1} \nabla|u| \in L^1(\mathbb{R}^n)$. So $v \in W^{1,1}(\mathbb{R}^n)$.
  - ⋆ Applying (#) in $W^{1,1}$ we get $\|v\|_{L^{\frac{n}{n-1}}} \leq C_n \|\nabla v\|_{L^1}$.
  - ⋆ On the left side, we have

$$\|v\|_{L^{\frac{n}{n-1}}} = \Big\{ \int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}}\, dx \Big\}^{\frac{n-1}{n}} = \|u\|^\gamma_{L^{\frac{n\gamma}{n-1}}}.$$

Proof of GNS's inequality

- Claim 2: If (#) holds for $p = 1$, then it holds for all $1 < p < n$.
  - ⋆ $\|v\|_{L^{\frac{n}{n-1}}} \leq C_n \|\nabla v\|_{L^1}$.
  - ⋆ On the left side, we have $\|v\|_{L^{\frac{n}{n-1}}} = \|u\|^{\gamma}_{L^{\frac{n\gamma}{n-1}}}$.
  - ⋆ On the right side, we use the inequality $|\nabla |u|| \leq |\nabla u|$ and compute using Hölder's inequality:

$$\|\nabla v\|_{L^1} \leq \int_{\mathbb{R}^n} \gamma |u|^{\gamma-1} |\nabla u|\,dx \leq \gamma \Big\{ \int_{\mathbb{R}^n} |u|^{(\gamma-1)p'}\,dx \Big\}^{\frac{1}{p'}} \Big\{ \int_{\mathbb{R}^n} |\nabla u|^p\,dx \Big\}^{\frac{1}{p}}$$
$$= \gamma \|u\|^{\gamma-1}_{L^{(\gamma-1)p'}} \|\nabla u\|_{L^p}.$$

  - ⋆ Now we select γ such that $(\gamma-1)p' = \frac{n\gamma}{n-1}$, i.e. $\gamma = \frac{(n-1)p}{n-p}$ and obtain

$$\|u\|^{\gamma}_{L^{p^*}} \leq C_n \gamma \|u\|^{\gamma-1}_{L^{p^*}} \|\nabla u\|_{L^p}.$$

    As $u \not\equiv 0$, we can divide both side by $\|u\|^{\gamma-1}_{L^{p^*}}$, and conclude Step 2.

Proof of GNS's inequality

- In view of Claim 1 and Claim 2, it thus remains to show GNS's inequality for smooth functions when $p = 1$. To better present the idea of the proof, I will only give the proof when $n = 2$, i.e.

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\diamondsuit)$$

(The case $n \geq 3$ can be dealt with in the same way (check this!).)

  - ⋆ The starting point is the estimate

$$|u(x)| = \Big| \int_{-\infty}^{x_1} \partial_{x_1} u(y_1, x_2)\, dy_1 \Big| \leq \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1.$$

Likewise,

$$|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\, dy_2.$$

Proof of GNS's inequality

- We are proving

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\diamondsuit)$$

  - ⋆ We have $|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1$ and $|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\, dy_2$.
  - ⋆ Multiplying the two inequalities gives

$$|u(x_1, x_2)|^2 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1 \Big\}\Big\{ \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\, dy_2 \Big\}.$$

  - ⋆ Now note that the first integral on the right hand side is independent of x_1, and if one integrates the second integral on the right hand side with respect to x_1, one gets $\|\nabla u\|_{L^1}$. Thus, by integrating both side in x_1, we get

$$\int_{-\infty}^{\infty} |u(x_1, x_2)|^2\, dx_1 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1 \Big\} \|\nabla u\|_{L^1}.$$

Proof of GNS's inequality

- We are proving

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\diamondsuit)$$

  - ⋆ We have shown

$$\int_{-\infty}^{\infty} |u(x_1, x_2)|^2\, dx_1 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1 \Big\} \|\nabla u\|_{L^1}$$

By the same line of argument, integrating the above in x_2 gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1, x_2)|^2\, dx_1\, dx_2 \leq \|\nabla u\|_{L^1}^2,$$

which gives exactly $(\diamondsuit)$ with $C = 1$.

An improved Gagliardo-Nirenberg's inequality

Remark

By inspection, note that when $p = 1$, we actually prove the following slightly stronger inequality:

$$\|u\|^n_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^n \|\partial_i u\|_{L^1(\mathbb{R}^n)}.$$

GNS's inequality for bounded domains

Theorem (Gagliardo-Nirenberg-Sobolev's inequality)

Assume that Ω is a bounded Lipschitz domain and $1 \leq p < n$. Then, for every $q \in [1, p^*]$, there exists $C_{n,p,q,\Omega}$ such that

$$\|u\|_{L^q(\Omega)} \leq C_{n,p,q,\Omega} \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

In particular, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Proof

- Let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then

$$\|u\|_{L^{p*}(\Omega)} \leq \|Eu\|_{L^{p*}(\mathbb{R}^n)} \leq C_{n,p} \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_{n,p} \|u\|_{W^{1,p}(\Omega)}.$$

- By Hölder inequality, we have $\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)} |\Omega|^{\frac{1}{q} - \frac{1}{p^*}}$.
- We conclude the proof with $C_{n,p,q,\Omega} = C_{n,p} |\Omega|^{\frac{1}{q} - \frac{1}{p^*}}$.

GNS's inequality – Can $p = n$?

- Consider now the case $p = n$. Does it hold that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_n \|\nabla u\|_{L^n(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n)? \qquad (\dagger)$$

  ⋆ When $n = 1$, this is true as

$$|u(x)| = \Big| \int_{-\infty}^{x} u'(s)\, ds \Big| \leq \int_{-\infty}^{\infty} |u'(s)|\, ds = \|u'\|_{L^1(\mathbb{R})}.$$

  ⋆ We next show that $(\dagger)$ does not hold when $n \geq 2$.

GNS's inequality – Can $p = n$?

- We know that if (†) holds then $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. Thus it suffices to exhibit a function $u \in W^{1,n}(\mathbb{R}^n) \setminus L^\infty(\mathbb{R}^n)$.
- It is enough to find $f \in W^{1,n}(B_2) \setminus L^\infty(B_1)$. The desired u then takes the form $u = f\zeta$ for any $\zeta \in C_c^\infty(B_2)$ with $\zeta \equiv 1$ in B_1.
- We impose that f is rotationally symmetric so that $f(x) = f(|x|) = f(r)$. Then we need to find a function $f : (0,2) \to \mathbb{R}$ such that

$$\int_0^2 [|f|^n + |f'|^n] \, r^{n-1} \, dr < \infty \text{ but } \operatorname*{ess\,sup}_{(0,1)} |f| = \infty.$$

GNS's inequality – Can $p = n$?

- Then we need to find a function $f : (0,2) \to \mathbb{R}$ such that

$$\int_0^2 \left[|f|^n + |f'|^n\right] r^{n-1}\, dr < \infty \text{ but } \operatorname*{ess\,sup}_{(0,1)} |f| = \infty.$$

- The fact that $|f'|^n r^{n-1}$ is integrable implies that, near $r = 0$, f' is 'smaller' than $\frac{1}{r}$, so f is 'smaller' than $\ln r$.
- If we try $f = (\ln \frac{4}{r})^\alpha$, then $|f'|^n r^{n-1} = \frac{\alpha^n}{r} (\ln \frac{4}{r})^{n(\alpha-1)}$ is integrable for $\alpha \leq \frac{n-1}{n}$. Also, $|f|^n r^{n-1}$ is continuous in $[0,2]$ and hence integrable. So $f \in W^{1,n}(B_2)$ when $\alpha \leq \frac{n-1}{n}$.
- On the other hand, if $\alpha > 0$, then $\operatorname{ess\,sup}_{(0,1)} |f| = \infty$.

Trudinger's inequality

Theorem (Trudinger's inequality)

There exists a small constant $c_n > 0$ *and a large constant* $C_n > 0$ *such that if* $u \in W^{1,n}(\mathbb{R}^n)$, *then* $\exp\left[\left(\frac{c_n|u|}{\|u\|_{W^{1,n}(\mathbb{R}^n)}}\right)^{\frac{n}{n-1}}\right] \in L^1_{loc}(\mathbb{R}^n)$ *and*

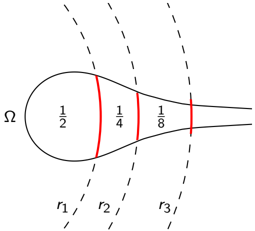
$$\sup_{x_0\in\mathbb{R}^n} \int_{B_1(x_0)} \exp\left[\left(\frac{c_n|u|}{\|u\|_{W^{1,n}(\mathbb{R}^n)}}\right)^{\frac{n}{n-1}}\right] dx \leq C_n.$$

A non-embedding theorem for unbounded domains

Fact

Suppose $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be an unbounded domain with finite volume. Then $W^{1,p}(\Omega)$ does not embed into $L^q(\Omega)$ whenever $q > p$.

Ideas



- We may assume $|\Omega| = 1$. We need to construct a function $f \in W^{1,p}(\Omega) \setminus L^q(\Omega)$.
- Let $r_0 = 0$ and select r_k such that $\Omega_k := \Omega \cap \{r_k \leq |x| < r_{k+1}\}$ has volume $\frac{1}{2^{k+1}}$.

A non-embedding theorem for unbounded domains

Sketch of proof

- The function f will be of the form $f(x) = f(|x|)$ which is increasing in $|x|$. If we let $b_k = f(r_k)$, then

$$\|f\|_{L^p}^p = \sum_k \int_{\Omega_k} |f|^p \, dx \leq \sum_k b_{k+1}^p |\Omega_k| = \sum_k b_{k+1}^p 2^{-k-1}.$$

  Likewise, $\|f\|_{L^q}^q \geq \sum_k b_k^q 2^{-k-1}$.

- To make $\|f\|_{L^q} = \infty$, we then require that $b_k = 2^{k/q}$ infinitely many times.
  If we also impose that $b_k \leq 2^{k/q}$ for all k, then

$$\|f\|_{L^p}^p \leq \sum_k 2^{-k(1-\frac{p}{q})} < \infty.$$

A non-embedding theorem for unbounded domains

Sketch of proof

- $b_k = 2^{k/q}$ infinitely many times $\Rightarrow \|f\|_{L^q} = \infty$,
  $b_k \leq 2^{k/q}$ for all $k \Rightarrow \|f\|_{L^p} < \infty$.
- Consider now $\|\nabla f\|_{L^p}$.
  - ⋆ On each Ω_k, we can arrange so that $|\nabla f| \sim \frac{b_{k+1} - b_k}{r_{k+1} - r_k}$.
  - ⋆ It is important to note that, for any fixed $\varepsilon > 0$, the inequality that $r_{k+1} - r_k > 2^{-\varepsilon k}$ must hold infinitely frequently. (As otherwise, $r_k \not\to \infty$.) Label them as $k_1 < k_2 < \dots$
  - ⋆ In Ω_{k_j}, we have $|\nabla f| \sim \frac{b_{k_j+1} - b_{k_j}}{r_{k_j+1} - r_{k_j}} \leq 2^{k_j(1/q+\varepsilon)}$.
  - ⋆ In Ω_k with $k \neq k_j$, we control $|\nabla f|$ by imposing $b_{k+1} = b_k$ so that $|\nabla f| = 0$.
  - ⋆ To meet the requirement in the first bullet point, we ask $b_{k_j} = 2^{k_j/q}$.

A non-embedding theorem for unbounded domains

Sketch of proof

- $\|f\|_{L^q} = \infty$ and $\|f\|_{L^p} < \infty$.
- Consider $\|\nabla f\|_{L^p}$.
  - ⋆ Putting things together, we have

$$\|\nabla f\|_{L^p}^p = \sum_j \int_{\Omega_{k_j}} |\nabla f|^p \, dx$$
$$\leq \sum_j 2^{k_j(1/q+\varepsilon)p} 2^{-k_j-1} \leq \sum_j 2^{-k_j(1-\frac{p}{q}-\varepsilon p)}.$$

Choosing $\varepsilon < \frac{1}{p} - \frac{1}{q}$, we see that this sum is finite.

- We conclude that $f \in W^{1,p}(\Omega)$ but $f \notin L^q(\Omega)$.