Infinite Groups

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About Mathematics

Mathematics knows no physical bounds.

It allows us to push forever outward in our exploration, taking the measure of objects and phenomena far removed from our immediate grasp.

Using a well, a sundial and geometry Erathostenes could measure the circumference of the Earth with an error of 50 miles, in 230 BC.

Example of ping-pong producing a free group

Example

For any real number $r \geqslant 2$ the matrices

$$g_1 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

generate a free subgroup of $SL(2, \mathbb{R})$.

The group $SL(2, \mathbb{R})$ acts on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ by linear fractional transformations

$$z \mapsto \frac{az + b}{cz + d}\,.$$

$g_1(z) = z + r,\ g_2(z) = \frac{z}{rz+1}$.

$I(z) = -\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z,\ g_2 = I \circ g_1^{-1} \circ I^{-1}$.

Why is $\langle g_1, g_2 \rangle$ free.

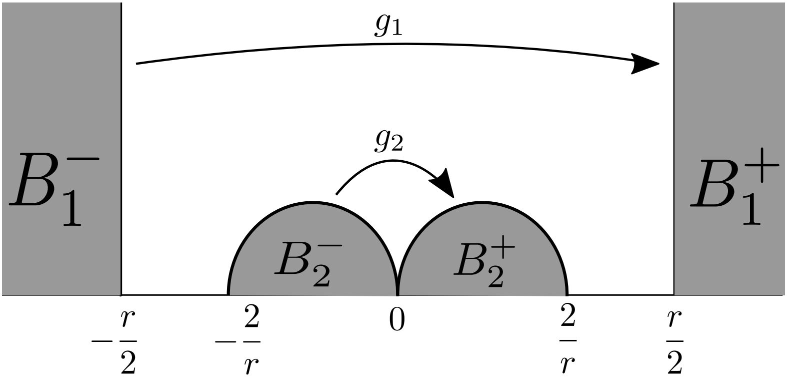
Define quarter-planes

$$B_1^+ = \{z \in \mathbb{H}^2 : \Re(z) \geqslant r/2\}, \quad B_1^- = \{z \in \mathbb{H}^2 : \Re(z) \leqslant -r/2\}$$

and half-disks

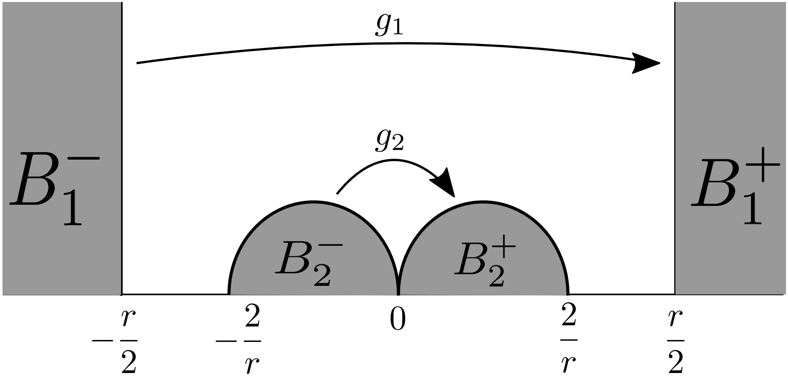
$$B_2^+ := \left\{ z \in \mathbb{H}^2 : |z - \frac{1}{r}| \leqslant \frac{1}{r} \right\}, \quad B_2^- := \left\{ z \in \mathbb{H}^2 : |z + \frac{1}{r}| \leqslant \frac{1}{r} \right\}.$$

Example of ping-pong.



$g_1(\mathbb{H}^2 \setminus B_1^-) \subset B_1^+$, $g_1^{-1}(\mathbb{H}^2 \setminus B_1^+) \subset B_1^-$.
$g_2 = I \circ g_1^{-1} \circ I^{-1}$. $I(B_1^+) = B_2^-$ and $I(B_1^-) = B_2^+$. Therefore
$g_2(\mathbb{H}^2 \setminus B_2^-) \subset B_2^+$ $g_2^{-1}(\mathbb{H}^2 \setminus B_2^+) \subset B_2^-$.

Example of ping-pong.



Take a reduced word in $\{g_1^{\pm 1}, g_2^{\pm 1}\}$, say $g_1 g_2^{-1} g_1 g_2$.

$B_2^+ \sqcup B_1^- \sqcup B_1^+ \xrightarrow{g_2} B_2^+ \xrightarrow{g_1} B_1^+ \xrightarrow{g_2^{-1}} B_2^- \xrightarrow{g_1} B_1^+$

General Ping-pong Lemma

Let $g_1, g_2 \in Bij(X)$ ("ping-pong partners") and $B_i^{\pm} \subset X, \quad i = 1, 2.$

Given $i \in \{1, 2\}$, let j be such that $\{i, j\} = \{1, 2\}$.
Define

$$C_i^+ := B_i^+ \cup B_j^- \cup B_j^+, C_i^- := B_i^- \cup B_j^- \cup B_j^+ .$$

Assume that:
$C_i^{\pm} \not\subset B_j^{\pm}$ and $C_i^{\pm} \not\subset B_j^{\mp}$ for all choices of i, j and $+, -$.

Theorem (Ping–pong, or table–tennis, lemma)

If

$$g_i^{\pm 1}(C_i^{\pm}) \subset B_i^{\pm}, \quad i = 1, 2,$$

then the bijections g_1, g_2 generate a free subgroup of $Bij(X)$.

Proof of General Ping-pong Lemma

Let w be a non-empty reduced word in $\{g_1, g_1^{-1}, g_2, g_2^{-1}\}$, of length at least 2.

w has the form

$$w = g_i^{\pm 1} u g_j^{\pm 1}.$$

We prove by induction on the length of w that

$$w(C_j^{\pm}) \subset B_i^{\pm}, \text{ hence } w \neq \mathrm{id}.$$

Length 2. $w = g_i^{\pm 1} g_j^{\pm 1}$.

$$C_j^{\pm} \xrightarrow{g_j^{\pm 1}} B_j^{\pm} \xrightarrow{g_i^{\pm 1}} B_i^{\pm}$$

The last transformation is true because the word is reduced, hence $B_j^{\pm} \neq B_i^{\mp}$, hence $B_j^{\pm}$ is contained in $C_i^{\pm}$.

Proof of General Ping-pong Lemma 2

Suppose it is true for all words w' of length n, we prove it for words w of length $n+1$.

Such a w has the form

$$w = g_i^{\pm 1} w', \quad \mathrm{length}(w') = n.$$

In its turn w' can be written as

$$w' = g_j^{\pm 1} u g_k^{\pm 1}, \quad g_j^{\pm 1} \neq g_i^{\mp 1}.$$

By the induction hypothesis

$$w'(C_k^{\pm}) \subset B_j^{\pm}.$$

Since $g_j^{\pm 1} \neq g_i^{\mp 1}$, we have that $B_j^{\pm} \neq B_i^{\mp}$,therefore $B_j^{\pm} \subset C_i^{\pm}$ and

$$w(C_k^{\pm})) = g_i^{\pm 1} w'(C_k^{\pm}) \subset g_i^{\pm 1}(C_i^{\pm}) \subset B_i^{\pm}.$$

Cayley graphs

Goal: Endow a group with a geometry, so first of all a metric.
Let $G = \langle S \rangle$, with $1 \notin S$ and $s^{-1} \in S$ for every $s \in S$.
We write the latter condition as $S^{-1} = S$.
The Cayley graph of G with respect to S is the directed/oriented graph $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ with

- set of vertices G;
- set of oriented edges (g, gs), with $s \in S$.

We label the oriented edge (g, gs) by s.
The underlying non-oriented graph $\mathrm{Cayley}(G, S)$ of $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ is the graph with

- set of vertices G;
- set of edges $\{g, h\}$ such that $h = gs$, with $s \in S$.

It is also called the Cayley graph of G with respect to S.
Occasionally we will use the notation $\overline{gh}$ and $[g, h]$ for the edge $\{g, h\}$.

Cayley graphs 2

- The definition of the graph makes sense for every $S \subset G$.
- $1 \notin S$ prevents edges from composing loops (monogons).
- $S^{-1} = S$ ensures that every edge in $\mathrm{Cayley}(G, S)$ appears in $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ with both orientations.
- By definition $\mathrm{Cayley}(G, S)$ is a simplicial graph if $1 \notin S$ (i.e. no monogons, no two edges with same endpoints).
- The valency of every vertex g in $\mathrm{Cayley}(G, S)$ (i.e. number of edges having g as an endpoint) is $k = \mathrm{card}(S)$.Thus $\mathrm{Cayley}(G, S)$ is k-regular (all vertices of same valency k).

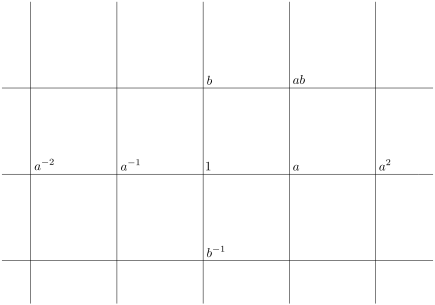
Lemma

$\mathrm{Cayley}(G, S)$ *is connected (i.e. every two vertices can be joined by an edge path) if and only if* S *generates* G.

Cayley graphs of $\mathbb{Z}^2$

Example

Consider $\mathbb{Z}^2$ *and* $S = \{a = (1,0), b = (0,1), a^{-1}, b^{-1}\}$.

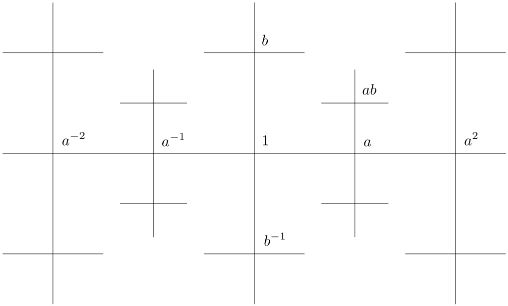


The Cayley graph of $\mathbb{Z}^2$ with respect to $\{\pm(1,0), \pm(1,1)\}$ has the same set of vertices as the above, but the vertical lines are replaced by diagonal lines.

Cayley graph of a free non-abelian group

Example

Let G be the free group on two generators a, b. Take $X = \{a, b\}$, $G = F(X)$ and $S = X \sqcup X^{-1}$. The Cayley graph $\mathrm{Cayley}(G, S)$ is the 4-valent tree.



A tree is a simplicial connected graph with no circuits. A k-valent tree is a k-regular tree.

Word metric

Convention

When talking about Cayley graphs, the groups G is always assumed to be finitely generated, and S is always assumed to be finite.

We endow $\mathrm{Cayley}(G, S)$ with a distance such that edges have length 1.

$$\mathrm{dist}_S(x, y) = \text{length of the shortest path joining } x, y.$$

The restriction of dist_S to $G \times G$ is called word metric.

Exercise

Prove that for every $g, h \in G$, $\mathrm{dist}_S(g, h)$ is the length k of the shortest word $w = s_1 \dots s_k$, where $s_i \in S, \forall i$, such that $g = hw$.

Word metric 2

Notation

- *We denote by* $|g|_S$ *the distance* $\mathrm{dist}_S(1, g)$, *that is the shortest word in* S *representing* g.
- *We denote by* $B_S(x, r)$ *the closed ball centred in* $x \in \mathrm{Cayley}(G, S)$ *and of radius* $r > 0$ *with respect to* dist_S.

Proposition

The action of G *on itself by multiplications to the left is an action by isometries, that is for every* $g \in G$

$$\mathrm{dist}(gx, gy) = \mathrm{dist}_S(x, y), \forall x, y \in G.$$

It extends to an action by isometries on $\mathrm{Cayley}(G, S)$

Word metric 3

Exercise

1. *Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that*

$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_{\bar{S}}(g, g') \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \quad (1)$$

2. *Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.*

Proposition

A finite index subgroup of a finitely generated group is finitely generated.

Proof. Let $G = \langle S \rangle$, S finite, and let $H \leqslant G$.

Finite index subgroup

We can write $G = H \sqcup \bigsqcup_{i=1}^{k} Hg_i$ for some elements $g_i \in G$.

Let $R = \max_{1 \leqslant i \leqslant k} |g_i|_S$.

$G = HB(1, R) \Leftrightarrow \forall g \in G, \exists h \in H$ such that $\mathrm{dist}_S(g, h) \leqslant R$.

We prove that $X = H \cap B(1, 2R + 1)$ is a generating set of H.

$\forall h \in H$, let $g_0 = 1, g_1, \ldots, g_n = h$ be consecutive vertices on a shortest path in $\mathrm{Cayley}(G, S)$ joining 1 and h.

For every $1 \leqslant i \leqslant n - 1$ there exist $h_i \in H$ such that $\mathrm{dist}_S(g_i, h_i) \leqslant R$.Set $h_0 = 1$ and $h_n = h$.

$\mathrm{dist}_S(h_i, h_{i+1}) \leqslant 2R + 1$, hence $h_{i+1} = h_i x_i$ for some $x_i \in X$, for every $0 \leqslant i \leqslant n - 1$.

It follows that $h = h_n = x_1 x_2 \cdots x_n$, whence X generates H. □