C4.3 Functional Analytic Methods for PDEs
Lectures 9-10

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In the last lecture

- Density results for Sobolev spaces.
- Extension theorems for Sobolev functions.
- Trace (boundary value) of Sobolev functions.
- Gagliardo-Nirenberg-Sobolev's inequality, $1 \leq p < n$.

This lecture

- Morrey's inequality, $n < p \leq \infty$.
- Friedrichs' inequality.
- Rellich-Kondrachov's compactness theorem.
- Poincaré's inequality.
- (Local behavior of Sobolev functions.)

Hölder and Lipschitz continuity

- Let D be a subset of $\mathbb{R}^n$.
- For $\alpha \in (0, 1]$, we say that a function $u : D \to \mathbb{R}$ is (uniformly) α-Hölder continuous in D if there exists $C \geq 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in D.$$

  The set of all α-Hölder continuous functions in D is denoted as $C^{0,\alpha}(D)$.
- When $\alpha = 1$, we use the term 'Lipschitz continuity' instead of '1-Hölder continuity'.

Hölder and Lipschitz continuity

- Note that, in our notation, when Ω is a bounded domain, $C^{0,\alpha}(\Omega) = C^{0,\alpha}(\bar{\Omega})$.
  In some text $C^{0,\alpha}(\Omega)$ is used to denote the set of continuous functions in Ω which is α-Hölder continuous on every compact subsets of Ω. In this course, we will use instead $C^{0,\alpha}_{loc}(\Omega)$ to denote this latter set, if such occasion arises.

$C^{0,\alpha}(D)$ is a Banach space

- For $u \in C^{0,\alpha}(D)$, let

$$[u]_{C^{0,\alpha}(D)} := \sup_{x,y\in D, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

and

$$\|u\|_{C^{0,\alpha}(D)} := \sup_D |u| + [u]_{C^{0,\alpha}(D)}.$$

Proposition

Let D be a subset of $\mathbb{R}^n$. Then $(C^{0,\alpha}(D), \|\cdot\|_{C^{0,\alpha}(D)})$ is a Banach space.

Hölder and Lipschitz continuity

Sketch of proof

- Piece 1: $\| \cdot \|_{C^{0,\alpha}(D)}$ is a norm.
  - ⋆ We will only give a proof for the statement that $[\cdot]_{C^{0,\alpha}(D)}$ satisfies the triangle inequality (i.e. it is a semi-norm). The rest is left as an exercise.
  - ⋆ Take $u, v \in C^{0,\alpha}(D)$. We want to show that $[u+v]_{C^{0,\alpha}(D)} \leq a + b$ where $a = [u]_{C^{0,\alpha}(D)}$ and $b = [v]_{C^{0,\alpha}(D)}$.
  - ⋆ Indeed, for any $x \neq y \in D$, we have $|u(x) - u(y)| \leq a|x-y|^\alpha$ and $|v(x) - v(y)| \leq b|x-y|^\alpha$. It follows that

$$|(u+v)(x) - (u+v)(y)| \leq (a+b)|x-y|^\alpha.$$

Divide both sides by $|x-y|^\alpha$ and take supremum we get

$$[u+v]_{C^{0,\alpha}(D)} = \sup_{x \neq y \in D} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq a + b,$$

as wanted.

$C^{0,\alpha}(D)$ is a Banach space

Sketch of proof

- Piece 2: $C^{0,\alpha}(D)$ is complete.
  - ⋆ Suppose that (u_m) is Cauchy in $C^{0,\alpha}(D)$.
  - ⋆ As $\|\cdot\|_{sup} \leq \|\cdot\|_{C^{0,\alpha}(D)}$, this implies that (u_m) is Cauchy in $C^0(\bar{D})$ and hence converges uniformly to some $u \in C^0(\bar{D})$.
  - ⋆ Claim: $u \in C^{0,\alpha}(D)$. Fix $\varepsilon > 0$. For every $x, y \in D$, we have

$$\begin{aligned} |(u_m - u_j)(x) - (u_m - u_j)(y)| &\leq [u_m - u_j]_{C^{0,\alpha}(D)}|x-y|^\alpha \\ &\leq \varepsilon|x-y|^\alpha \text{ for large } m, j. \end{aligned}$$

Sending $j \to \infty$, we thus have

$$|(u_m - u)(x) - (u_m - u)(y)| \leq \varepsilon|x-y|^\alpha \text{ for large } m.$$

Choose one such m we arrive at

$$|u(x) - u(y)| \leq \Big([u_m]_{C^{0,\alpha}(D)} + \varepsilon\Big)|x-y|^\alpha.$$

So $u \in C^{0,\alpha}(D)$.

$C^{0,\alpha}(D)$ is a Banach space

Sketch of proof

- Piece 2: $C^{0,\alpha}(D)$ is complete.
  - ⋆ Finally, we show that $u_m \to u$ in $C^{0,\alpha}(D)$. As u_m converges to u uniformly, it remains to show that $[u_m - u]_{C^{0,\alpha}(D)} \to 0$.
  - ⋆ Fix $\varepsilon > 0$. Recall from the previous slide that, for $x, y \in D$, we have

$$|(u_m - u)(x) - (u_m - u)(y)| \leq \varepsilon |x - y|^\alpha \text{ for large } m.$$

    Divide both sides by $|x - y|^\alpha$ and take supremum, we have

$$[u_m - u]_{C^{0,\alpha}(D)} \leq \varepsilon \text{ for large } m.$$

  - ⋆ As ε is arbitrary, we conclude that $[u_m - u]_{C^{0,\alpha}(D)} \to 0$.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

An integral mean value inequality

Lemma

Let Ω be a domain in $\mathbb{R}^n$ and suppose $u \in C^1(\Omega)$. Then

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy \text{ for all } B_r(x) \subset \Omega.$$

Proof

- It suffices to consider the case $x = 0$. We write $y = s\theta$ where $s \in [0, r)$ and $\theta \in \mathbb{S}^{n-1} \in \mathbb{R}^n$.
- By the fundamental theorem of calculus, we have

$$u(s\theta) - u(0) = \int_0^s \frac{d}{dt} u(t\theta)\, dt = \int_0^s \theta_i \partial_i u(t\theta)\, dt.$$

It follows that $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\, dt.$

An integral mean value inequality

Proof

- $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\, dt.$
- Integrating over θ and using Tonelli's theorem, we get

$$\begin{aligned}
\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta &\leq \int_0^s \int_{\partial B_1(0)} |\nabla u(t\theta)|\, d\theta\, dt \\
&= \int_0^s \int_{\partial B_t(0)} |\nabla u(y)| \frac{dS(y)}{t^{n-1}}\, dt \\
&= \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.
\end{aligned}$$

An integral mean value inequality

Proof

- $\displaystyle\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta \leq \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$
- Multiplying both sides by s^{n-1} and integrating over s, we get

$$\begin{aligned}\int_{B_r(0)} |u(y) - u(0)|\, dy &= \int_0^r \int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta s^{n-1} ds\\ &\leq \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy \int_0^r s^{n-1}\, ds\\ &= \frac{1}{n} r^n \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.\end{aligned}$$

This gives the desired integral mean value inequality.

A corollary of the integral mean value inequality

Corollary

Suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ for some $p > n$. Then

$$\int_{B_r(x)} |u(y) - u(x)|\, dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} r^{\frac{n(p-1)}{p}+1} \text{ for all } B_r(x) \subset \Omega,$$

where the constant $C_{n,p}$ depends only on n and p.

Proof

- As in the previous proof, we assume without loss of generality that $x = 0$. We start with the integral mean value inequality:

$$\int_{B_r(0)} |u(y) - u(0)|\, dy \leq \frac{r^n}{n} \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$$

A corollary of the integral mean value inequality

Proof

- By Hölder's inequality this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq \frac{r^n}{n} \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_{B_r(0)} \frac{1}{|y|^{(n-1)p'}}\,dy \Big\}^{1/p'}$$
$$= C_n r^n \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_0^r s^{-(n-1)(p'-1)}\,ds \Big\}^{1/p'}.$$

- As $p > n$, we have that $p' < \frac{n}{n-1}$ and so $(n-1)(p'-1) < 1$. Hence the integral in the curly braces converges to $C_{n,p} r^{-(n-1)(p'-1)+1}$. After a simplification, this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(0))} r^{\frac{n}{p'}+1},$$

which is the conclusion.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Proof when $p < \infty$. The case $p = \infty$ will be dealt with later.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Suppose that (*) holds for functions in $C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. We show that this implies the theorem.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Let $u \in W^{1,p}(\mathbb{R}^n)$. As $p < \infty$, we can find $u_m \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Applying (*) to $u_m - u_j$ we have

$$\|u_m - u_j\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)} \stackrel{m,j\to\infty}{\longrightarrow} 0.$$

    This means that (u_m) is Cauchy in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$, and hence converges in $C^{0,1-\frac{n}{p}}$ to some $u_* \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.
  - ⋆ On the other hand, as $u_m \to u$ in L^p, a subsequence of it converges a.e. in $\mathbb{R}^n$ to u.
  - ⋆ It follows that $u = u_*$ a.e. in $\mathbb{R}^n$, i.e. u has a continuous representative.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ We may thus assume henceforth that u is continuous, so that u_m converges to u in both $W^{1,p}$ and $C^{0,1-\frac{n}{p}}$.
  - ⋆ Now, applying (*) to u_m we have

$$\|u_m\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

Sending $m \to \infty$, we hence have

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

as wanted.

Morrey's inequality

Proof when $p < \infty$.

- Step 2: Proof of the C^0 bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

  - ⋆ By triangle inequality, we have

$$|B_1(x)||u(x)| \leq \int_{B_1(x)} |u(y) - u(x)|\, dy + \int_{B_1(x)} |u(y)| dy.$$

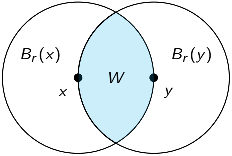
  - ⋆ By Hölder's inequality, the last integral is bounded by $C_{n,p}\|u\|_{L^p(B_1(x))}$.
  - ⋆ On the other hand, by the corollary to the integral mean value inequality, the first integral on the right hand side is bounded by $C_{n,p}\|\nabla u\|_{L^p(B_1(x))}$. The inequality (**) follows.

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}|x-y|^{1-\frac{n}{p}} \text{ for all } x, y \in \mathbb{R}^n. \quad (***)$$



- ⋆ If $x = y$, there is nothing to show. Suppose henceforth that $r = |x-y| > 0$ and let $W = B_r(x) \cap B_r(y)$.
- ⋆ Let a be the average of u in W, i.e. $a = \frac{1}{|W|}\int_W u(z)\,dz$. Then

$$|u(x) - u(y)| \leq |u(x) - a| + |u(y) - a|.$$

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*).
  - ⋆ We estimate $|u(x) - a|$ as follows:

$$|u(x) - a| \leq \frac{1}{|W|} \int_W |u(x) - u(z)| dz$$
$$\leq \frac{C_n}{r^n} \int_{B_r(x)} |u(x) - u(z)| dz.$$

By the corollary to the mean value inequality, the right hand side is bounded by $C_{n,p} \|\nabla u\|_{L^p(B_r(x))} \, r^{1-\frac{n}{p}}$. So

$$|u(x) - a| \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} \, r^{1-\frac{n}{p}}$$

  - ⋆ Similarly, $|u(y) - a| \leq C_{n,p} \|\nabla u\|_{L^p(B_r(y))} \, r^{1-\frac{n}{p}}$.
  - ⋆ Putting these together and recalling that $r = |x - y|$, we arrive at (***).

Morrey's inequality on domain for $n < p < \infty$

Theorem (Morrey's inequality)

Suppose that $n < p < \infty$ and Ω is a bounded Lipschitz domain. Then every $u \in W^{1,p}(\Omega)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative and

$$\|u\|_{C^{0,1-\frac{n}{p}}(\Omega)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.$$

Indeed, let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then Eu has a continuous representative and

$$\begin{aligned}\|Eu\|_{C^{0,1-\frac{n}{p}}(\Omega)} &\leq \|Eu\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)}\\ &\leq C_{n,p}\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

An improved integral mean value inequality

Lemma

Suppose $u \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ *for some* $p > n$. *Then, for every ball* $B_r(x) \subset \mathbb{R}^n$, *there holds*

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy.$$

Proof

- Replacing p by any $\tilde{p} \in (n, p)$, we may assume that p is finite. Then we can find $u_m \in C^\infty(B_R(0)) \cap W^{1,p}(B_R(0))$ such that $u_m \to u$ in $W^{1,p}$. Furthermore, arguing as in Step 1 in the proof of Morrey's inequality, we also have that $u_m \to u$ in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.

An improved integral mean value inequality

Proof

- $u_m \to u$ in $W^{1,p}(B_R(0))$ and in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.
- By the integral mean value inequality for C^1 functions, we have

$$\int_{B_r(x)} |u_m(y) - u_m(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u_m(y)|}{|y-x|^{n-1}}\, dy.$$

- The left hand side converges to $\int_{B_r(x)} |u(y) - u(x)| dy$ since $u_m \to u$ uniformly.
- The right hand side converges to $\frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy$ since $\nabla u_m \to \nabla u$ in L^p and since the function $y \mapsto \frac{1}{|y-x|^{n-1}}$ belongs to $L^{p'}$ (as noted in the proof of the corollary to the integral mean value inequality). The conclusion follows.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Note that when $p = \infty$ we can no longer use the previous proof, as $C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ is not dense in $W^{1,\infty}(\mathbb{R}^n)$.

Morrey's inequality

Proof when $p = \infty$.

- Suppose $u \in W^{1,\infty}(\mathbb{R}^n)$. Then $u \in W^{1,s}(B_R)$ for all $s < \infty$ and all ball B_R. By Morrey's inequality in the case of finite p, we thus have that u has a continuous representative, which we will assume to be u itself.
- By the improved integral mean value inequality, we have

$$\int_{B_r(x)} |u(y) - u(x)|dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy.$$

- Step 2 and Step 3 of the proof in the case $p < \infty$ can now be repeated to get

$$|u(x)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

and

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}|x-y| \text{ for all } x, y \in \mathbb{R}^n. \qquad (***)$$

Morrey's inequality

Proof when $p = \infty$.

- It follows that

$$\|u\|_{C^{0,1}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}$$

  and we are done.

Morrey's inequality on domains

We make a couple of remarks:

- If Ω and p are such that there exists a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ (in particular $Eu = u$ a.e. in Ω for all $u \in W^{1,p}(\Omega)$), then

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\Omega).$$

- The same proof on the whole space work on balls without establishing the existence of an extension operator. (Check this!)
- For general domains, one only has

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}_{loc}(\Omega).$$

(Revisit the example of the disk in $\mathbb{R}^2$ with a line segment removed.)

More on $W^{1,\infty}$

We have the following important theorem for the space $W^{1,\infty}(\Omega)$:

Theorem

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then

$$W^{1,\infty}(\Omega) = C^{0,1}(\Omega).$$

Friedrichs' inequality

Theorem (Friedrichs' inequality)

Assume that Ω is a bounded open set and $1 \leq p < \infty$. Then, there exists $C_{p,\Omega}$ such that

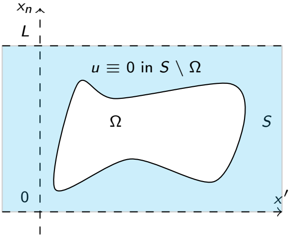
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Note that

- Only the derivatives of u appear on the right hand side.
- The function u belongs to $W_0^{1,p}(\Omega)$. The inequality is false for $u \in W^{1,p}(\Omega)$.
- By Friedrichs' inequality, when Ω is bounded, if we define $|||u||| = \|\nabla u\|_{L^p(\Omega)}$, then $||| \cdot |||$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\| \cdot \|_{W^{1,p}(\Omega)}$.
- In some text, Friedrichs' inequality is referred to as Poincaré's inequality.

Friedrichs' inequality

Proof



- We may assume that Ω is contain in the slab $S := \{(x', x_n) : 0 < x_n < L\}$.
- As usual, using the density of $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, it suffices to prove

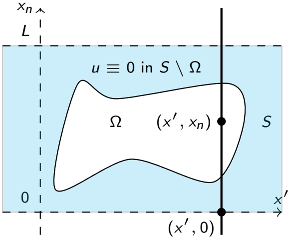
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)}$$

for $u \in C_c^\infty(\Omega)$.

- Take an arbitrary $u \in C_c^\infty(\Omega)$ and extend u by zero outside of Ω so that $u \in C_c^\infty(S)$.

Friedrichs' inequality

Proof



- Now, for every fixed x', we have

$$|u(x', x_n)| \leq \int_0^{x_n} |\partial_n u(x', t)|\, dt \leq \Big\{ \int_0^{x_n} |\partial_n u(x', t)|^p\, dt \Big\}^{1/p} x_n^{1/p'}$$
$$\leq \Big\{ \int_0^{L} |\partial_n u(x', t)|^p\, dt \Big\}^{1/p} x_n^{\frac{p-1}{p}}.$$

Friedrichs' inequality

Proof

- $|u(x', x_n)| \leq \left\{ \int_0^L |\partial_n u(x', t)|^p \, dt \right\}^{1/p} x_n^{\frac{p-1}{p}}$.
- It follows that

$$\int_0^L |u(x', x_n)|^p \, dx_n \leq \frac{1}{p} L^p \int_0^L |\partial_n u(x', t)|^p \, dt.$$

- Integrating over x' then gives

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &= \int_{\mathbb{R}^{n-1}} \int_0^L |u(x', x_n)|^p \, dx_n \, dx' \\ &\leq \frac{1}{p} L^p \int_{\mathbb{R}^{n-1}} \int_0^L |Du(x', t)|^p \, dt \, dx' = \frac{1}{p} L^p \|\nabla u\|_{L^p(\Omega)}^p. \end{aligned}$$

We are done.

Friedrichs' inequality

Theorem (Friedrichs' inequality)

Assume that Ω is a bounded open set and $1 \le p < \infty$. Then, there exists $C_{p,\Omega}$ such that

$$\|u\|_{L^p(\Omega)} \le C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Friedrichs-type inequality

Theorem (Friedrichs-type inequality)

Assume that Ω is a bounded open set and $1 \leq p < \infty$. Suppose that $1 \leq q \leq p^*$ if $p < n$, $1 \leq q < \infty$ if $p = n$, and $1 \leq q \leq \infty$ if $p > n$. Then there exists $C_{n,p,q,\Omega}$ such that

$$\|u\|_{L^q(\Omega)} \leq C_{n,p,q,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Proof

- Extend u by zero to $\mathbb{R}^n$.
- If $p < n$, we have by Gagliardo-Nirenberg-Sobolev's inequality, that

$$\|u\|_{L^{p^*}(\Omega)} = \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} = C\|\nabla u\|_{L^p(\Omega)}.$$

  As Ω has finite measure, $\|u\|_{L^q(\Omega)} \leq C\|u\|_{L^{p^*}(\Omega)}$, and so we're done in this case.

Friedrichs-type inequality

Proof

- Note that, as Ω has finite measure, $W^{1,n}(\Omega) \hookrightarrow W^{1,\hat{p}}(\Omega)$ for any $\hat{p} < p$. The case $p = n$ thus follows from the previous case.
- When $p > n$, we have by Morrey's inequality that

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} = C\|u\|_{W^{1,p}(\Omega)}.$$

  By Friedrichs' inequality, we have $\|u\|_{W^{1,p}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$. Also, as Ω has finite measure, $\|u\|_{L^q(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}$. Putting these together we're also done in this case.

Rellich-Kondrachov's theorem

Theorem (Rellich-Kondrachov's compactness theorem)

Let Ω be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Suppose $1 \leq q < p^*$ when $p < n$, $1 \leq q < \infty$ when $p = n$, and $1 \leq q \leq \infty$ when $p > n$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, i.e. every bounded sequence in $W^{1,p}(\Omega)$ contains a subsequence which converges in $L^q(\Omega)$.

Critical embedding is not compact

Remark

For $1 \leq p < n$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact.

Example by 'concentration'

- This example is by scaling. It is related to the argument in Lecture 7 to inspect for which p and q the space $W^{1,p}(\mathbb{R}^n)$ is embedded $L^q(\mathbb{R}^n)$.
- We may assume that the origin lies inside Ω and $B_{r_0} \subset \Omega$. Take an arbitrary non-zero function $u \in C_c^\infty(\mathbb{R}^n)$ with $Supp(u) \subset B_{r_0}$. We define, for $\lambda > 0$, $u_\lambda(x) = u(\lambda x)$.
- We knew that

$$\|u_\lambda\|_{L^q} = \lambda^{-n/q}\|u\|_{L^q} \text{ and } \|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p}\|\nabla u\|_{L^p}.$$

Critical embedding is not compact

Example by ‘concentration’

- Hence, if we let $\hat{u}_\lambda = \lambda^{-1+n/p} u_\lambda$, then

$$\|\hat{u}_\lambda\|_{L^p} = \lambda^{-1} \|u\|_{L^p},$$
$$\|\hat{u}_\lambda\|_{L^{p^*}} = \|u\|_{L^{p^*}},$$
$$\|\nabla \hat{u}_\lambda\|_{L^p} = \|\nabla u\|_{L^p}.$$

  In particular, as $\lambda \to \infty$,

$$\|\hat{u}_\lambda\|_{W^{1,p}} \leq \|u\|_{W^{1,p}} \text{ and } \|\hat{u}_\lambda\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0.$$

Critical embedding is not compact

Example by 'concentration'

- Now if the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ was compact, then as $(\hat{u}_\lambda)$ is bounded in $W^{1,p}$, we could select a sequence $\lambda_k \to \infty$ such that $(\hat{u}_{\lambda_k})$ converges in $L^{p^*}(\Omega)$ to some limit $u_* \in L^{p^*}(\Omega)$.
- This would imply that

$$\|u_*\|_{L^{p^*}} = \lim_{k\to\infty} \|\hat{u}_{\lambda_k}\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0.$$

- On the other hand, $Supp(\hat{u}_\lambda) \subset B_{r_0/\lambda}$ and so $\hat{u}_\lambda \to 0$ a.e. in Ω as $\lambda \to \infty$. This would give that $u_* = 0$ a.e. which contradicts the above.

Critical embedding is not compact

Remark

For $1 \leq p < n$, the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is not compact.

Example by 'translations'

- Take again an arbitrary non-zero function $u \in C_c^\infty(\mathbb{R}^n)$ and fix some unit vector e. Let $u_s(x) = u(x + se) = \tau_{se} u(x)$.
- Then $\|u_s\|_{W^{1,p}} = \|u\|_{W^{1,p}}$, $\|u_s\|_{L^{p^*}} = \|u\|_{L^{p^*}}$. Also $Supp(u_s) = \{x - se : x \in Supp(u)\}$ and so $u_s \to 0$ a.e. on $\mathbb{R}^n$ as $s \to \infty$.
- By the same reasoning, there is no sequence $s_k \to \infty$ such that u_{s_k} is convergent in L^{p^*}.

Pre-compactness criterion in $L^p(\Omega)$

Let us now do some preparation for the proof of Rellich-Kondrachov's theorem. Recall:

Theorem (Kolmogorov-Riesz-Fréchet's theorem)

Let $1 \le p < \infty$ and Ω be an open bounded subset of $\mathbb{R}^n$. Suppose that a sequence (f_i) of $L^p(\Omega)$ satisfies

1. *(Boundedness)* $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
2. *(Equi-continuity in L^p) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where $\tilde{f}_i$ is the extension by zero of f_i to all of $\mathbb{R}^n$.*

Then, there exists a subsequence (f_{i_j}) which converges in $L^p(\Omega)$.

In the case we are considering, boundedness follows from the embedding theorems. Let us now consider equi-continuity.

Continuity of translation operators in $W^{1,p}$

Lemma

Let $1 \le p < \infty$. For every $v \in W^{1,p}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, it holds that

$$\|\tau_y v - v\|_{L^p(\mathbb{R}^n)} \le |y| \|\nabla v\|_{L^p(\mathbb{R}^n)}.$$

Proof

- Using the density of $C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$ for $p < \infty$, it suffices to consider $v \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
- By the mean value theorem and Hölder's inequality, we have

$$|v(y+x) - v(x)| \le \int_0^1 |\frac{d}{dt} v(ty+x)|\,dt = \int_0^1 |y_i \partial_i v(ty+x)|\,dt$$
$$\le |y| \Big\{ \int_0^1 |\nabla v(ty+x)|^p\,dt \Big\}^{1/p}.$$

Continuity of translation operators in $W^{1,p}$

Proof

- $|v(y+x) - v(x)|^p \leq |y|^p \int_0^1 |\nabla v(ty+x)|^p \, dt.$
- Integrating over x gives

$$\begin{aligned}
\|\tau_y v - v\|_{L^p}^p &= \int_{\mathbb{R}^n} |v(y+x) - v(x)|^p \, dx \\
&\leq |y|^p \int_{\mathbb{R}^n} \int_0^1 |\nabla v(ty+x)|^p \, dt \, dx \\
&= |y|^p \int_0^1 \int_{\mathbb{R}^n} |\nabla v(ty+x)|^p \, dx \, dt \\
&= |y|^p \|\nabla v\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned}$$

So we have $\|\tau_y v - v\|_{L^p} \leq |y| \|\nabla v\|_{L^p(\mathbb{R}^n)}$ as wanted.

Continuity of translation operators in $W^{1,p}$

Remark

We remarked in Lecture 3 that the map $h \mapsto \tau_h$ is not a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$.
The above lemma implies that $h \mapsto \tau_h$ is a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$.

Proof

- Let $X = \mathscr{L}(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. The statement amounts to $\tau_y \to Id$ in X as $y \to 0$. So we need to show that

$$0 = \lim_{y\to 0} \|\tau_y - Id\|_X = \lim_{y\to 0} \sup_{u\in W^{1,p}(\mathbb{R}^n):\|u\|_{W^{1,p}}\le 1} \|\tau_y u - u\|_{L^p}.$$

- By the lemma, we have $\|\tau_y u - u\|_{L^p} \le |y| \|\nabla u\|_{L^p} \le |y|$ whenever $\|u\|_{W^{1,p}} \le 1$. So the point above is clear.

Characterisation of $W^{1,p}$ using translation operators

Theorem

Assume that $1 < p < \infty$ *and* $v \in L^p(\mathbb{R}^n)$*. Suppose that there exist small* $r > 0$ *and large* C *such that*

$$\|\tau_y v - v\|_{L^p(\mathbb{R}^n)} \leq C|y| \text{ for all } |y| \leq r.$$

Then

$$v \in W^{1,p}(\mathbb{R}^n) \text{ and } \|\nabla v\|_{L^p(\mathbb{R}^n)} \leq C.$$

Sketch of proof

- Fix a direction e_i. By hypothesis $q_t := \frac{1}{t}[\tau_{te_i} v - v]$ is bounded in L^p for $|t| \leq r$. By the weak sequential compactness property in L^p, we have along a sequence $t_k \to 0$ that q_{t_k} converges weakly in L^p to some $w_i \in L^p(\mathbb{R}^n)$.

Characterisation of $W^{1,p}$ using translation operators

Sketch of proof

- $q_{t_k} = \frac{1}{|t_k|}[\tau_{t_k e_i} v - v] \rightharpoonup w_i$ in L^p.
- The key point is the following identity

$$\int_{\mathbb{R}^n} [\tau_{t_k e_i} v - v]\varphi\, dx = -\int_{\mathbb{R}^n} v[\varphi - \tau_{-t_k e_i}\varphi]\, dx.$$

- Now divide both side by t_k and sending $k \to \infty$, we then get

$$\int_{\mathbb{R}^n} w_i \varphi\, dx = -\int_{\mathbb{R}^n} v \partial_i \varphi\, dx \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

This proves $\partial_i v = w_i \in L^p(\mathbb{R}^n)$. The conclusion follows.

Rellich-Kondrachov's theorem

Theorem (Rellich-Kondrachov's compactness theorem)

Let Ω be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Suppose $1 \leq q < p^*$ when $p < n$, $1 \leq q < \infty$ when $p = n$, and $1 \leq q \leq \infty$ when $p > n$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, i.e. every bounded sequence in $W^{1,p}(\Omega)$ contains a subsequence which converges in $L^q(\Omega)$.

We reiterate that, when $p < n$, the endpoint embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact.
When $p > n$, we have $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\Omega)$, so the above is a consequence of Ascoli-Arzelà's theorem. (Check this!)

Rellich-Kondrachov's theorem

Proof of the case $q = p \leq n$.

- Suppose that (u_m) is bounded in $W^{1,p}(\Omega)$. We need to construct a subsequence (u_{m_j}) which converges in $L^p(\Omega)$.
- As (u_m) is bounded in $L^p(\Omega)$, we would be done by Kolmogorov-Riesz-Fréchet's theorem if (u_m) is equi-continuous in L^p sense.
- To make use of the continuity property of translation operators in $W^{1,p}(\mathbb{R}^n)$, we let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be a bounded linear extension operator. Then the family (Eu_m) is bounded in $L^p(\mathbb{R}^n)$ and is equi-continuous in $L^p(\mathbb{R}^n)$ sense. But as $\mathbb{R}^n$ is unbounded, we cannot apply Kolmogorov-Riesz-Fréchet's theorem to this family.

Rellich-Kondrachov's theorem

Proof of the case $q = p \leq n$.

- We proceed as follows: Take a large ball B_R containing $\bar{\Omega}$ and select a cut-off function $\zeta \in C_c^\infty(B_R)$ such that $\zeta \equiv 1$ in Ω. Let

$$v_m = \zeta E u_m.$$

  Clearly $v_m = u_m$ a.e. in Ω, $Supp(v_m) \subset B_R$ and (v_m) is bounded in $W^{1,p}(\mathbb{R}^n)$.
- We aim to apply Kolmogorov-Riesz-Fréchet's theorem to $(v_m|_{B_R})$.
  - ⋆ It is clear that $(v_m|_{B_R})$ is bounded in $L^p(B_R)$.
  - ⋆ Also, by the continuity of translation operators in $W^{1,p}$, we have

$$\|\tau_y v_m - v_m\|_{L^p(\mathbb{R}^n)} \leq |y| \|Dv_m\|_{L^p(\mathbb{R}^n)} \leq |y| \|v_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

    Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y v_m - v_m\|_{L^p(B_R)} \leq \varepsilon$ for all m and all $|y| < \delta$, i.e. $(v_m|_{B_R})$ is equi-continuous in L^p sense. We're done.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- Suppose that $1 \leq q < p^*$ if $p < n$, $1 \leq q < \infty$ if $p = n$. By the embedding theorems, we know that there exists $\hat{q} > q$ such that $W^{1,p}(\Omega) \hookrightarrow L^{\hat{q}}(\Omega)$.
- Suppose that (u_m) is bounded in $W^{1,p}(\Omega)$. We need to construct a subsequence (u_{m_j}) which converges in $L^q(\Omega)$.
- We knew from the previous case that there is a subsequence (u_{m_j}) which converges in $L^p(\Omega)$ to some $u \in L^p(\Omega)$. Passing to a subsequence if necessary, we may also assume that (u_{m_j}) converges to u a.e. in Ω.
- To conclude, we show that $u \in L^q(\Omega)$ and (u_{m_j}) converges in $L^q(\Omega)$ to u.
- If $q \leq p$, the above follows from Hölder's inequality. We assume henceforth that $q > p$.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- We now show that $u \in L^q(\Omega)$. In fact, we show that $u \in L^{\hat{q}}(\Omega)$.
  - ⋆ By the embedding $W^{1,p}(\Omega) \hookrightarrow L^{\hat{q}}(\Omega)$, we have that u_m is bounded in $L^{\hat{q}}(\Omega)$.
  - ⋆ By Fatou's lemma, we have

$$\int_\Omega |u|^{\hat{q}}\,dx \leq \liminf_{j\to\infty} \int_\Omega |u_{m_j}|^{\hat{q}}\,dx < \infty.$$

Hence $u \in L^{\hat{q}}(\Omega)$.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- Finally, we show that $u_{m_j} \to u$ in $L^q(\Omega)$.
  - We observe that $u_{m_j} - u$ converges to 0 in $L^p(\Omega)$ and is bounded in $L^{\hat{q}}(\Omega)$ with $p < q < \hat{q}$.
  - Now we write, for $\theta \in (0,1)$ to be fixed

$$\|u_{m_j} - u\|_{L^q}^q = \int_\Omega |u_{m_j} - u|^q \, dx = \int_\Omega |u_{m_j} - u|^{q\theta} |u_{m_j} - u|^{q(1-\theta)} \, dx$$

and apply Hölder's inequality with some pair of conjugate exponents r and r' to be fixed:

$$\|u_{m_j} - u\|_{L^q}^q \leq \Big\{ \int_\Omega |u_{m_j} - u|^{q\theta r} \, dx \Big\}^{1/r} \Big\{ \int_\Omega |u_{m_j} - u|^{q(1-\theta)r'} \, dx \Big\}^{1/r'}.$$

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- ...we are showing that $u_{m_j} \to u$ in $L^q(\Omega)$.
  - $u_{m_j} - u \to 0$ in $L^p(\Omega)$ and $u_{m_j} - u$ is bounded in $L^{\hat{q}}(\Omega)$ with $p < q < \hat{q}$.
  - $\|u_{m_j} - u\|_{L^q} \leq \|u_{m_j} - u\|_{L^{q\theta r}}^{\theta} \|u_{m_j} - u\|_{L^{q(1-\theta)r'}}^{1-\theta}$.
  - Now, if we can chose $\theta \in (0,1)$ and $r > 1$ such that $q\theta r = p$ and $q(1-\theta)r' = \hat{q}$, then the first factor on the right hand side goes to zero and the second factor remains bounded, and so $u_{m_j} \to u$ in $L^q(\Omega)$ as wanted.
  - To solve for θ and r, we first eliminate r to obtain

$$1 = \frac{1}{r} + \frac{1}{r'} = \theta\frac{p}{q} + (1-\theta)\frac{\hat{q}}{q}.$$

    As $\frac{p}{q} < 1 < \frac{\hat{q}}{q}$, we can certainly select $\theta \in (0,1)$ satisfying the above. The exponent r is given by $r = \frac{q}{p\theta}$. This concludes the proof.

Poincaré's inequality

Theorem (Poincaré's inequality)

Suppose that $1 \le p \le \infty$ and Ω is a bounded Lipschitz domain. There exists a constant $C_{n,p,\Omega} > 0$ such that

$$\|u - \bar{u}_\Omega\|_{L^p(\Omega)} \le C_{n,p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\bar{u}_\Omega$ is the average of u in Ω:

$$\bar{u}_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x)\, dx.$$

When $p = \infty$, the theorem is a consequence of the fact that $W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$. (Check this!)

Poincaré's inequality

Proof for $p < \infty$.

- We argue by contradiction. Suppose the conclusion is not true. Then there exists a sequence $(u_m) \subset W^{1,p}(\Omega)$ such that

$$\|u_m - \bar{u}_m\|_{L^p} > m\|\nabla u_m\|_{L^p},$$

  where $\bar{u}_m$ is the average of u_m in Ω.
- Replacing u_m by $u_m - \bar{u}_m$, we may assume that u_m has zero average, so that $\|u_m\|_{L^p} > m\|\nabla u_m\|_{L^p}$.
- Replacing u_m by $\frac{1}{\|u_m\|_{L^p}} u_m$, we may assume that $\|u_m\|_{L^p} = 1$.
- The above implies that $\|\nabla u_m\|_{L^p} \leq \frac{1}{m}$ and so (u_m) is bounded in $W^{1,p}(\Omega)$.
- By Rellich-Kondrachov's compactness theorem, we can find a subsequence (u_{m_j}) which converges in $L^p(\Omega)$, say to u.

Poincaré's inequality

Proof for $p < \infty$.

- By the strong convergence of u_{m_j} to u, we have that

$$\|u\|_{L^p} = \lim_{j\to\infty} \|u_{m_j}\|_{L^p} = 1,$$

  and

$$\int_\Omega u\,dx = \lim_{j\to\infty} \int_\Omega u_{m_j}\,dx = 0.$$

- On the other hand, as $\|\nabla u_m\|_{L^p} < \frac{1}{m}$, we have for every $\varphi \in C_c^\infty(\Omega)$ that

$$\int_\Omega u\partial_i\varphi\,dx = \lim_{j\to\infty} \int_\Omega u_{m_j}\partial_i\varphi\,dx = -\lim_{j\to\infty} \int_\Omega \partial_i u_{m_j}\varphi\,dx = 0.$$

  Hence u is weakly differentiable and $\nabla u = 0$ in Ω. In Sheet 2, we show that this implies u is constant.

- As u has zero average, we must then have $u = 0$ in Ω, which contradicts the assertion that $\|u\|_{L^p} = 1$.

Local differentiability of Sobolev functions

Theorem

Suppose Ω is a domain in $\mathbb{R}^n$ and $n < p \leq \infty$. Assume that $u \in W^{1,p}(\Omega) \cap C(\Omega)$. Then u is differentiable a.e. in Ω and its derivatives equal its weak derivatives a.e. in Ω.

Proof

- We will only consider the case $p < \infty$. The case $p = \infty$ is a consequence.
- By Lebesgue's differentiation theorem, there is a set $Z \subset \Omega$ of measure zero such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy = 0 \text{ for all } x \in \Omega \setminus Z.$$

We aim to show that u is differentiable at those $x \in \Omega \setminus Z$.

Local differentiability of Sobolev functions

Proof

- Fix some $x \in \Omega \setminus Z$ and consider the function

$$v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x) \text{ for } y \in \Omega.$$

  Then $v \in W^{1,p}(\Omega) \cap C(\Omega)$, $v(x) = 0$ and $\nabla v(y) = \nabla u(y) - \nabla u(x)$.

- By Morrey's inequality, we have for every ball $B_r(x) \in \Omega$ and $y \in \partial B_r(x)$ that

$$\begin{aligned} |v(y)| = |v(y) - v(x)| &\leq [v]_{C^{0,1-\frac{n}{p}}(B_r(x))} |x - y|^{1-\frac{n}{p}} \\ &\leq C r^{1-\frac{n}{p}} \|\nabla v\|_{L^p(B_r(x))} \\ &= C r^{1-\frac{n}{p}} \Big\{ \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dx \Big\}^{1/p}. \end{aligned}$$

Local differentiability of Sobolev functions

Proof

- So we have
  - $\star$ $\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p\,dy = 0$, and
  - $\star$ $|v(y)| \leq C r^{1-\frac{n}{p}} \Big\{ \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p\,dy \Big\}^{1/p}$ whenever $|y-x| = r$.

  Putting the two together, we see that

$$\lim_{y\to x} \frac{1}{|y-x|} |u(y) - u(x) - \nabla u(x)\cdot(y-x)| = \lim_{y\to x} \frac{1}{|y-x|} |v(y)| = 0.$$

  This means that u is differentiable at x and its classical gradient at x is the same at its weak gradient at x.

L^p differentiability of Sobolev functions

Theorem

Suppose Ω *is a domain in* $\mathbb{R}^n$ *and* $1 \leq p < n$. *Assume that* $u \in W^{1,p}(\Omega)$. *Then for almost all* $x \in \Omega$ *it holds that*

$$\lim_{r \to 0} \frac{1}{r^{1+\frac{n}{p}}} \Big\{ \int_{B_r(x)} |u(y) - u(x) - \nabla u(x) \cdot (y - x)|^p \, dy \Big\}^{1/p} = 0.$$

Discussion of proof

- As in the case $p > n$, we start by picking a set $Z \subset \Omega$ of measure zero such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy = 0 \text{ for all } x \in \Omega \setminus Z.$$

L^p differentiability of Sobolev functions

Discussion of proof

- We consider again the function

$$v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x) \text{ for } y \in \Omega,$$

  so that $v \in W^{1,p}(\Omega)$ and $\nabla v(y) = \nabla u(y) - \nabla u(x)$. Note that however the meaning of $v(x) = 0$ is rather obscure since v does not have enough regularity.
- If we have the Poincaré-type inequality

$$\|v\|_{L^p(B_r(x))} \leq Cr\|\nabla v\|_{L^p(B_r(x))}, \qquad (*)$$

  then, by recalling that $r^{-n}\|\nabla v\|^p_{L^p(B_r(x))} \to 0$ as $r \to 0$, we can obtain the conclusion as in the case $p > n$ considered previously. However, (*) is general not valid for arbitrary functions $v \in W^{1,p}$.

L^p differentiability of Sobolev functions

Discussion of proof

- The proof is actually much more involved and goes through approximation of u by smooth functions.
- It should be clear that the conclusion hold when $u \in C^1(\Omega)$ as

$$u(y) - u(x) - \nabla u(x) \cdot (y - x) = o(|y - x|) \text{ as } y \to x.$$