Infinite Groups

Cornelia Druțu

University of Oxford

Part C course MT 2022

Free group= Cayley graph a tree

Proposition

The group G is a free group generated by Y if and only if $\mathrm{Cayley}(G; Y)$ is a tree.

Proof of "only if". Assume $\mathrm{Cayley}(G; Y)$ contains a circuit.
We choose an orientation on the circuit and read the label on the thus oriented circuit: it is a word $s_1 \dots s_k$ equal to 1 in G.
The word is then not reduced: for some i we have $s_{i+1} = s_i^{-1}$.
Thus the oriented circuit contains two consecutive edges $e_i = (x_i, y_i)$ and $e_{i+1} = (y_i, x_i)$, contradiction.
Proof of "if". $\mathrm{Cayley}(G; Y)$ connected $\Rightarrow G = \langle Y \rangle$.
By Universal Property there exists $\varphi : F(Y) \to G$ onto homomorphism.
Assume there exists $s_1 \dots s_k$ reduced word in Y, equal to 1 in G.
Assume k is minimal. A path labeled by $s_1 \dots s_k$ in $\mathrm{Cayley}_{\mathrm{dir}}(G; Y)$ is a loop. Minimality of k implies it is simple.

Group acting freely on a tree

Corollary

A free group acts freely on a tree.

free action = the stabilizer of every point is $\{1\}$.

The converse of the above is also true, that is

Theorem

A group is free if and only if it acts freely on a simplicial tree.

The proof of this is part of the "Geometric Group Theory" course.

Corollary

Every subgroup of a free group is free.

Rank of a free group

We mention a few other results, without proof.

Proposition

$F(X)$ is isomorphic to $F(Y)$ if and only if $\mathrm{card}(X) = \mathrm{card}(Y)$.

Notation

We denote by F_n the group $F(X)$ with $\mathrm{card}(X) = n$*, unique up to isomorphism by the above.*

Proposition

The rank of $F(X)$ is $\mathrm{card}(X)$.

NB $F(X) \leqslant F(Y)$ does not imply $\mathrm{card}(X) \leqslant \mathrm{card}(Y)$.

Exercise

Every free group of countable rank can be embedded as a subgroup of F_2 (Exercise 7, Sheet 1).

Presentations of groups

How to fully describe a group?

- Table of multiplication if G is finite;
- Free groups.

Answer in general case: by generators and relations.

Example

$\mathbb{Z}^2$ is the group generated by two elements a, b satisfying the relation

$$ab = ba \Leftrightarrow [a, b] = 1.$$

We write $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1 \rangle$ or simply $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$.

Presentations of groups 2

In general let $G = \langle S \rangle$. By Universal property, $\exists$ an onto homomorphism

$$\pi_S : F(S) \to G$$

whence G isomorphic to $F(S)/\ker(\pi_S)$.
The elements of $\ker(\pi_S)$ are called relators or relations for G and the generating set S.
We are interested in minimal subsets R of $\ker(\pi_S)$ such that $\ker(\pi_S)$ is normally generated by R.
$N \lhd G$ is normally generated by $R \subset N$ (or N normal closure of R) if one of the following equivalent properties is satisfied:

- N is the smallest normal subgroup of G containing R;
- $N = \bigcap_{R \subset K \lhd G} K$;
- $N = \{r_1^{x_1} \cdots r_n^{x_n} \mid n \in \mathbb{N}, r_i \in R \cup R^{-1}, x_i \in G\} \cup \{1\}$.

Notation

$a^b = bab^{-1}$, $A^B = \{a^b \mid a \in A, b \in B\}$. $N = \langle\langle R \rangle\rangle$.

Presentation of groups 3

Let $R \subset \ker(\pi_S)$ be such that $\ker(\pi_S) = \langle\langle R \rangle\rangle$.
We say that the elements $r \in R$ are defining relators.
The pair (S, R) defines a presentation of G.
We write $G = \langle S \mid r = 1, \forall r \in R \rangle$ or simply $G = \langle S \mid R \rangle$.
Formally, it means G is isomorphic to $F(S)/\langle\langle R \rangle\rangle$.
Equivalently:

- $\forall g \in G$, $g = s_1 \cdots s_n$, for some $n \in \mathbb{N}$ and $s \in S \cup S^{-1}$;
- $w \in F(S)$ satisfies $w =_G 1$ if and only if in $F(S)$

$$w = \prod_{i=1}^{m} r_i^{x_i}, \text{ for some } m \in \mathbb{N}, r_i \in R, x_i \in F(S).$$

Examples of group presentations

1. $\langle a_1, \ldots, a_n \mid [a_i, a_j], 1 \leqslant i, j \leqslant n \rangle$ is a finite presentation of $\mathbb{Z}^n$;
2. $\langle x, y \mid x^n, y^2, yxyx \rangle$ is a presentation of the finite dihedral group D_{2n};
3. $\langle x, y \mid x^3, y^2, [y, x] \rangle$ is a presentation of the cyclic group $\mathbb{Z}_6$.
4. the Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \ldots, x_n, y_1, \ldots, y_n, z\,;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n \rangle.$$

The Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \dots, x_n, y_1, \dots, y_n, z\,;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n\rangle.$$

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \; ; \; x_i, y_j, z \in \mathbb{Z} \right\}$$

Generalization of the Universal Property

Proposition

Let $G = \langle S|R\rangle$. Let H be a group and $\psi : S \to H$ be a map s.t. $\psi(r) = 1$ for every $r \in R$.
Then ψ has an unique extension to a group homomorphism $\Phi : G \to H$.

Proof Universal property of free groups $\Rightarrow \psi$ extends to $\tilde{\psi} : F(S) \to H$.

$\langle\langle R\rangle\rangle = \langle R^{F(S)}\rangle$ is generated by elements of the form grg^{-1}, where $g \in F(S), r \in R$.
$\tilde{\psi}(grg^{-1}) = 1 \Rightarrow \langle\langle R\rangle\rangle \leqslant \ker(\tilde{\psi}) \Rightarrow \tilde{\psi}$ defines $\Phi : F(S)/\langle\langle R\rangle\rangle \to H$.

Uniqueness: because every homomorphism is entirely determined by its restriction to a generating set. □

Finite presentability is independent of the generating set

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

Proof. $\forall s \in S$ $\exists a_s(X)$ word in X s.t. $s = a_s(X)$ in G. (involves a choice)
The map $i_{SX} : S \to F(X)$, $i_{SX}(s) = a_s(X)$ extends to a unique homomorphism $p : F(S) \to F(X)$ (rewriting homomorphism).
We have that $\pi_S = \pi_X \circ p$.
Likewise, $\forall x \in X$ $\exists b_x(S)$ in S s.t. $x = b_x(S)$.
The map $i_{XS} : X \to F(S)$, $i_{XS}(x) = b_x(S)$, extends to homomorphism $q : F(X) \to F(S)$ (another rewriting homomorphism).
As previously $\pi_S \circ q = \pi_X$.

Finite presentability is independent of the generating set 2

For every $x \in X$,

$$\pi_X(p(q(x))) = \pi_S(q(x)) = \pi_X(x).$$

Whence for every $x \in X$, $x^{-1}p(q(x)) \in \ker(\pi_X)$.
Let N be the normal subgroup of $F(X)$ normally generated by

$$\{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

We have that $N \leqslant \ker(\pi_X)$. Goal: prove equality.
There is a natural projection

$$Q : F(X)/N \to F(X)/\ker(\pi_X).$$