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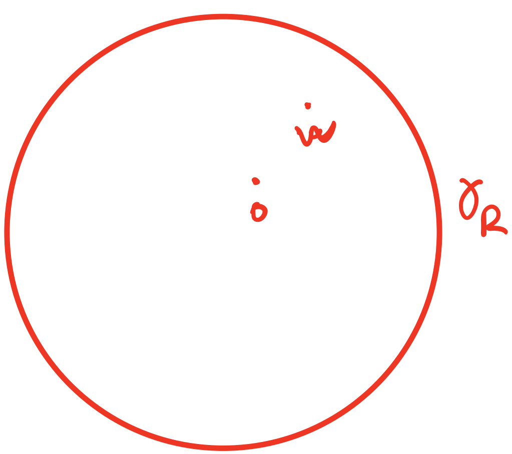

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Thus as $R \to \infty$ we get $|f(w) - f(0)| = 0$, so f is constant.

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Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ *is a non-constant polynomial where* $a_k \in \mathbb{C}$ *and* $a_n \neq 0$. *Then there is a* $z_0 \in \mathbb{C}$ *for which* $p(z_0) = 0$.

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We note that f is bounded on any disc $\bar{B}(0, R)$, so it suffices to show that $|f(z)| \to 0$ as $z \to \infty$, that is, to show that $|p(z)| \to \infty$ as $z \to \infty$.

$$|p(z)| = |z^n + \sum_{k=0}^{n-1} a_k z^k| = |z^n|\big(|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}|\big) \geq |z^n| \cdot (1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}).$$

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Since $\frac{1}{|z|^m} \to 0$ as $|z| \to \infty$ for any $m \geq 1$ it follows that for sufficiently large $|z|$, say $|z| \geq R$, we will have

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Thus for $|z| \geq R$ we have $|p(z)| \geq \frac{1}{2}|z|^n$. Since $|z|^n \to \infty$ as $|z| \to \infty$ it follows $|p(z)| \to \infty$ so $f(z)$ is constant and hence $p(z)$ is constant. □

Theorem

(Morera's theorem) Suppose that $f\colon U \to \mathbb{C}$ is a continuous function on a domain $U \subseteq \mathbb{C}$. If for any closed path $\gamma\colon [a,b] \to U$ we have $\int_\gamma f(z)dz = 0$, then f is holomorphic.

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But this follows from our proof of Cauchy's theorem for starlike domains as $B(a,r)$ is convex.

Theorem

(Riemann's removable singularity theorem): Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$. If $f: U \backslash \{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then f extends to a holomorphic function on all of U.

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$$h(z) = \begin{cases} (z - z_0)^2 f(z), & z \neq z_0; \\ 0, & z = z_0 \end{cases}$$

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At $z = z_0$:

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If we chose $r > 0$ s.t. $\bar{B}(z_0, r) \subset U$, then $h(z)$ is equal to its Taylor series centred at z_0, thus

$$h(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

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But this is equal to $f(z)$ on $B(z_0, r)\backslash\{z_0\}$, so by redefining $f(z_0) = a_2$, we can extend f to a holomorphic function on all of U. □

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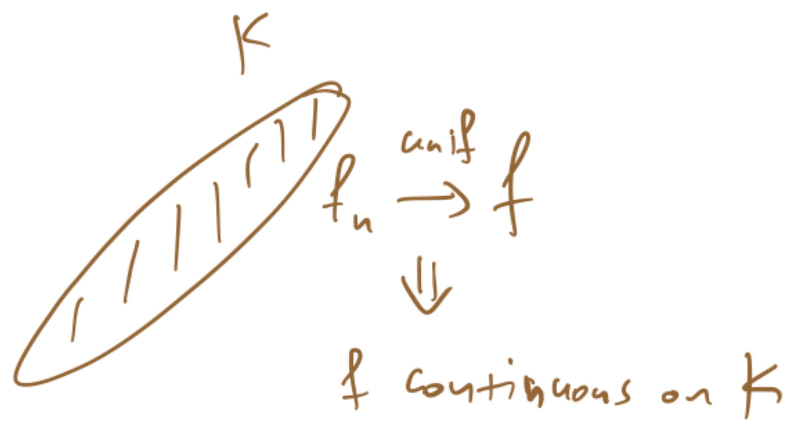
Let U be an open subset of $\mathbb{C}$. If (f_n) is a sequence of functions defined on U, we say $f_n \to f$ uniformly on compacts if for every compact subset K of U, the sequence $(f_{n|K})$ converges uniformly to $f_{|K}$.



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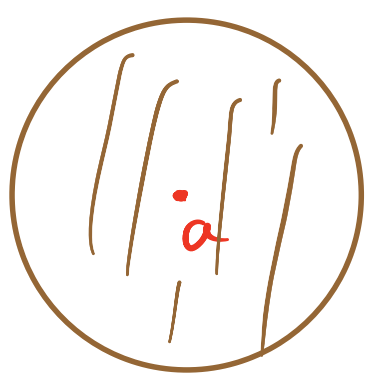
Note that in this case f is continuous if the f_n are: Let $a \in U$. Since U is open, $\bar{B}(a, r) \subseteq U$ for some r. $K = \bar{B}(a, r)$ is compact and $f_n \to f$ uniformly on K, so f is continuous on K, hence it is continuous at a.



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Example

Power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$.
If R is the radius of convergence of $f(z)$ the partial sums $s_n(z)$ of the power series converge uniformly on compacts in $B(0,R)$ as they converge uniformly on $B(0,r)$ for $r < R$.

Proposition

Suppose that U is a domain and the sequence of holomorphic functions $f_n \colon U \to \mathbb{C}$ converges to $f \colon U \to \mathbb{C}$ uniformly on compacts in U. Then f is holomorphic.

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But $\gamma^* = \gamma([a, b])$ is a compact subset of U, hence $f_n \to f$ uniformly on γ^*. It follows that

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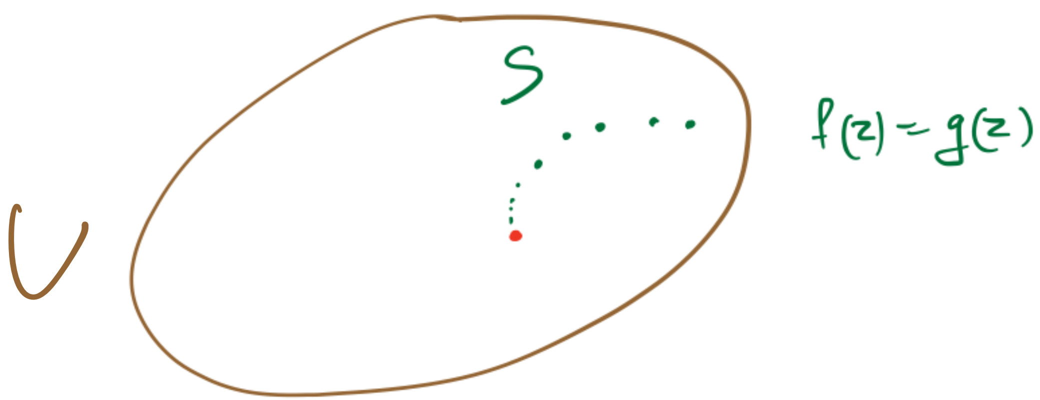
So f has a primitive F on $B(w, r)$. F is differentiable, hence infinitely differentiable, so f is differentiable on $B(w, r)$. □

The Identity Theorem

Let f, g be two holomorphic functions defined on a domain U and let $S = \{z \in U : f(z) = g(z)\}$ be the locus on which they are equal. Then if S has a limit point in U we have actually $f(z) = g(z)$, $\forall z$!



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Proposition

Let U be an open set and suppose that $g \colon U \to \mathbb{C}$ is holomorphic on U. Let $S = \{z \in U : g(z) = 0\}$. If $z_0 \in S$ then either z_0 is isolated in S (so that g is non-zero in some disk about z_0 except at z_0 itself) or $g = 0$ on a neighbourhood of z_0. In the former case there is a unique integer $k > 0$ and holomorphic function g_1 such that $g(z) = (z - z_0)^k g_1(z)$ where $g_1(z_0) \neq 0$.

Proof. Let $z_0 \in U$ with $g(z_0) = 0$. Since U is open and g is analytic at z_0, there is an $r > 0$ such that

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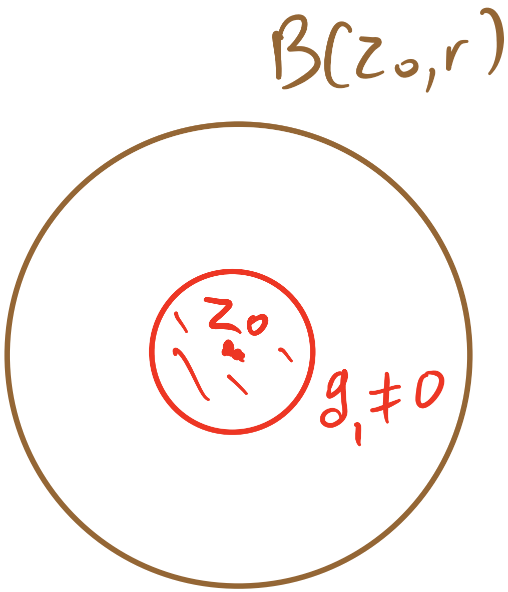

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There is an $\epsilon > 0$ such that $g_1(z) \neq 0$ for all $z \in B(z_0, \epsilon)$. Since $g(z) = (z - z_0)^k g_1(z)$, z_0 is isolated.

To see that *k is unique*, suppose that $g(z) = (z - z_0)^k g_1(z) = (z - z_0)^l g_2(z)$ say with $g_1(z_0)$ and $g_2(z_0)$ both nonzero.

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The integer k in the previous proposition is called the multiplicity of the zero of g at $z = z_0$ (or sometimes the order of vanishing).

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Theorem

(Identity theorem): Let U be a domain and suppose that f_1, f_2 are holomorphic functions defined on U. Then if $S = \{z \in U : f_1(z) = f_2(z)\}$ has a limit point in U, we must have $S = U$, that is $f_1(z) = f_2(z)$ for all $z \in U$.

Proof.

Let $g = f_1 - f_2$, so $S = \{z : g(z) = 0\}$.

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Denote by T the set of limit points of S in U. We note that since g is continuous $T \subseteq S$. We will show that T is both closed and open. Since it is non-empty and U is connected $T = U$, hence $S = U$.

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T is closed in U:

If $z_n \to a \in U$ with $z_n \in T$ then $g(a) = 0$. So $a \in T$, hence T is closed. $\square$

Remark

The requirement in the theorem that S have a limit point lying in U is essential: For example take $U = \mathbb{C} \setminus \{0\}$ *and* $f_1 = \sin(1/z)$ *and* $f_2 = 0$*.*

Now the zeros of f_1 *have a limit point at* $0 \notin U$ *since* $f_1(1/(\pi n)) = 0$ *for all* $n \in \mathbb{N}$*, but certainly* f_1 *is not identically zero on U!*

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Also the connectedness of U is necessary: if U is a union of two disjoint open discs D_1, D_2 *we may define* $f = 0$ *on* D_1 *and* $f = 1$ *on* D_2. *f is holomorphic on U but not equal to* 0.

Example

Show that there is no holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $f(x) = \log x$ for all $x \in \mathbb{R}_+$.

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However then $f(z)$ is not continuous on $(-\infty, 0]$.

Isolated Singularities

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Definition

If f is a function that is holomorphic on $B(z_0, r)\backslash\{z_0\}$ for some $r > 0$ but is not holomorphic at z_0, then we say that z_0 is an isolated singularity of f. It is possible that f is not defined at z_0 or that it is defined but it is not holomorphic at z_0.

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If f has an isolated singularity at z_0 which is not removable nor a pole, we say that z_0 is an essential singularity.

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So $(1/f)(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$ and $m \in \mathbb{Z}_{>0}$. We say that m is the order of the pole of f at z_0.

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We have then $f(z) = (z - z_0)^{-m} \cdot (1/g)$ near z_0, where $1/g$ is holomorphic near z_0. If $m = 1$ we say that f has a simple pole at z_0.

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$h(z)$ is not bounded at 0 and $\frac{1}{\exp(1/z)}$ is not continuous at 0, so not holomorphic. For example note that $h(1/n) \to \infty$ while $h(1/2\pi in) = \exp(2\pi in) = 1$. So we have an essential singularity.

Lemma

Let f be a holomorphic function with a pole of order m at z_0. Then there is an $r > 0$ such that for all $z \in B(z_0, r) \setminus \{z_0\}$ we have

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

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Near z_0, $h(z)$ is equal to its Taylor series at z_0, and multiplying this by $(z - z_0)^{-m}$ gives a series of the required form for $f(z)$. □

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A function on an open set U which has only isolated singularities all of which are poles is called a meromorphic function on U.

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Suppose that f has an isolated singularity at a point z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

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On the other hand, if $|f(z)| \to \infty$ as $z \to z_0$, then $1/f(z) \to 0$ as $z \to z_0$, so that $1/f(z)$ has a removable singularity and f has a pole at z_0. □

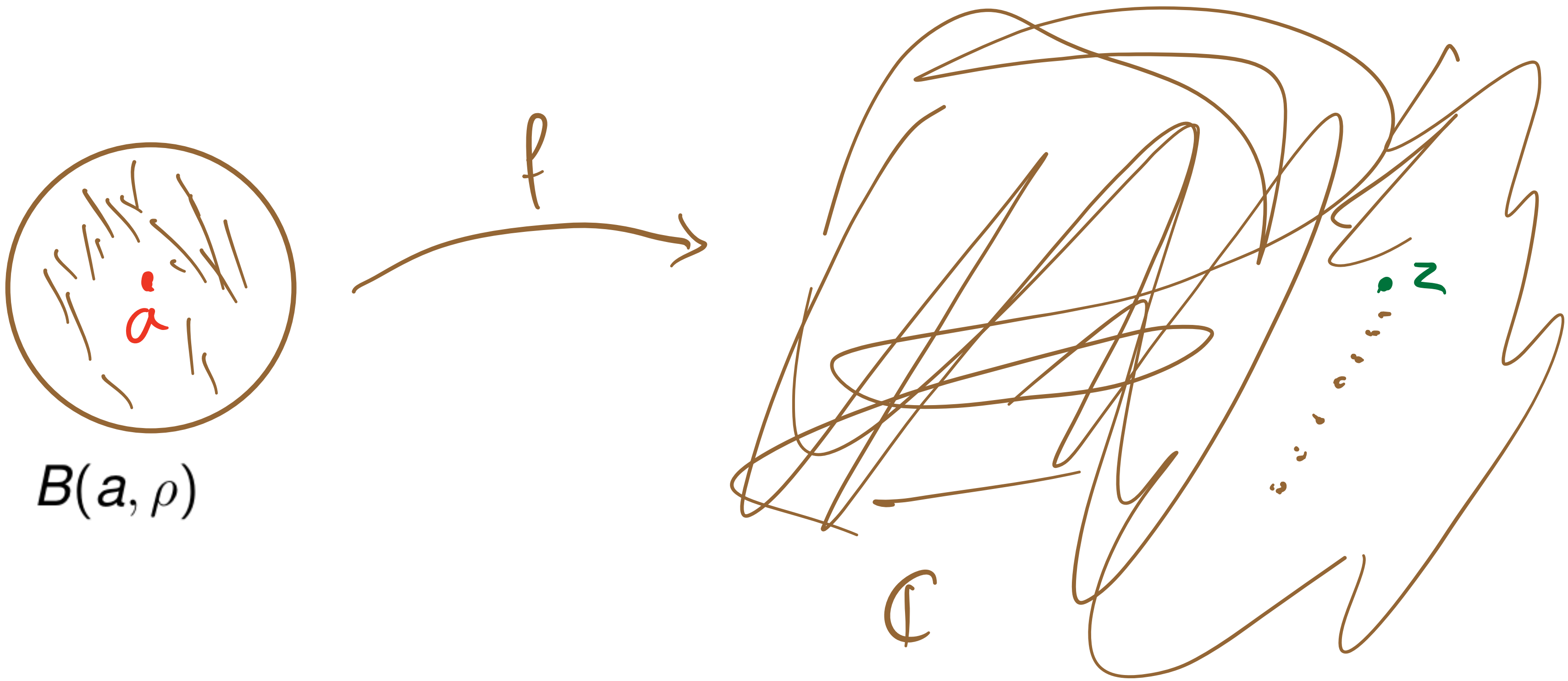
Remark

The previous Lemma can be rephrased to say that f has a pole at z_0 precisely when f extends to a continuous function $f\colon U \to \mathbb{C}_\infty$ with $f(z_0) = \infty$.

Essential singularities.

Theorem

(Casorati-Weierstrass): Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f : U \backslash \{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a, \rho) \subseteq U$, the set $f(B(a, \rho) \backslash \{a\})$ is dense in $\mathbb{C}$, that is, the closure of $f(B(a, \rho) \backslash \{a\})$ is all of $\mathbb{C}$.



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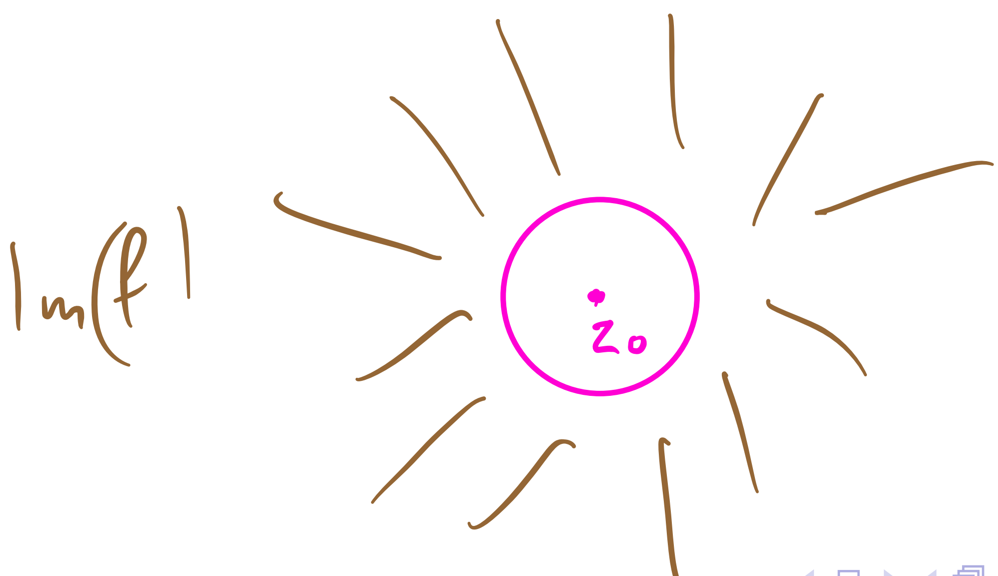

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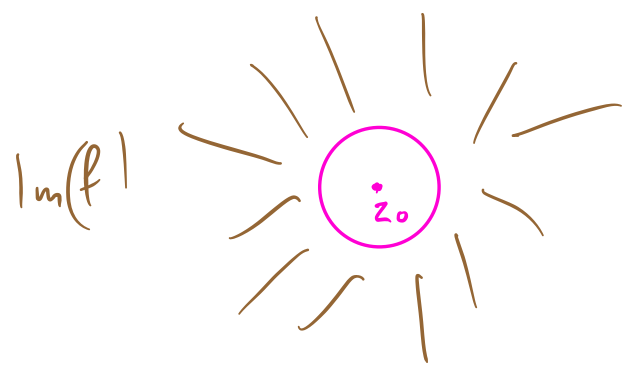

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If $g(a) = 0$, $|1/g(z)| \to \infty$ as $z \to a$, so $|f(z)| \to \infty$ as $z \to a$, and f has a pole at a, a contradiction. □

Remark

In fact Picard showed that if f has an isolated essential singularity at z_0 then in any open disk about z_0 the function f takes ***every*** *complex value infinitely often with at most one exception.*

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$f(z) = exp(1/z)$, *has an essential singularity at* $z = 0$ *and* $f(z) \neq 0$ *for all* $z \neq 0$ *so this result is best possible.*

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The residue of f at z_0 is defined to be the coefficient c_{-1} and denoted $\text{Res}_{z_0}(f)$.

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Then for each $z_0 \in S$ we have the principal part $P_{z_0}(f)$ of f at z_0, a rational function which is holomorphic everywhere on $\mathbb{C}\backslash\{z_0\}$.

It turns out that the residue is 'all that counts' when we calculate integrals along closed paths.

Say $f\colon U \to \mathbb{C}_\infty$ is a meromorphic function with poles at a finite set $S \subseteq U$.

Then for each $z_0 \in S$ we have the principal part $P_{z_0}(f)$ of f at z_0, a rational function which is holomorphic everywhere on $\mathbb{C}\backslash\{z_0\}$.

The difference

$$g(z) = f(z) - \sum_{z_0 \in S} P_{z_0}(f),$$

is holomorphic on all of U.

Thus if U is starlike and $\gamma\colon [0,1] \to U$ is any closed path in U with $\gamma^* \cap S = \emptyset$, we have

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz + \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz = \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz.$$

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This is the residue theorem for meromorphic functions on a starlike domain.

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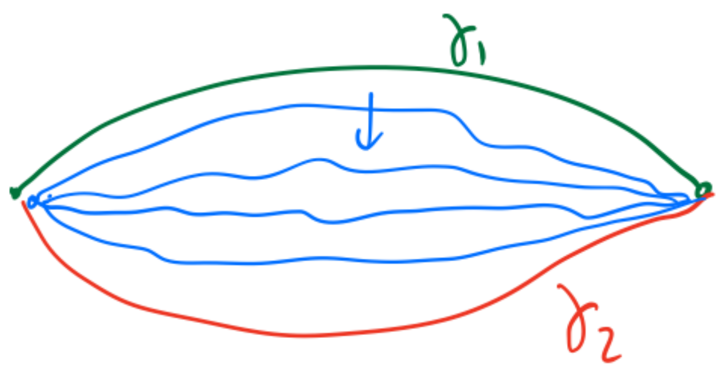

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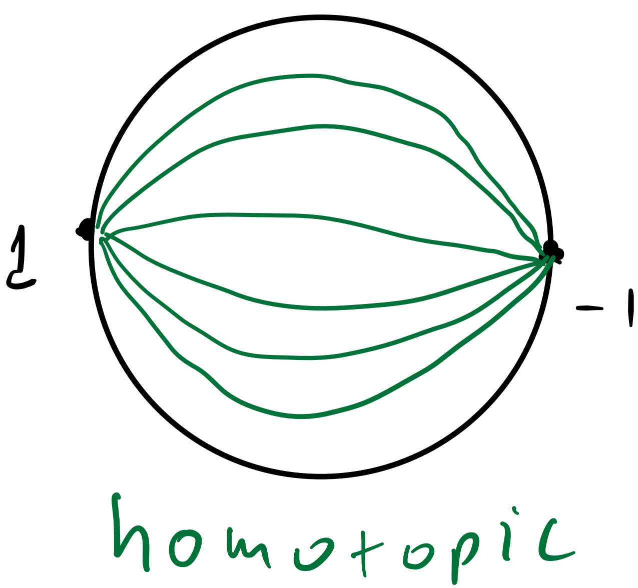

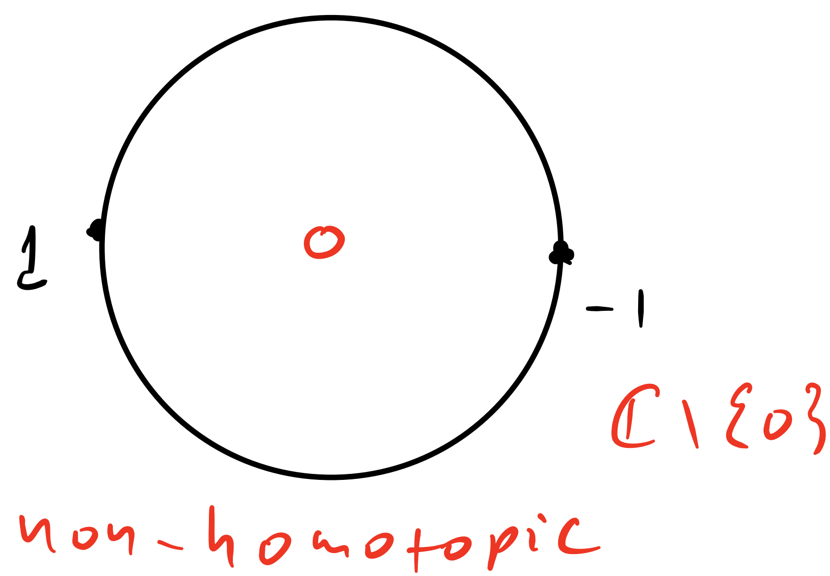

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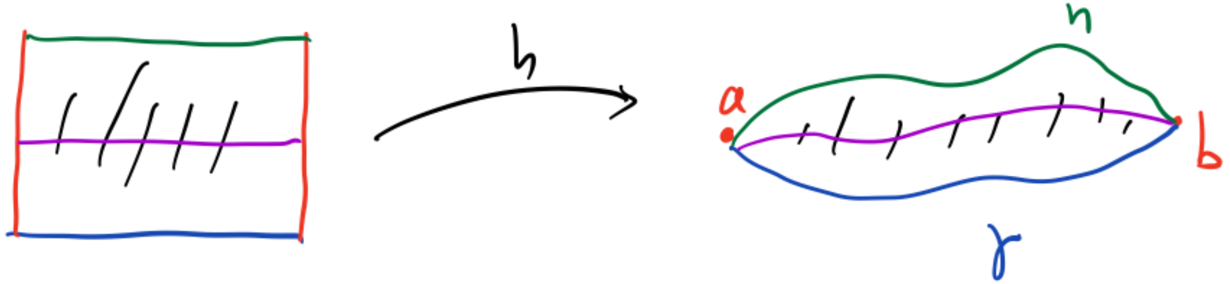
What does it mean 'continuously deform'? We will need a function of two variables to express this.

Definition

Suppose that U is an open set in $\mathbb{C}$ and $a, b \in U$ and that $\eta\colon [0,1] \to U$ and $\gamma\colon [0,1] \to U$ are paths in U such that $\gamma(0) = \eta(0) = a$ and $\gamma(1) = \eta(1) = b$. We say that γ and η are homotopic in U if there is a continuous function $h\colon [0,1] \times [0,1] \to U$ such that

$$h(0,s) = a, \quad h(1,s) = b$$
$$h(t,0) = \gamma(t), \quad h(t,1) = \eta(t).$$



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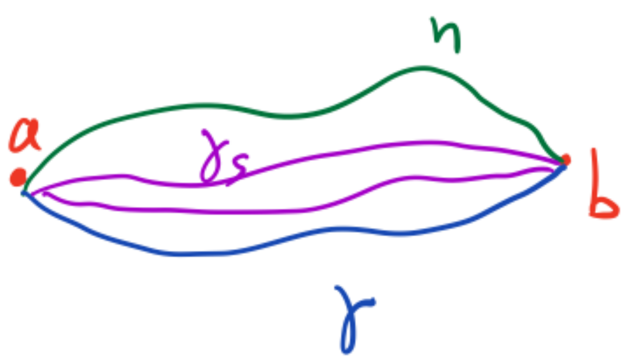
One should think of h as a family of paths in U indexed by the second variable s which continuously deform γ into η.



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Definition

Suppose that U is a domain in $\mathbb{C}$. We say that U is simply connected if for every $a, b \in U$, any two paths from a to b are homotopic in U.

Lemma

Let U be a convex open set in $\mathbb{C}$. Then U is simply connected. Moreover if U_1 and U_2 are homeomorphic, then U_1 is simply connected if and only if U_2 is.

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Proof.

Suppose that $\gamma\colon [0,1] \to U$ and $\eta\colon [0,1] \to U$ are paths starting and ending at a and b respectively for some $a, b \in U$. Then for $(s,t) \in [0,1] \times [0,1]$ let

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Then h is continuous and gives the required homotopy.

If $f\colon U_1 \to U_2$ is a homeomorphism then f and γ, η with common endpoints in U_2 then $f^{-1}(\gamma), f^{-1}(\eta)$ are paths with common endpoints in U_1. If h is a homotopy between them in U_1 then $f \circ h$ is a homotopy between γ, η. So if U_1 is simply connected then U_2 is too.

□

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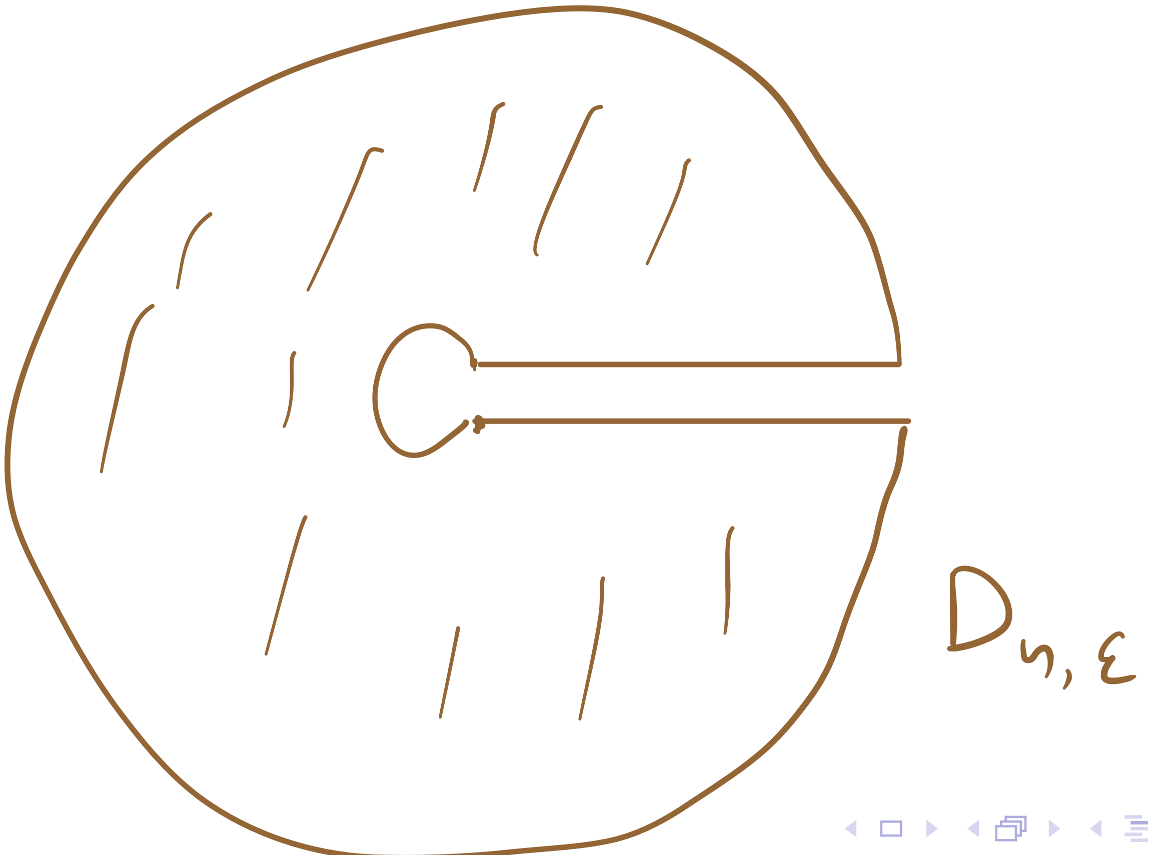
If D is star-like with respect to $z_0 \in D$, then if $\gamma : [0,1] \to D$ is a closed path with $\gamma(0) = \gamma(1) = z_0$, it follows $h(s,t) = z_0 + s(\gamma(t) - z_0)$ gives a homotopy between γ and the constant path c_{z_0}.

Example

Consider the domain

$$D_{\eta,\epsilon} = \{z \in \mathbb{C} : z = re^{i\theta} : \eta < r < 1, 0 < \theta < 2\pi(1-\epsilon)\},$$

where $0 < \eta, \epsilon < 1/10$ say. We claim that it is simply connected.



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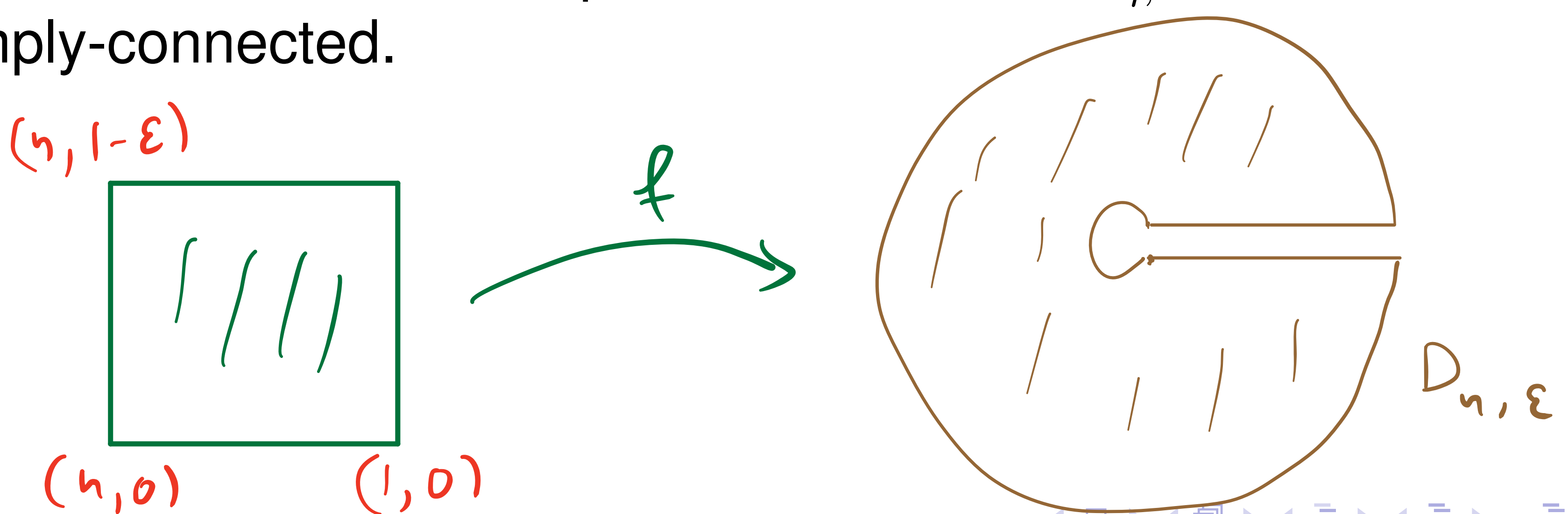
Indeed it is the image of the convex set $(\eta, 1) \times (0, 1-\epsilon)$ under the map $(r, \theta) \mapsto re^{2\pi i\theta}$. Since this map has a continuous inverse, it is a homeomorphism so it follows $D_{\eta,\epsilon}$ is simply-connected.



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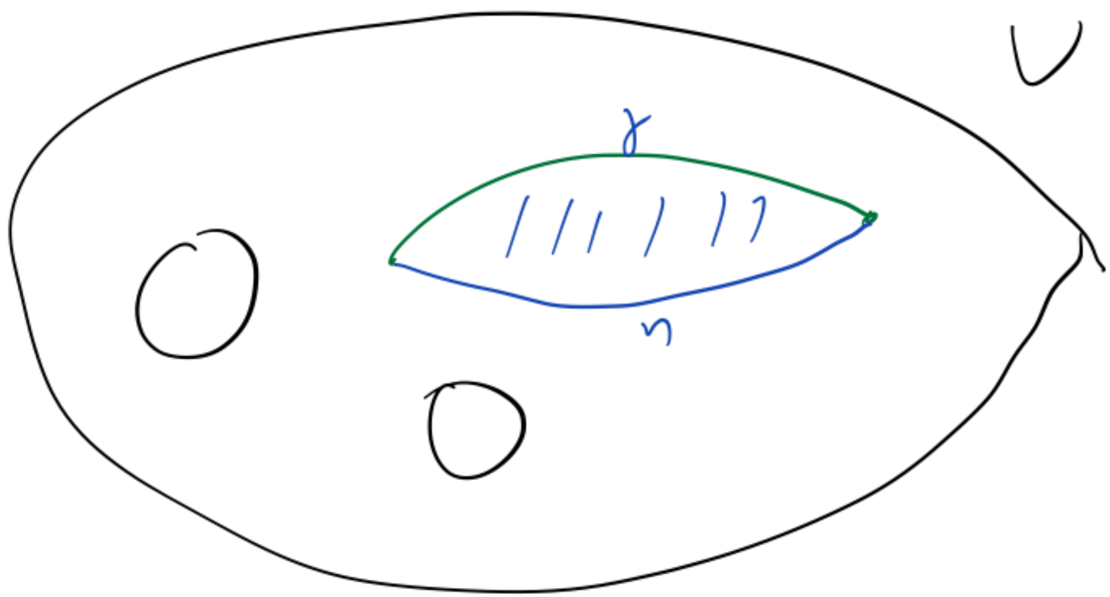
When η and ϵ are small, the boundary of this set, oriented anti-clockwise, is a version of what is called a key-hole contour.

Theorem

(Homotopy form of Cauchy's Theorem)
Let U be a domain in $\mathbb{C}$ and $a, b \in U$. Suppose that γ and η are paths from a to b which are homotopic in U and $f\colon U \to \mathbb{C}$ is a holomorphic function. Then

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$



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Remark

One significance of the homotopy form of Cauchy's theorem is that it applies to domains U even when there is no primitive for f on U-while in the earlier version of this theorem our proof proceeded by showing that f has a primitive in a star-like domain.

Theorem

Suppose that U is a simply-connected domain, let $a, b \in U$, and let $f\colon U \to \mathbb{C}$ be a holomorphic function on U. Then if γ_1, γ_2 are paths from a to b we have

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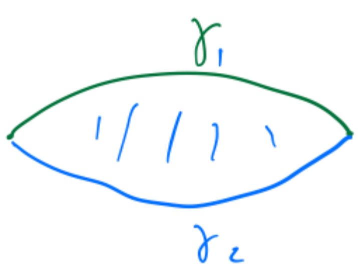

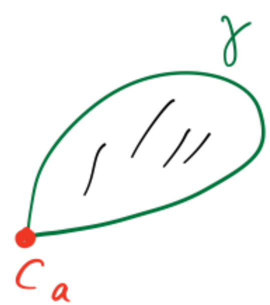

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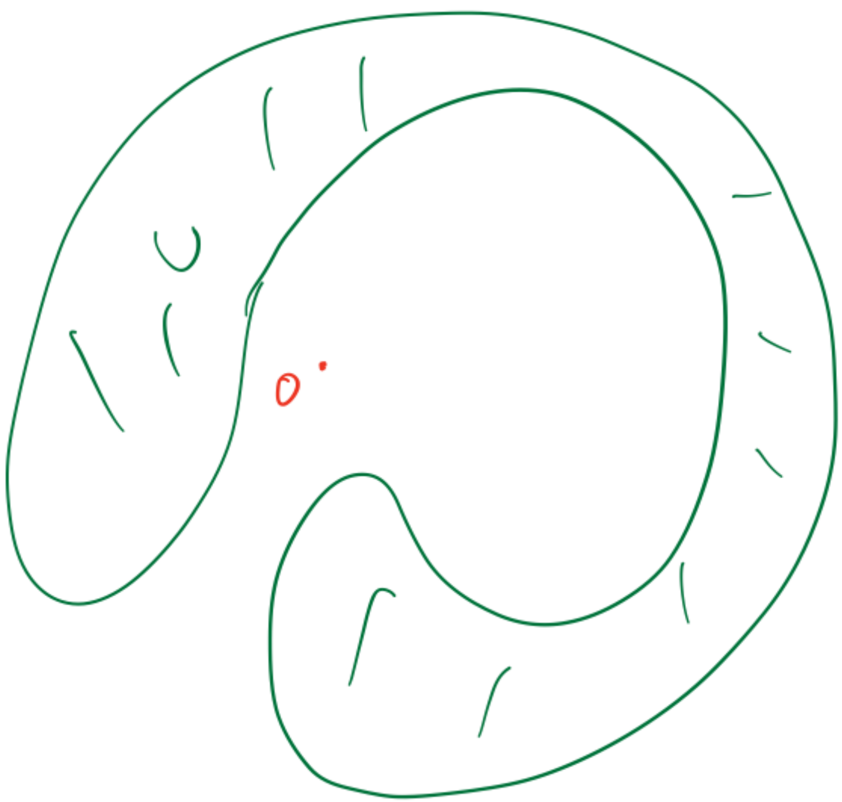
The final assertion then follows as vanishing of all these integrals implies that f has a primitive. □

Example

If $U \subseteq \mathbb{C}\backslash\{0\}$ is simply-connected, the previous theorem implies that there is a holomorphic branch of $[\mathrm{Log}(z)]$ defined on all of U.



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So by definition of the logarithm f is a holomorphic branch of $[\mathrm{Log}(z)]$ in U.

Remark

In previous lectures we called a domain D in the complex plane primitive if every holomorphic function $f : D \to \mathbb{C}$ on it had a primitive. Cauchy's Theorem shows that any simply-connected domain is primitive. In fact the converse is also true – any primitive domain is necessarily simply-connected. Thus the term "primitive domain" is in fact another name for a simply-connected domain.

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Let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma\colon [0,1] \to U$ be a closed path whose inside lies entirely in U, that is $I(\gamma, z) = 0$ for all $z \notin U$. Then we have, for all $z \in U \backslash \gamma^*$,

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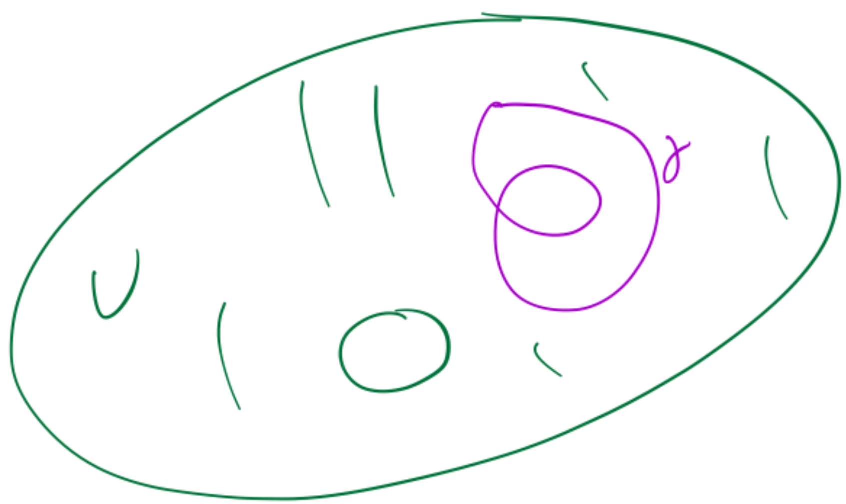

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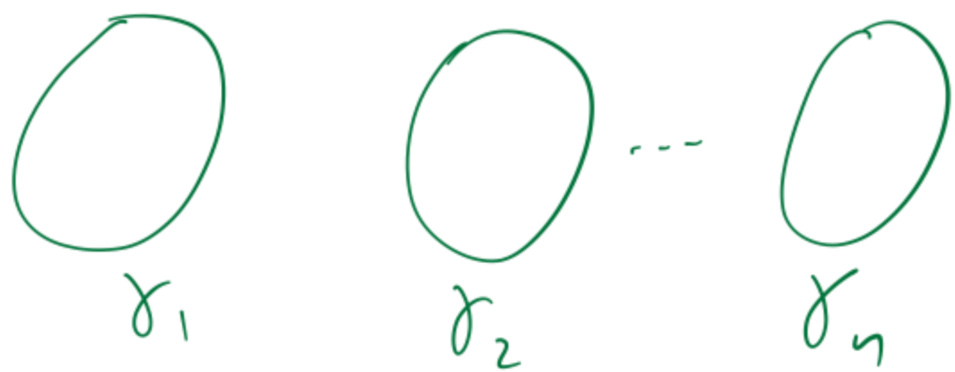
Moreover, if U is simply-connected and $\gamma: [a,b] \to U$ is any closed path, then $I(\gamma, z) = 0$ for any $z \notin U$, so the above identities hold for all closed paths in such U.

Remark

The "moreover" statement follows from the fact that a simply-connected domain is primitive: if D is a domain and $w \notin D$, then the function $1/(z - w)$ is holomorphic on all of D, and hence has a primitive on D. It follows $I(\gamma, w) = 0$ for any path γ with $\gamma^* \subseteq D$.

Remark. The homology version of Cauchy's theorem has a natural extension: instead of integrating over a single closed path, one can integrate over formal sums of closed paths.
A cycle is a formal sum $\Gamma = \sum_{i=1}^{k} a_i \gamma_i$ where $a_1, \ldots, a_k \in \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_k$ are closed paths.
We define the integral of a function f along the cycle Γ to be

$$\int_\Gamma f(z)dz = \sum_{i=1}^{k} a_i \int_{\gamma_i} f(z)dz.$$



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$I(\Gamma, z) = \sum_{i=1}^{k} a_i I(\gamma_i, z)$. If we write $\Gamma^* = \gamma_1^* \cup \ldots \cup \gamma_k^*$ then $I(\Gamma, z)$ is defined for all $z \notin \Gamma^*$.
We define the inside of a cycle to be the set of $z \in \mathbb{C}$ for which $I(\Gamma, z) \neq 0$.

Theorem

(Cauchy's Theorem, Homology version) Let $f : U \to \mathbb{C}$ be a holomorphic function and let Γ be a cycle whose inside lies entirely in U, that is $I(\Gamma, z) = 0$ for all $z \notin U$. Then we have, for all $z \in U \backslash \Gamma^*$,

$$\int_\Gamma f(\zeta) d\zeta = 0; \quad \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i \cdot I(\Gamma, z) f(z).$$

Note that if z is inside $\Gamma = \sum_{i=1}^{k} a_i\gamma_i$ then it must be the case that z is inside some γ_i, but the converse is not necessarily the case: it may be that z lies inside some of the γ_i but does not lie inside Γ.

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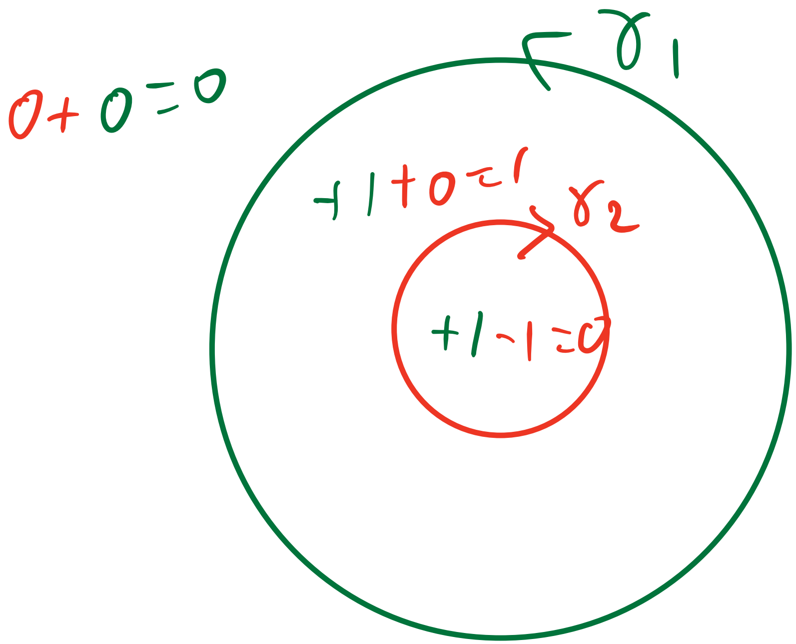
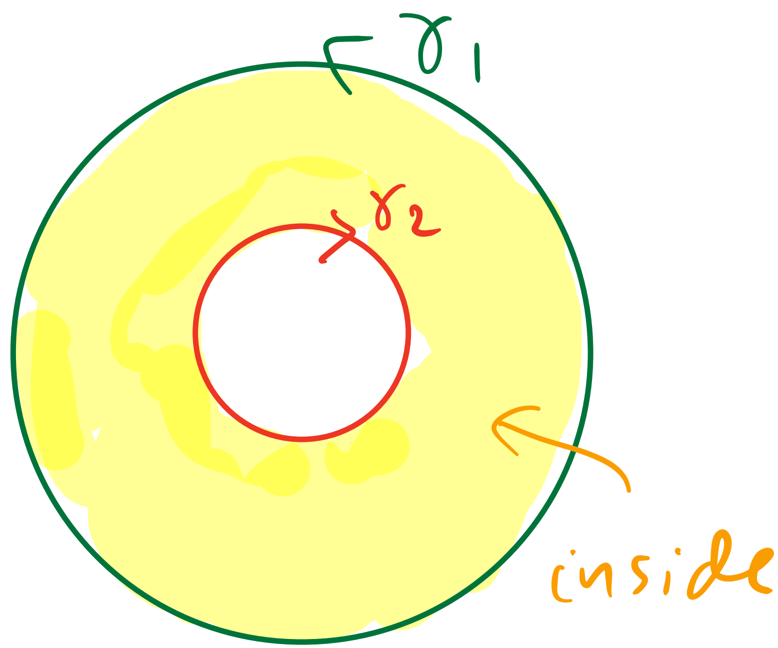
For example take Γ to be the sum of two concentric circles with opposite orientation. Then the center is not inside Γ. In this case the set of points 'inside' Γ is the annulus between the two circles.





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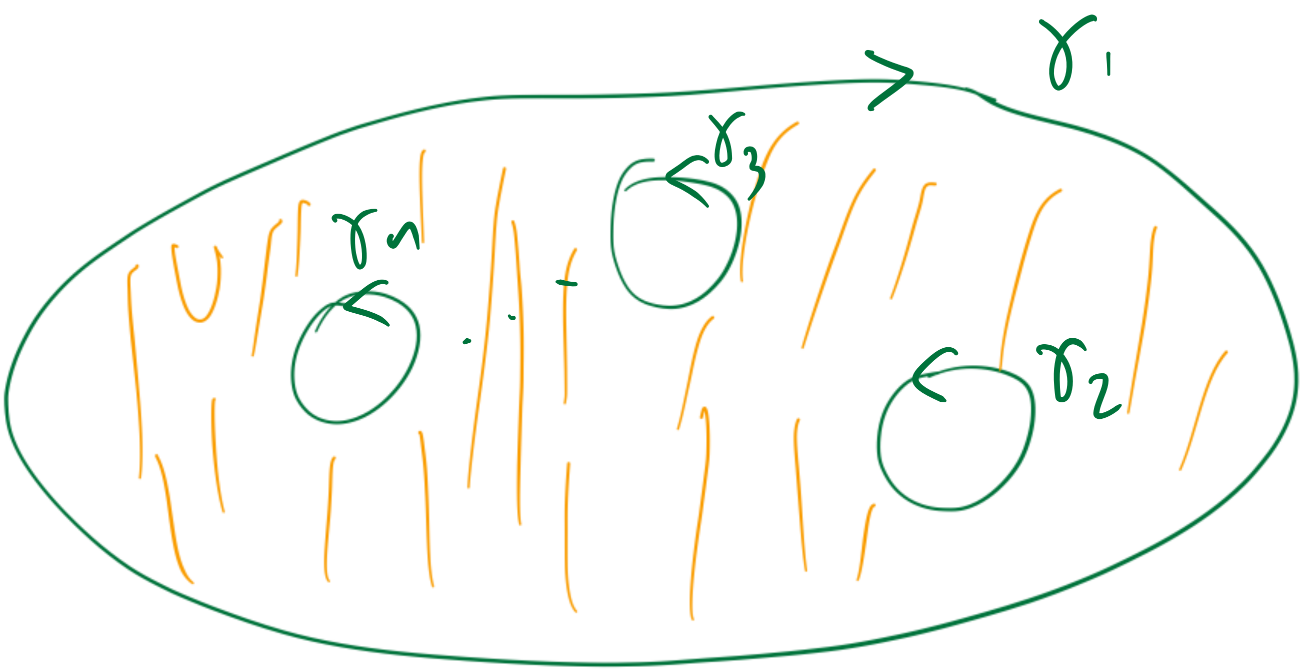

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More generally cycles appear naturally as follows.
Let D be a non-simply connected domain such that ∂D is a union of continuous simple closed curves $\gamma_1, ..., \gamma_n$. Then if γ_1 is the boundary of the unbounded component of $\mathbb{C} \setminus D$ and we give $\gamma_2, ..., \gamma_n$ the same orientation as γ_1 then the inside of the cycle

$$\Gamma = \gamma_1 - \gamma_2 - .... - \gamma_n$$

is exactly the domain D.

Laurent series

Definition

By a *Laurent series (or Laurent expansion)* around z_0 we mean a series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

We say that this series converges absolutely (uniformly) on a set $A \subset \mathbb{C}$ if the two series

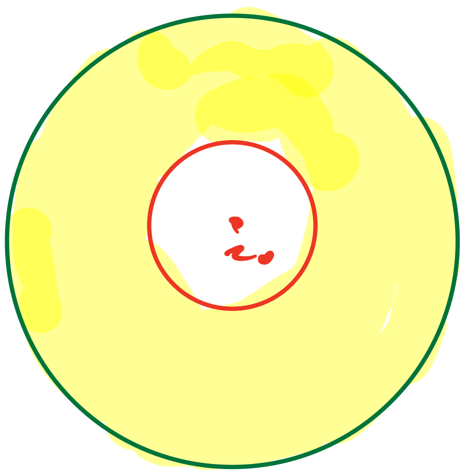
$$f^+(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad f^-(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^{-n},$$

converge absolutely (uniformly) on A. Then the sum of the Laurent series is the function $f(z) = f^+(z) + f^-(z)$.

Definition

Let $0 \leq r < R$ be real numbers and let $z_0 \in \mathbb{C}$. An open annulus is a set

$$A = A(r, R, z_0) = B(z_0, R) \setminus \bar{B}(z_0, r) = \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$



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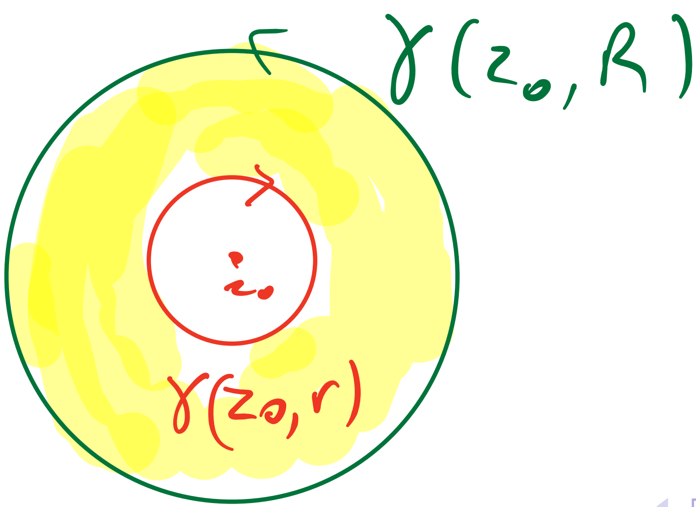
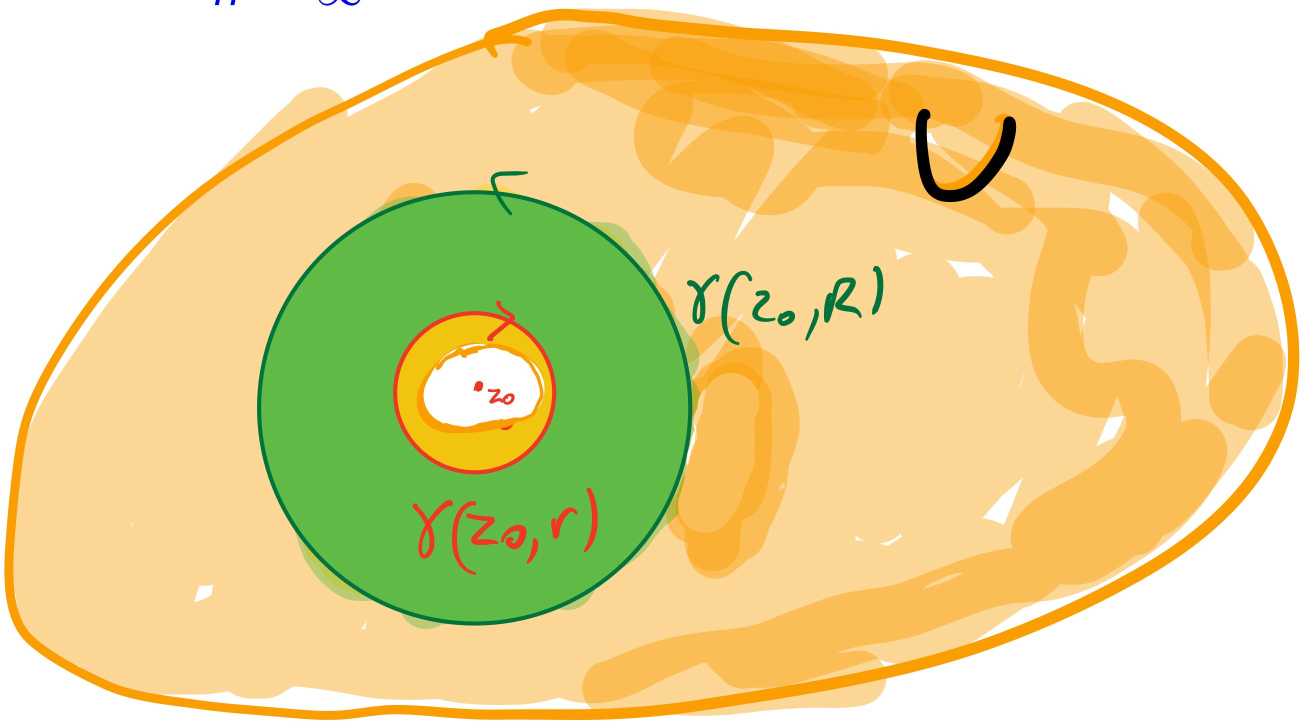
If we write (for $s > 0$) $\gamma(z_0, s)$ for the closed path $t \mapsto z_0 + se^{2\pi it}$ then notice that the inside of the cycle $\Gamma_{r,R,z_0} = \gamma(z_0, R) - \gamma(z_0, r)$ is precisely A, since for any s, $I(\gamma(z_0, s), z)$ is 1 precisely if $z \in B(z_0, s)$ and 0 otherwise.



Theorem

Suppose that $0 < r < R$ *and* $A = A(r, R, z_0)$ *is an annulus centred at* z_0. *If* $f : U \to \mathbb{C}$ *is holomorphic on an open set* U *which contains* $\bar{A}$, *then there exist* $c_n \in \mathbb{C}$ *such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \forall z \in A.$$



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$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \forall z \in A.$$

Moreover, the c_n *are unique and are given by the following formulae:*

$$c_n = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where $s \in [r, R]$ *and for any* $s > 0$ *we set* $\gamma_s(t) = z_0 + se^{2\pi it}$*.*

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3.This applies in particular to power series. For example if $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R then it converges uniformly on compact sets in $B(0,R)$. So if γ is a C^1-path in $B(0,R)$

$$\int_\gamma \sum_{n=0}^{\infty} a_n z^n dz = \sum_{n=0}^{\infty} \int_\gamma a_n z^n.$$

5. Note that if $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R then $\sum_{n=-1}^{-\infty} a_n z^n$ converges absolutely for $|z| > r = 1/R$ so it is holomorphic in $\mathcal{C} \setminus \bar{B}(0, r)$.

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6. $\frac{1}{1-z} = 1 + z + z^2 + ...$ and the convergence is uniform for $|z| < r < 1$. More generally we have

$$\frac{1}{w-z} = \frac{1}{w(1-z/w)} = 1/w + z/w^2 + z^2/w^3 + ...$$

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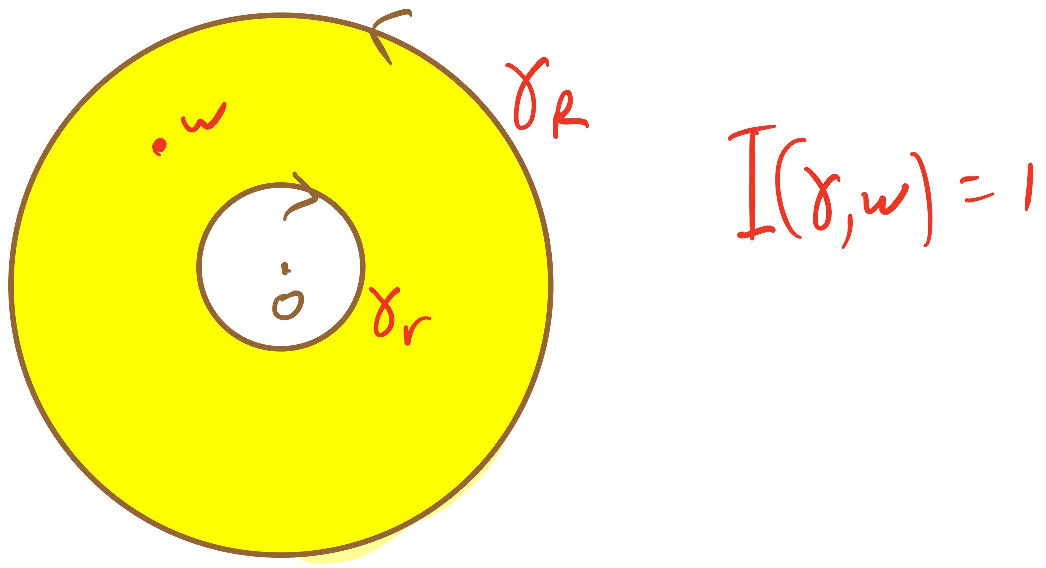
7. Cauchy's integral formula. Here we will need the general winding number version of this.

Proof By translation we may assume that $z_0 = 0$. Since A is the inside of the cycle Γ_{r,R,z_0} it follows from the winding number form of Cauchy's integral formula that for $w \in A$ we have

$$2\pi i f(w) = \int_{\gamma_R} \frac{f(z)}{z-w}\,dz - \int_{\gamma_r} \frac{f(z)}{z-w}\,dz$$



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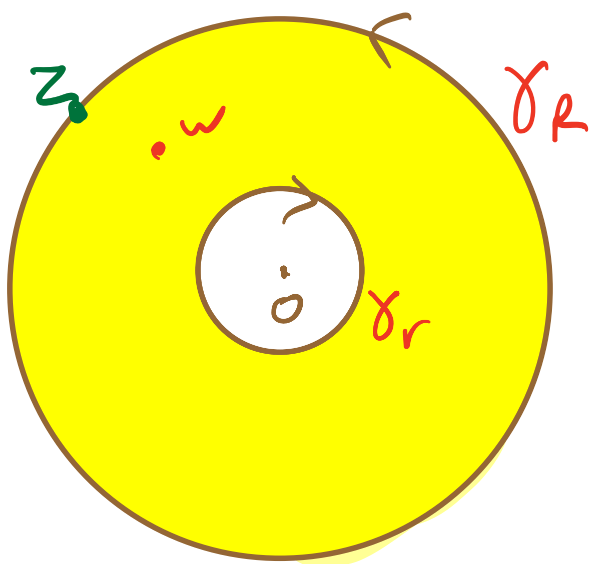
If we fix w, then, for $|w| < |z|$ we have $\dfrac{1}{z-w} = \sum_{n=0}^{\infty} w^n/z^{n+1}$, converging uniformly in z for $|z| > |w| + \epsilon$ for any $\epsilon > 0$.



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It follows that

$$\int_{\gamma_R} \frac{f(z)}{z-w} dz = \int_{\gamma_R} \sum_{n=0}^{\infty} \frac{f(z) w^n}{z^{n+1}} dz = \sum_{n \geq 0} \left(\int_{\gamma_R} \frac{f(z)}{z^{n+1}} dz \right) w^n.$$

for all $w \in A$.

(if $w \in A$, $|w| < R$)

Similarly since for $|z| < |w|$ we have

$$\frac{1}{z-w} = \frac{1}{w(z/w-1)} = -\sum_{n\geq 0} z^n/w^{n+1} = -\sum_{n=-1}^{-\infty} w^n/z^{n+1},$$

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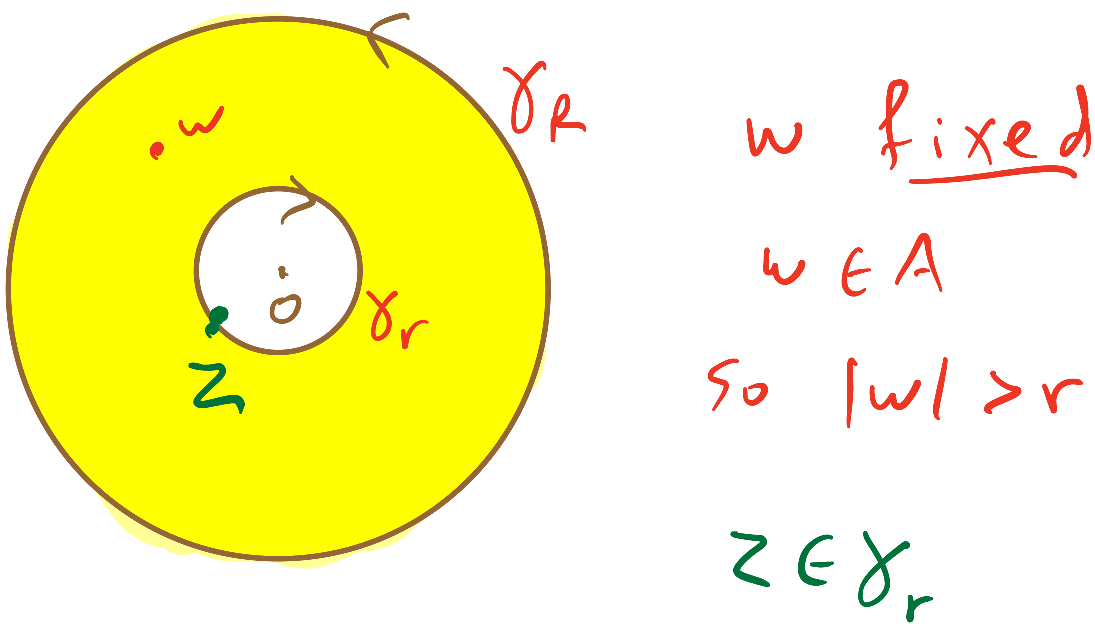

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taking $(c_n)_{n\in\mathbb{Z}}$ as in the statement of the theorem, we see that

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z-w} dz = \sum_{n\in\mathbb{Z}} c_n w^n,$$

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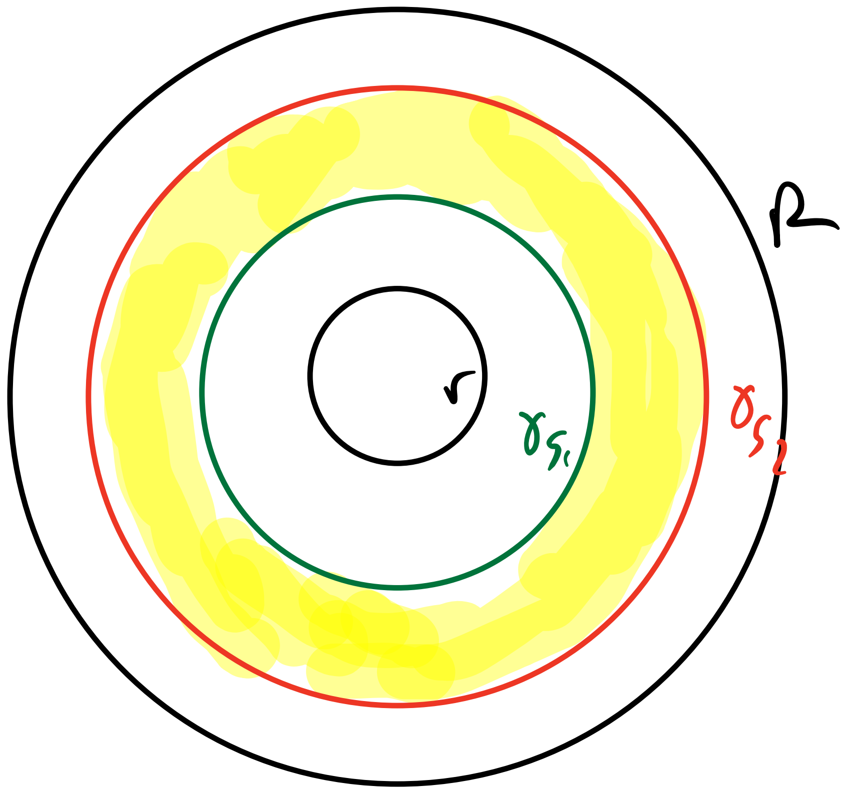

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It follows that γ_r in $\int_{\gamma_r} \frac{f(z)}{z^{n+1}}dz$ can be replaced by γ_{s_1} and similarly γ_R can be replaced by γ_{s_2}. But s_1, s_2 can take any values in $[r, R]$. Hence we obtain

$$c_n = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

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Uniqueness: Let $\sum_{n\in\mathbb{Z}} d_n z^n$ be any series expansion for $f(z)$ on A. By the integral formulae above (for $z_0 = 0$):

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Since $\sum_{n\in\mathbb{Z}} d_n z^n$ converges uniformly on compact sets in A to $f(z)$ we have that

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so $d_n = c_n$.

Remark

Note that

$$\int_{\gamma_R} \frac{f(z)}{z-w}\,dz$$

is a holomorphic function of w *in* $B(z_0, R)$ *and*

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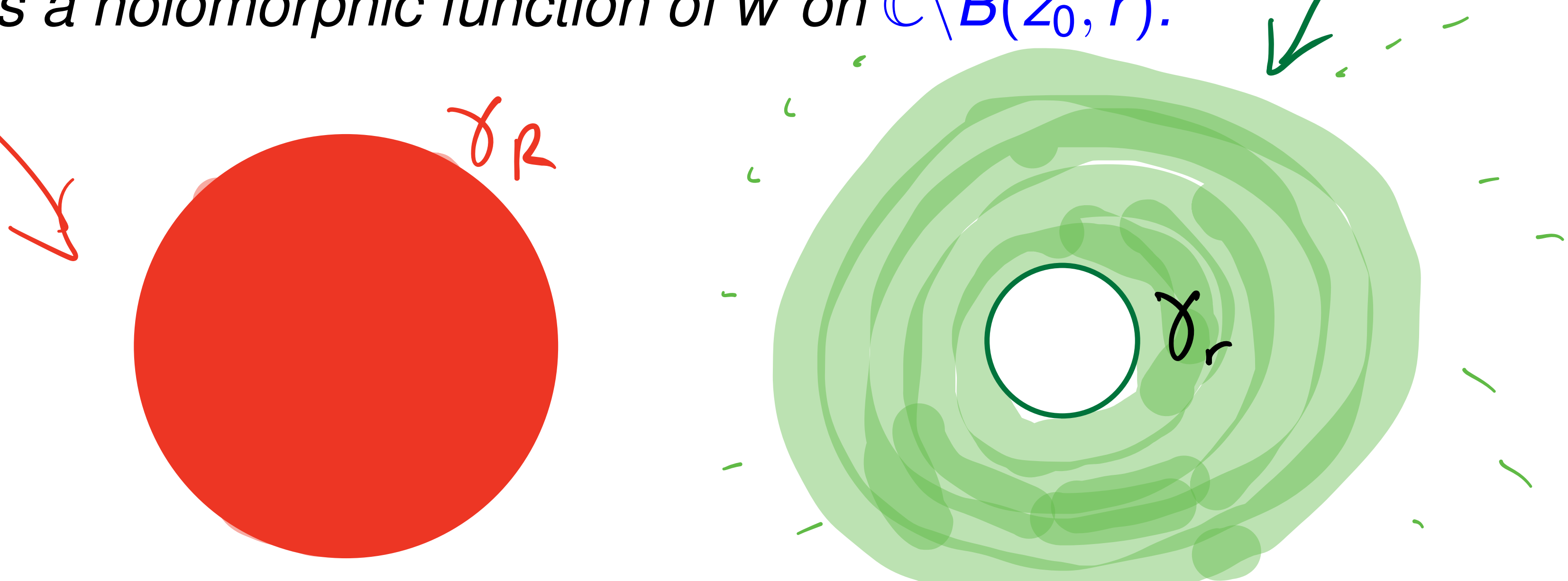

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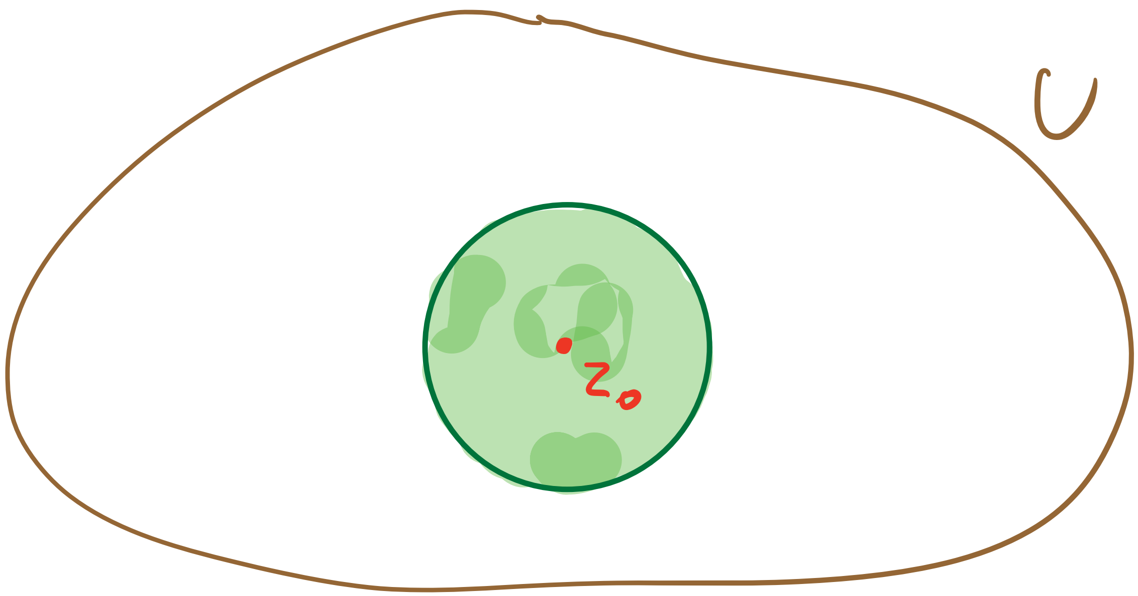
Thus we have actually expressed $f(w)$ on A as the difference of two functions which are holomorphic on $B(z_0, R)$ and $\mathbb{C} \backslash \bar{B}(z_0, r)$ respectively.

Corollary

If $f\colon U \to \mathbb{C}$ is a holomorphic function on an open set U containing an annulus $A = A(r, R, z_0)$ then f has a Laurent expansion on A. In particular, if f has an isolated singularity at z_0, then it has a Laurent expansion on a punctured disc $B(z_0, r)\backslash\{z_0\}$ for sufficiently small $r > 0$.



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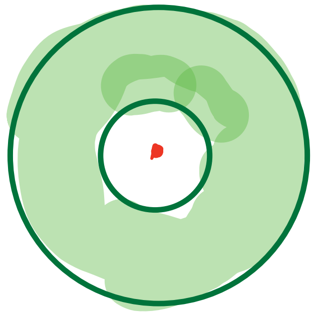
Proof.

This follows from the previous Theorem and the fact that for any $0 \leq r \leq R$ we have

$$A(r, R, z_0) = \bigcup_{r<r_1<R_1<R} \overline{A(r_1, R_1, z_0)}.$$



The final sentence follows from the fact that $B(z_0, r) \setminus \{z_0\} = A(0, r, z_0)$. □

Definition

Let $f\colon U \backslash S \to \mathbb{C}$ holomorphic on a domain U except at a discrete set $S \subseteq U$. Then for any $a \in S$ by the previous corollary for $r > 0$ sufficiently small, we have

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n, \quad \forall z \in B(a,r) \backslash \{a\}.$$

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This generalizes the previous definition we gave for the principal part of a meromorphic function. Note that the series $P_a(f)$ is uniformly convergent on $\mathbb{C}\backslash B(a,r)$ for all $r > 0$, and hence defines a holomorphic function on $\mathbb{C}\backslash\{a\}$.

Example. Calculate the Laurent series for

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For $|z| > 1$ we write

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{(1-1/z)} = \frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots)$$

so $f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \ldots.$