Infinite Groups

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Mathematics

The Ancient Greeks believed that the world was ordered following a mathematical pattern, and that by discovering new theorems truths about the universe would be unlocked.

"Mathematics" comes from the Greek "mathematike", derived from "mathanein" which means "learn".

Galileo:

The Book of Nature is written in the language of Mathematics.

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

Proof. $\forall s \in S\ \exists a_s(X)$ word in X s.t. $s = a_s(X)$ in G. (involves a choice)
The map $i_{SX} : S \to F(X)$, $i_{SX}(s) = a_s(X)$ extends to a unique homomorphism $p : F(S) \to F(X)$ (rewriting homomorphism).
We have that $\pi_S = \pi_X \circ p$.
Likewise, $\forall x \in X\ \exists b_x(S)$ in S s.t. $x = b_x(S)$.
The map $i_{XS} : X \to F(S)$, $i_{XS}(x) = b_x(S)$, extends to homomorphism $q : F(X) \to F(S)$ (another rewriting homomorphism).
As previously $\pi_S \circ q = \pi_X$.

Finite presentability is independent of the generating set 2

For every $x \in X$,

$$\pi_X(p(q(x))) = \pi_S(q(x)) = \pi_X(x).$$

Whence for every $x \in X$, $x^{-1}p(q(x)) \in \ker(\pi_X)$.
Let N be the normal subgroup of $F(X)$ normally generated by

$$\{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

We have that $N \leqslant \ker(\pi_X)$. Goal: prove equality.
There is a natural projection

$$Q : F(X)/N \to F(X)/\ker(\pi_X).$$

Finite presentability is independent of the generating set 3

Let $\bar{p} : F(S) \to F(X)/N$ be the homomorphism induced by p.
$\bar{p}(r) = 1$ for all $r \in R \Rightarrow \bar{p}$ induces a homomorphism

$$Q' : F(S)/\ker(\pi_S) \to F(X)/N.$$

Note that the domain of Q' is isomorphic to G, and Q' is onto:
$F(X)/N$ is generated by $xN = p(q(x))N$, and the latter is the image under Q' of $q(x)\ker(\pi_S)$.
Consider the homomorphism

$$Q \circ Q' : F(S)/\ker(\pi_S) \to F(X)/\ker(\pi_X)$$

.
The isomorphism $G \to F(S)/\ker(\pi_S)$ sends every $x \in X$ to $q(x)\ker(\pi_S)$.
The isomorphism $G \to F(X)/\ker(\pi_X)$ sends every $x \in X$ to $x\ker(\pi_X)$.
Note that $Q \circ Q'\,(q(x)\ker(\pi_S)) = Q(xN) = x\ker(\pi_X)$, whence $Q \circ Q'$ isomorphism $\Rightarrow Q'$ injective $\Rightarrow Q'$ isomorphism $\Rightarrow Q$ isomorphism $\Rightarrow N = \ker(\pi_X)$.

Finite presentability is independent of the generating set 4

In particular, $\ker(\pi_X)$ is normally generated by the finite set of relators

$$\mathfrak{R} = \{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

Since $\mathfrak{R} \subset \langle\langle T \rangle\rangle$, every relator $\rho \in \mathfrak{R}$ can be written as a product

$$\prod_{i \in I_\rho} t_i^{v_i}$$

with $v_i \in F(X), t_i \in T$ and I_ρ finite.
Whence $\ker(\pi_X)$ normally generated by the finite subset

$$T_0 = \bigcup_{\rho \in \mathfrak{R}} \{t_i \mid i \in I_\rho\}$$

of T.

Reformulation of Proposition:

Given a short exact sequence

$$1 \to N \to F(X) \to G \to 1$$

with X finite and G finitely presented, N is normally generated by finitely many elements $n_1, \ldots, n_k$.

This reformulation has a generalization to arbitrary short exact sequences that will appear later on.

Commutators

Notation

$[x, y] = xyx^{-1}y^{-1}$ = *the commutator of the elements* x, y *in a group* G

For subsets A, B *in a group* G, $[A, B]$= *the subgroup of* G *generated by all the commutators* $[a, b], a \in A, b \in B$.

For every $x_1, \ldots, x_n$ *in a group* G *we denote by* $[x_1, \ldots, x_n]$ *the n-fold left-commutator*

$$[[[x_1, x_2], \ldots, x_{n-1}], x_n].$$

For subsets $A_1, \ldots A_n$ *in a group* G, $[A_1, \ldots, A_n]$= *the subgroup of* G *generated by all the commutators* $[a_1, \ldots, a_n], a_i \in A_i$.

Nilpotent Groups: first definition

There are two ways of defining nilpotent groups and of measuring "how far they are from being abelian".

First definition: from the group downwards.
The lower central series of a group G,

$$C^1G \trianglerighteq C^2G \trianglerighteq \ldots \trianglerighteq C^nG \trianglerighteq \ldots,$$

is defined inductively by:

$$C^1G = G\,,\; C^{n+1}G = [C^nG, G]\,.$$

Each C^kG is a characteristic subgroup of G. (for every automorphism $\varphi : G \to G$, $\varphi(C^kG) = C^kG$).
$C^2G = [G, G] = G'$ is the commutator subgroup, or the derived subgroup, of G.

Nilpotent Groups 2

Definition

G is k-step nilpotent if $C^{k+1}G = \{1\}$. The minimal k for which G is k-step nilpotent is called the (nilpotency) class of G.

Examples

1. Every non-trivial abelian group is nilpotent of class 1.
2. The group $\mathcal{U}_n(\mathbb{K})$ of upper triangular $n \times n$ matrices with 1 on the diagonal and entries in a ring $\mathbb{K}$, is nilpotent of class $n-1$ (see Exercise Sheet 2).
3. The integer Heisenberg group $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.