Infinite Groups

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Nilpotent Groups: first definition

There are two ways of defining nilpotent groups and of measuring "how far they are from being abelian".

First definition: from the group downwards.
The lower central series of a group G,

$$C^1G \trianglerighteq C^2G \trianglerighteq \ldots \trianglerighteq C^nG \trianglerighteq \ldots,$$

is defined inductively by:

$$C^1G = G\,,\; C^{n+1}G = [C^nG, G]\,.$$

Each C^kG is a characteristic subgroup of G. (for every automorphism $\varphi : G \to G$, $\varphi(C^kG) = C^kG$).
$C^2G = [G, G] = G'$ is the commutator subgroup, or the derived subgroup, of G.

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Definition

G is k-step nilpotent if $C^{k+1}G = \{1\}$. The minimal k for which G is k-step nilpotent is called the (nilpotency) class of G.

Examples

1. Every non-trivial abelian group is nilpotent of class 1.
2. The group $\mathcal{U}_n(\mathbb{K})$ of upper triangular $n \times n$ matrices with 1 on the diagonal and entries in a ring $\mathbb{K}$, is nilpotent of class $n-1$ (see Exercise Sheet 2).
3. The integer Heisenberg group $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

Basic properties

Lemma

If $G = \langle S \rangle$ (S not necessarily finite, G not necessarily nilpotent), then $\forall k$ the subgroup $C^k G$ is generated by the k-fold left commutators in S, together with $C^{k+1} G$.

Proof Induction on k and two formulas:

- $[x, yz] = [x, y]\,[y, [x, z]]\,[x, z]\,;$
- $[xy, z] = [x, [y, z]]\,[y, z]\,[x, z] = [y, z]^x [x, z]\,.$

Corollary

If $G = \langle S \rangle$ is nilpotent, then $C^n G$ is generated by all the k-fold left commutators in S, where $k \geqslant n$. In particular, if G is finitely generated, each subgroup $C^n G$ is finitely generated.

Nilpotent Groups: second definition

Second definition: from $\{1\}$ upwards.
The center $Z(H)$ of a group H is composed of all $z \in H$ s.t. $zh = hz, \forall h \in H$.

Given a group G, define inductively an increasing sequence of normal subgroups $Z_i(G) \lhd G$ by:

- $Z_0(G) = \{1\}$.
- If $Z_i(G) \lhd G$ is defined and $\pi_i : G \to G/Z_i(G)$ is the quotient map, then
$$Z_{i+1}(G) = \pi_i^{-1}\left(Z(G/Z_i(G))\right).$$

Note that $Z_{i+1}(G)$ is normal in G, as the inverse image of a normal subgroup of a quotient of G.
In particular,
$$Z_{i+1}(G)/Z_i(G) \cong Z(G/Z_i(G)).$$

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Proposition

G is k-step nilpotent if and only if $Z_k(G) = G$.

Proof Assume that G is nilpotent of class k.
We prove by induction on $i \geqslant 0$ that $C^{k+1-i}G \leqslant Z_i(G)$.
For $i = 0$ we have equality.
Assume that

$$C^{k+1-i}G \leqslant Z_i(G).$$

For every $g \in C^{k-i}G$ and every $x \in G$, $[g, x] \in C^{k+1-i}G \leqslant Z_i(G)$, whence $gZ_i(G)$ is in the center of $G/Z_i(G)$, i.e. $g \in Z_{i+1}(G)$.

For $i = k$ the inclusion becomes $C^1G = G \leqslant Z_k(G)$, hence $Z_k(G) = G$.

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Conversely, assume $Z_k(G) = G$.
We prove by induction on $j \geqslant 1$ that $C^j G \leqslant Z_{k+1-j}(G)$.
For $j = 1$ the two are equal.
Assume the inclusion true for j.
$C^{j+1}G$ generated by $[c, g]$ with $c \in C^j G$ and $g \in G$.
Since $c \in C^j G \leqslant Z_{k+1-j}(G)$, by the definition of $Z_{k+1-j}(G)$, the element c commutes with g modulo $Z_{k-j}(G)$, equivalently $[c, g] \in Z_{k-j}(G)$.
This implies that $[c, g] \in Z_{k-j}(G)$. It follows that $C^{j+1}G \leqslant Z_{k-j}(G)$.
For $j = k + 1$ this gives $C^{k+1}G \leqslant Z_0(G) = \{1\}$, hence G is k-step nilpotent.

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Definition

The ascending series

$$Z_0(G) = \{1\} \lhd Z_1(G) \lhd \dots \lhd Z_i(G) \lhd Z_{i+1}(G) \lhd \dots$$

of normal subgroups of G is called the upper central series of G.

A group G is nilpotent if and only if there exists i such that $Z_i(G) = G$, and its nilpotency class is the minimal k such that $Z_k(G) = G$.

The following example shows that the difference between lower and upper central series of groups can be quite substantial:

Example

We start with the integer Heisenberg group H; it is 2-step nilpotent, $C^2H = H' = Z(H) \cong \mathbb{Z}$.
Take $G = H \times \mathbb{Z}$, 2-step nilpotent. $C^2G = C^2H \cong \mathbb{Z}$, while $Z(G) \cong \mathbb{Z}^2$.

Nilpotent Groups: properties

Lemma

1. *Every subgroup of a nilpotent group is nilpotent.*
2. *If G is nilpotent and $N \lhd G$ then G/N is nilpotent.*
3. *The direct product of a family of nilpotent groups is again nilpotent.*

NB (3) not true for semidirect products. Not even for $\mathbb{Z}^n \rtimes \mathbb{Z}$.

Nilpotent Groups: a key property

Theorem

Every subgroup H of a finitely generated nilpotent group G is finitely generated.

Proof by induction on the class of nilpotency k of G.
For $k = 1$ G is abelian finitely generated. Assume the assertion true for k, let G be a nilpotent group of class $k+1$ and let $H \leqslant G$.
By the induction hypothesis $H_1 = H \cap C^2 G$ is finitely generated.
$H_2 = H/(H \cap C^2 G)$ is finitely generated because subgroup of $G/C^2 G$, abelian finitely generated.
Thus H fits in the short exact sequence

$$1 \to H_1 \to H \xrightarrow{\pi} H_2 \to 1,$$

where H_1, H_2 are finitely generated.
Therefore H is also finitely generated. □

Nilpotent Groups

We show that the lower central series is graded with respect to commutators, that is:

Proposition

Let $C^k G$ be the k-th group in the lower central series of G. Then for every $i, j \geqslant 1$

$$\left[C^i G, C^j G\right] \leqslant C^{i+j} G\,. \tag{1}$$

First, note that $[a, b]^{-1} = [b, a]$, whence $[A, B] = [B, A]$.

Lemma

If A, B, C normal subgroups in G, then $[A, B, C] \lhd G$ and it is generated by $[a, b, c]$ with $a \in A, b \in B, c \in C$.