Infinite Groups

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About Maths again

Aristotle in "Metaphysics":

The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree.

William Thurston: "Experience has shown repeatedly that a mathematical theory with a rich internal structure generally turns out to have significant implications for the understanding of the real world, often in ways no one could have envisioned before the theory was developed."

We show that the lower central series is graded with respect to commutators, that is:

Proposition

Let $C^k G$ be the k-th group in the lower central series of G. Then for every $i, j \geqslant 1$

$$\left[C^i G, C^j G\right] \leqslant C^{i+j} G\,. \tag{1}$$

First, note that $[a, b]^{-1} = [b, a]$, whence $[A, B] = [B, A]$.

Lemma

If A, B, C normal subgroups in G, then $[A, B, C] \lhd G$ and it is generated by $[a, b, c]$ with $a \in A, b \in B, c \in C$.

Generation of $[A, B, C]$

Proof. $[A, B, C] \lhd G$ follows from $[x, y]^g = [x^g, y^g]$.

$[A, B, C]$ generated by $[k, c]$, $c \in C$, k product of n commutators $[a, b]$ or inverses.

We prove by induction on n, that $[k, c]$ is a product of finitely many $[a, b, c]$ and inverses.

$n = 1$: consider the case $[t^{-1}, c]$, where $t = [a, b]$.

$[t^{-1}, c] = [c, t]^{t^{-1}} = [c^{t^{-1}}, t] = [c', t] = [t, c']^{-1} = [a, b, c']^{-1}$.

Generation of $[A, B, C]$ 2

Assume the statement is true for n, let $k = k_1 t$, where t is $[a, b]$ or $[a, b]^{-1} = [b, a]$, and k_1 product of n commutators.

$$[k_1 t, c] = [t, c]^{k_1} [k_1, c] \, .$$

Both $[t, c]^{k_1}$ and $[k_1, c]$ are products of commutators $[a, b, c]$ and inverses, by the induction assumption and the fact that A, B, C are normal subgroups. □

Exercise

Prove the same result for $[H_1, \dots, H_n]$, where all H_i are normal subgroups of G.

Second Key Lemma

Lemma

Assume that A, B, C are normal subgroups in G. Then

$$[A, B, C] \leqslant [B, C, A][C, A, B]\,. \tag{2}$$

Proof Uses the previous Lemma and the Hall identity:

$$\left[x^{-1}, y, z\right]^{x} \left[z^{-1}, x, y\right]^{z} \left[y^{-1}, z, x\right]^{y} = 1\,. \tag{3}$$

The latter identity implies

$$[a, b, c]^{a^{-1}} \leqslant [B, C, A][C, A, B].$$

$\square$

Proof of Proposition

We prove by induction on $i \geqslant 1$ that for every $j \geqslant 1$,

$$\left[C^i G, C^j G\right] \leqslant C^{i+j} G\,. \tag{4}$$

For $i = 1$, by definition of $C^{i+1}G$ we have equality. Assume true for i and prove for $i+1$.
Consider $j \geqslant 1$ arbitrary.

$$[C^{i+1}G, C^jG] = [C^iG, G, C^jG] \leqslant [G, C^jG, C^iG][C^jG, C^iG, G] \leqslant$$

$$[C^{j+1}G, C^iG][C^{j+i}G, G] = [C^iG, C^{j+1}G][C^{j+i}G, G] \leqslant C^{j+i+1}G\,,$$

since $[C^iG, C^{j+1}G] \leqslant C^{j+i+1}G$ by the induction hypothesis.

$\square$

Back to a more global picture

We study the following classes of groups:

Nilpotent finitely generated $\subset$ Polycyclic $\subset$ Solvable

Definition

Given a class $\mathcal{X}$ of groups, a group G is said to be poly-$\mathcal{X}$ if it admits a subnormal descending series:

$$G = G_0 \rhd G_1 \rhd \ldots \rhd G_k \rhd G_{k+1} = \{1\},$$

such that each G_i/G_{i+1} belongs to the class $\mathcal{X}$.

Polycyclic if $\mathcal{X} =$ all cyclic groups.

Poly-C_∞ if $\mathcal{X} = \{\mathbb{Z}\}$.

Solvable if $\mathcal{X} =$ all abelian groups.

Differences and similarities

Nilpotent finitely generated groups= the only groups with polynomial growth.
Polycyclic (hence also nilpotent f.g.) groups = finitely presented, linear (therefore residually finite) while solvable groups are not necessarily finitely presented, linear or residually finite.
A different behaviour of the torsion:

$$\mathrm{Tor}\, G = \{g \in G \mid \exists n \geq 1 \text{ s.t. } g^n = 1\}.$$

When G nilpotent, $\mathrm{Tor}\, G$ is a characteristic subgroup of G.

When G nilpotent and moreover f.g., $\mathrm{Tor}\, G$ is a finite characteristic subgroup of G.

When G polycyclic, $\mathrm{Tor}\, G$ not necessarily a subgroup of G, nor a finite subset of G. Examples in Ex. Sheet 2.

Torsion for nilpotent groups

Theorem

When G is nilpotent (not necessarily finitely generated), $\mathrm{Tor}\, G$ is a characteristic subgroup.

Proof by induction on the nilpotency class.

A key result for this induction:

Lemma

Let G be nilpotent of class k. For every $x \in G$ the subgroup H generated by x and $C^2 G$ is a normal subgroup, nilpotent of class $\leqslant k-1$.

Proof that $H = \langle x, C^2G \rangle$ of class $\leqslant k-1$.

Since $C^2G \lhd G$,

$$H = \{x^m c \mid m \in \mathbb{Z},\ c \in C^2G\}.$$

H normal: $\forall g \in G$, and $h \in H$, $h = x^m c$, $ghg^{-1} = x^m[x^{-m}, g]gcg^{-1}$.
The last two factors are in $C^2G \Rightarrow$ the product is in H.
We prove $C^2H \leqslant C^3G$ (implying H is of class $\leqslant k-1$).
Let $h = x^m c_1$, $h' = x^n c_2$ with $c_i \in C^2G$. By Lema 5.1 in the Lecture Notes

$$[h, h'] = [h, x^n c_2] = [h, x^n]\,[x^n, [h, c_2]]\,[h, c_2].$$

The last term is in C^3G, hence the middle term is in C^4G.
The first term, using Lema 5.1 in the Lecture Notes, can be rewritten as

$$[h, x^n] = [x^m c_1, x^n] = [x^m, [c_1, x^n]][c_1, x^n].$$

The last term is in C^3G and the first in C^4G. $\square$