Summary sheet on Sobolev spaces

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Integration by parts formula

Let Ω be a bounded Lipschitz domain and ν the outward normal to $\partial\Omega$. If $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_\Omega \partial_i u v \, dx = \int_{\partial\Omega} u v \nu_i \, dS(x) - \int_\Omega u \partial_i v \, dx.$$

Here the values of u and v on $\partial\Omega$ are understood in the sense of trace.

Density results

(i) Let $k \geq 0$ and $1 \leq p < \infty$. Then $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Proof: Convolution.

(ii) Let $k \geq 0$ and $1 \leq p < \infty$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. In particular $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

(iii) (Meyers-Serrin) Let $k \geq 0$, $1 \leq p < \infty$ and Ω be an open subset of $\mathbb{R}^n$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

(iv) Let $k \geq 0$, $1 \leq p < \infty$ and Ω be an open bounded subset of $\mathbb{R}^n$ satisfying the segment condition. Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Extension

(i) Let $k \geq 0$, $1 \leq p < \infty$ and Ω be an open subset of $\mathbb{R}^n$. If $u \in W_0^{k,p}(\Omega)$, then its extension by zero $\bar{u}$ to $\mathbb{R}^n$ belongs to $W_0^{k,p}(\mathbb{R}^n)$.

(ii) (Stein) Let Ω be a bounded Lipschitz domain. Then there exists a linear operator sending functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ such that for every $k \geq 0$, $1 \leq p < \infty$ and $u \in W^{k,p}(\Omega)$ it hold that $Eu = u$ a.e. in Ω and

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,p,\Omega} \|u\|_{W^{k,p}(\Omega)}.$$

Traces

Let $1 \leq p < \infty$ and Ω be a bounded Lipschitz domain. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.

Characterisation via translations

(i) Let $1 \le p < \infty$ and $v \in W^{1,p}(\mathbb{R}^n)$. Then

$$\|\tau_y v - v\|_{L^p} \le C_{n,p}|y|\|\nabla v\|_{L^p(\mathbb{R}^n)}.$$

(ii) Let $1 < p < \infty$, Ω be a bounded Lipschitz domain. If $v \in L^p(\Omega)$ and if $C > 0$ such that

$$\|\tau_y v - v\|_{L^p(\omega)} \le C|y| \text{ for any } \omega \Subset \Omega, |y| < dist(\omega, \partial\Omega),$$

then $v \in W^{1,p}(\Omega)$.

Embeddings

Unless otherwise stated, let Ω be a bounded Lipschitz domain

(i) (Gagliardo-Nirenberg-Sobolev: Gagliardo-Nirenberg for $p = 1$ and Sobolev for short in general) Let $1 \le p < n$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ continuously:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C_{n,p}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Proof: First prove the embedding inequality for $p = 1$ and smooth functions, by Newton-Leibnitz along lines parallel to axes and multiplying them all together. Then prove for $W^{1,1}$ functions using density, where in passing to limit one needs to use Fatou's lemma on one side of the inequality. Then prove for $W^{1,p}$ functions by applying the case $p = 1$ to a power.

Reason for the exponent p^*: Scaling.

(ii) $W^{1,n}(\mathbb{R}^n)$ does not embed in $L^\infty(\mathbb{R}^n)$.

(iii) (Gagliardo-Nirenberg-Sobolev) Let $1 \le p < n$. Then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ continuously for any $1 \le q \le p^*$:

$$\|u\|_{L^q(\Omega)} \le C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.$$

Proof: Via extension.

(iv) (Morrey) Let $n < p \le \infty$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ continuously:

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof for finite p: Reduce to smooth case via density. Compare the value of u at a point by its average on a ball using the integral mean value inequality:

$$\int_{B_r(x)} |u(y) - u(x)|\, dx \le \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} dx.$$

(v) (Morrey) Let $n < p \leq \infty$. Then $W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\mathbb{R}^n)$ continuously for any $0 < \beta < 1 - \frac{n}{p}$.

Proof: Via extension.

(vi) $W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$.

(vii) (Rellich-Kondrachov) Let $1 \leq p < n$. Then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ compactly for any $1 \leq q < p^*$. The limit case $q = p^*$ is non-compact.

Proof: First prove for $q = p$ using Komolgorov-Riesz-Fréchet. This implies the case $q < p$. For $p < q < p^*$, use interpolation knowing the convergence in L^p and boundedness in L^{p^*}.

Reason for non-compactness in the critical case: Scaling, translations.

(viii) $W^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ compactly for any $1 \leq q < \infty$. This follows from Rellich-Kondrachov.

(ix) Let $p > n$. Then $W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$ compactly for any $0 < \beta < 1 - \frac{n}{p}$. The limit case $\beta = 1 - \frac{n}{p}$ is non-compact.

Proof: Use Ascoli-Arzèla and Morrey.

Reason for non-compactness in the critical case: Scaling, translations.

(x) (Friedrichs) Let $1 \leq p < \infty$ and Ω be a bounded open set. Then

$$\|u\|_{L^p(\Omega)} \leq C_{n,p,\Omega} \|Du\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

(xi) (Friedrichs-type) Let $1 \leq p < n$, $1 \leq q \leq p^*$ and Ω be a bounded open set. Then

$$\|u\|_{L^q(\Omega)} \leq C_{n,p,\Omega} \|Du\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

(xii) (Poincaré) Let $1 \leq p \leq \infty$. Then

$$\|u - \bar{u}_\Omega\|_{L^p(\Omega)} \leq C_{n,p,\Omega} \|Du\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

Proof: Argue by contradiction and appeal to Rellich-Kondrachov.

(xiii) (Poincaré-Sobolev) Let $1 \leq p < n$ and $1 \leq q \leq p^*$. Then

$$\|u - \bar{u}_\Omega\|_{L^q(\Omega)} \leq C_{n,p,\Omega} \|Du\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

Proof: Apply Gagliardo-Nirenberg-Sobolev inequality, then appeal to Poincaré inequality. (Mimicking the proof of Poincaré only gives the case $q < p^*$.)