Infinite Groups

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Inspirational quotations about Mathematics again

William Thurston: "As one reads mathematics, one needs to have an active mind, asking questions, forming mental connections between the current topic and other ideas from other contexts, so as to develop a sense of the structure, not just familiarity with a particular tour through the structure."

Carl Friedrich Gauss: "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Polycyclic groups

Definition

Let $\mathcal{X}$ be a class of groups.
G is poly-$\mathcal{X}$ if it admits a subnormal descending series:

$$G = N_0 \triangleright N_1 \triangleright \ldots \triangleright N_k \triangleright N_{k+1} = \{1\}, \tag{1}$$

such that each N_i/N_{i+1} belongs to $\mathcal{X}$, up to isomorphism.

Polycyclic if $\mathcal{X} =$ all cyclic groups.
Poly-C_∞ if $\mathcal{X} = \{\mathbb{Z}\}$.
Cyclic series of G= a series as in (1) with $\mathcal{X}$ set of cyclic groups. Its length is the number of non-trivial groups.
The length $\ell(G)$ of a polycyclic group is the least length of a cyclic series of G.
C_∞ series of G= a series as in (1) with $\mathcal{X} = \{\mathbb{Z}\}$.
By convention, $\{1\}$ is poly-C_∞.

Further properties of polycyclic groups

Proposition

1. *Any subgroup H of a polycyclic group G is polycyclic (hence, finitely generated).*
2. *If $N \lhd G$, then G/N is polycyclic.*
3. *If $N \lhd G$ and both N and G/N are polycyclic then G is polycyclic.*
4. *Properties (1) and (3) hold with 'polycyclic' replaced by 'poly-C_∞', but not (2): $\mathbb{Z}_k$ is a quotient of $\mathbb{Z}$.*

Proof. (1). Given a cyclic series for G as above, the intersections $H \cap N_i$ define a cyclic series for H.

Properties of polycyclic groups 4

(2). Proof is by induction on the length $\ell(G) = n$.
For $n = 1$, G is cyclic and any quotient of G is also cyclic.
Assume the statement is true for all $k \leqslant n$, and consider a group G with $\ell(G) = n + 1$.
Let N_1 be the first term distinct from G in this cyclic series. By the induction hypothesis, $N_1/(N_1 \cap N) \simeq N_1N/N$ is polycyclic. The subgroup N_1N/N is normal in G/N and $(G/N)/(N_1N/N) \simeq G/N_1N$ is cyclic, as it is a quotient of G/N_1. It follows that G/N is polycyclic.

(3) Consider the cyclic series

$$G/N = Q_0 \geqslant Q_1 \geqslant \cdots \geqslant Q_n = \{\bar{1}\}$$

and

$$N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_k = \{1\}\,.$$

Given $\pi : G \to G/N$ and $H_i := \pi^{-1}(Q_i)$, a cyclic series for G is:

$$G \geqslant H_1 \geqslant \ldots \geqslant H_n = N = N_0 \geqslant N_1 \geqslant \ldots \geqslant N_k = \{1\}\,.$$

Two key examples of polycyclic groups

Proposition

Every finitely generated nilpotent group is polycyclic.

Proof Consider the finite descending series with terms $C^k G$.

- For every $k \geqslant 1$, $C^k G / C^{k+1} G$ is finitely generated abelian, hence there exists a finite subnormal descending series

$$C^k G = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_n \geqslant A_{n+1} = C^{k+1} G$$

  such that every quotient A_i / A_{i+1} is cyclic.
- By inserting all these finite descending series into the one defined by the $C^k G$'s, we obtain a finite subnormal cyclic series for G. □

Proposition

Given any homomorphism $\varphi : \mathbb{Z}^n \to \mathrm{Aut}\,(\mathbb{Z}^m)$, the semidirect product $\mathbb{Z}^m \rtimes_\varphi \mathbb{Z}^n$ is poly-C_∞.

Two key properties of polycyclic groups

Proposition

Polycyclic groups are finitely presented and residually finite.

Finite presentation is proved using a general property:

Proposition

Let $N \lhd G$. If both N and G/N are finitely presented then G is finitely presented.

Proof Let $N = \langle X \mid r_1, \dots, r_k \rangle$, and $G/N = \langle \overline{Y} \mid \rho_1, \dots, \rho_m \rangle$ be finite presentations, where Y is a finite subset of G s. t. $\overline{Y} = \{yN \mid y \in Y\}$.
G is generated by $S = X \cup Y$. S satisfies the following relations:

$$r_i(X) = 1\,,\ 1 \leqslant i \leqslant k\,,\ \rho_j(Y) = u_j(X)\,,\ 1 \leqslant j \leqslant m\,, \tag{2}$$

$$x^y = v_{xy}(X)\,,\ x^{y^{-1}} = w_{xy}(X)\,. \tag{3}$$

We denote the above finite set of relations by T.

Proof continued

We claim that $G = \langle S \mid T \rangle$. Let $K = \langle\langle T \rangle\rangle$ in $F(S)$.
The epimorphism $\pi_S : F(S) \to G$ defines an epimorphism $\varphi : F(S)/K \to G$. Goal: to prove φ is an isomorphism.
Let wK be an element in $\ker(\varphi)$, w word in S.
Relations (3) imply that there exist a word $w_1(X)$ in X and a word $w_2(Y)$ in Y, such that $wK = w_1(X)w_2(Y)K$.
Applying the projection $\pi : G \to G/N$, we see that $\pi(\varphi(wK)) = 1$, i.e. $\pi(\varphi(w_2(Y)K)) = 1$.
Therefore $w_2(Y)$ is a product of finitely many conjugates of $\rho_i(Y)$, hence $w_2(Y)K$ is a product of finitely many conjugates of $u_j(X)K$, by the set of relations in (2).
This and the relations (3) imply that $w_1(X)w_2(Y)K = v(X)K$ for some word $v(X)$ in X.
Then the image $\varphi(wK) = \varphi(v(X)K)$ is in N; therefore, $v(X)$ is a product of finitely many conjugates of relators $r_i(X)$. This implies that $v(X)K = K$. □

Finite presentation continued

Remark

G finitely presented does not imply $H \leqslant G$ finitely presented or G/N finitely presented, for $N \lhd G$.

Proposition

Let G be a group, and $H \leq G$ such that $|G : H|$ is finite. Then G is FP if and only if H is FP.