Multifunctions

Panos Papazoglou

e-mail : papazoglou@math.ox.ac.uk

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These are examples of multifunctions as eg one can take $\log(-1) = i\pi$ or $\log(-1) = -i\pi$.

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But f is not continuous on the whole plane:

For $\theta \to 0$, $\quad re^{i\theta}, re^{i(2\pi-\theta)} \to r$, but

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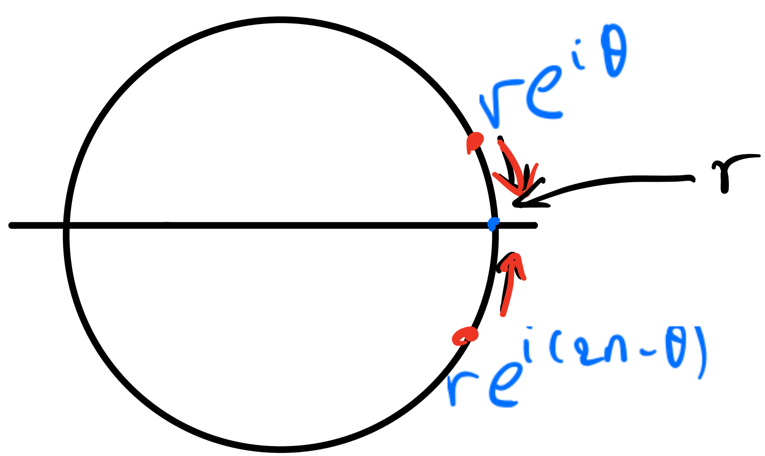

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$f(z)$ is holomorphic on $\mathbb{C} \backslash R$:

$$\frac{f(a+h) - f(a)}{h} = \frac{f(a+h) - f(a)}{\underbrace{f^2(a+h)}_{=a+h} - \underbrace{f^2(a)}_{=a}} = \frac{1}{f(a+h) + f(a)} \to \frac{1}{2f(a)}$$

as $h \to 0$.

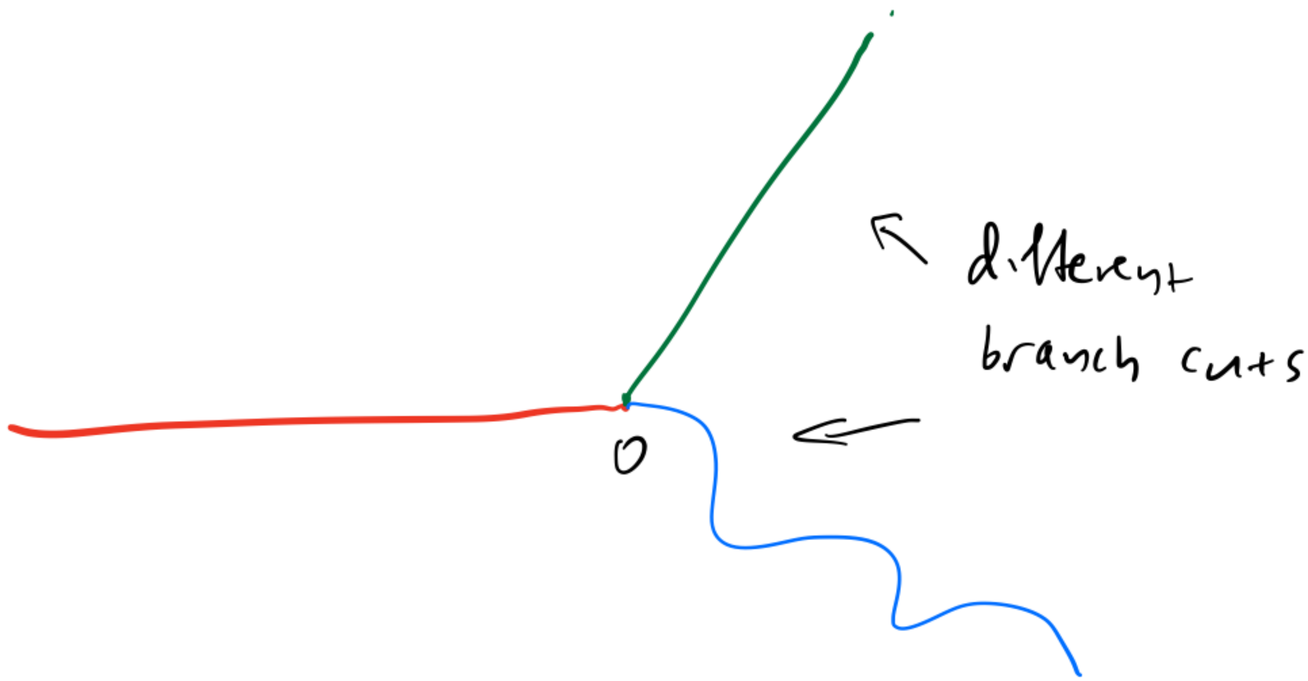
Multifunctions

The positive real axis is called a branch cut for the *multi-valued function* $z^{1/2}$.

If we set

$$g(z) = g(re^{i\theta}) = r^{1/2}e^{i(\frac{\theta}{2}+\pi)} = -r^{1/2}e^{i\theta/2}.$$

we obtain another branch of $z^{1/2}$ on $\mathbb{C}\backslash R$.



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Definition

A *multi-valued function* or multifunction on a subset $U \subseteq \mathbb{C}$ is a map $f\colon U \to \mathcal{P}(\mathbb{C})$ assigning to each point in U a subset of the complex numbers. A branch of f on a subset $V \subseteq U$ is a function $g\colon V \to \mathbb{C}$ such that $g(z) \in f(z)$, for all $z \in V$. If g is continuous (or holomorphic) on V we refer to it as a continuous, (respectively holomorphic) branch of f.

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Notation: $[f(z)]$ so eg $[\text{Log}(z)] = \{w \in \mathbb{C} : e^w = z\}$.

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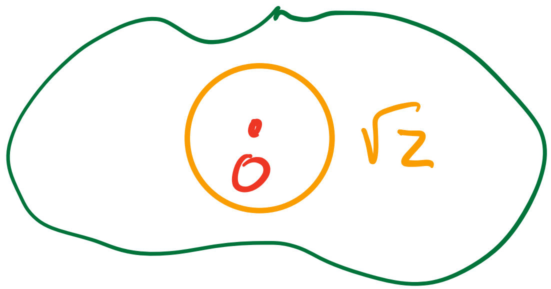
This is because it is not possible to choose a continuous branch of $[z^{1/2}]$ on any open set containing 0.



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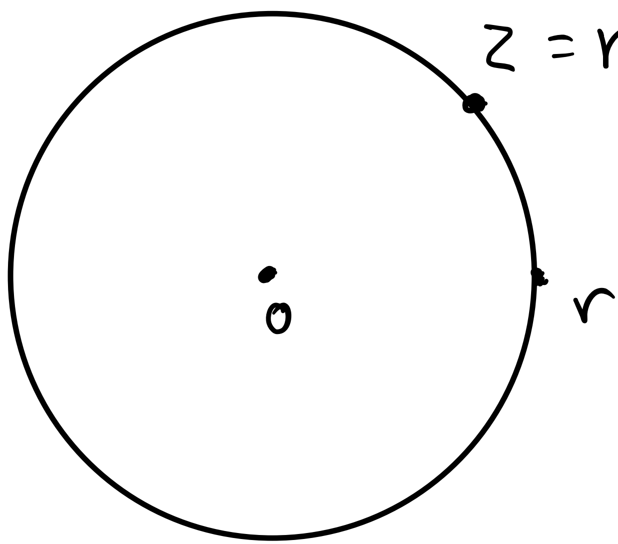

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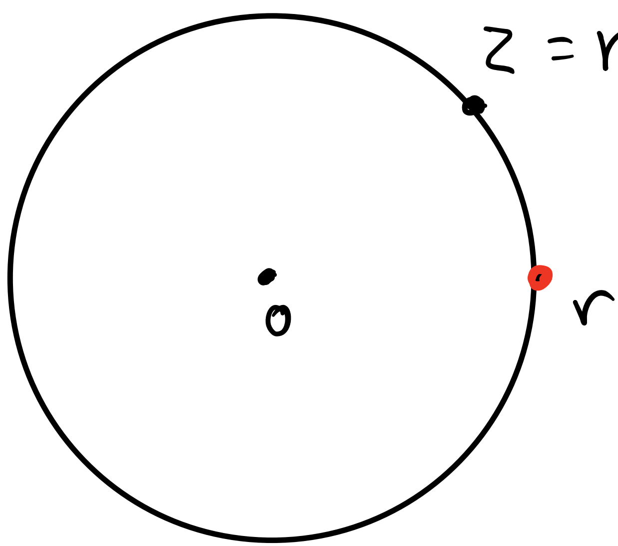
Then $f(0) = \pm\sqrt{r}$. Consider the function $g : [0, 1) \to \mathbb{C}$, $g(t) = \sqrt{r}e^{\pi i t}$. Then g is continuous.



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So the quotient f/g is a continuous function defined on $[0,1)$ and $f(t)/g(t) = \pm 1$ for any $t \in [0,1)$. Since $[0,1)$ is connected f/g is necessarily constant, so $f = \pm g$.

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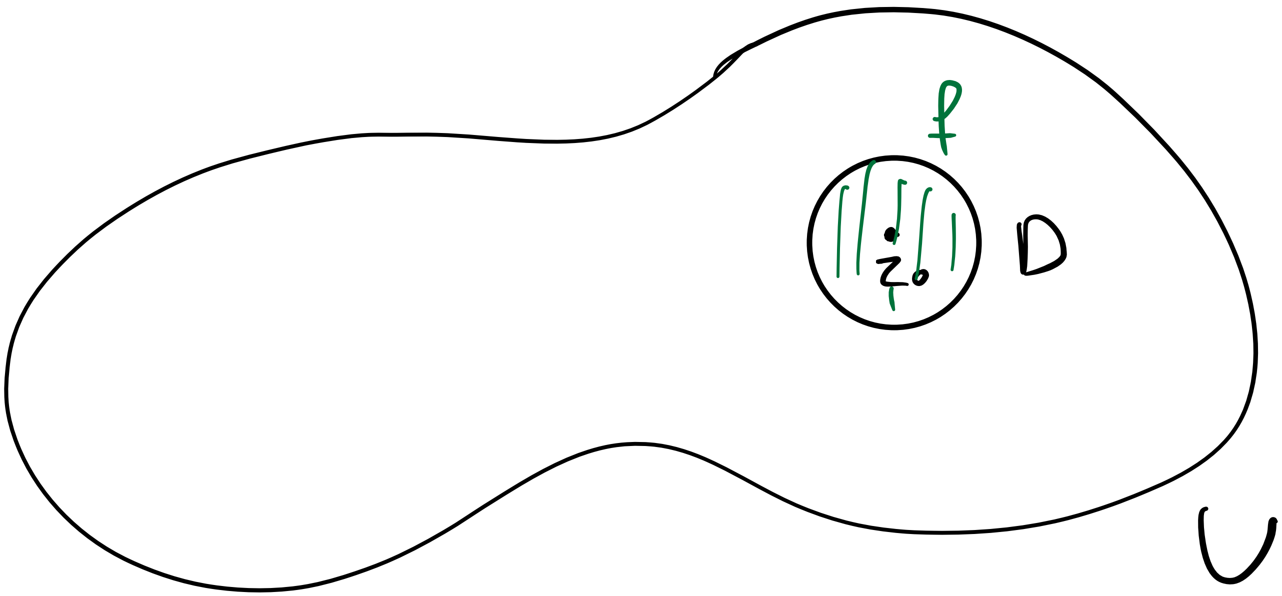
So $f(0) = \sqrt{r} \neq f(1) = \sqrt{r}e^{\pi i} = -\sqrt{r}$, however $re^{2\pi i\cdot 0} = re^{2\pi i\cdot 1}$, and similarly we arrive at a contradiction if $f(t) = -g(t)$.

Definition

Suppose that $f\colon U \to \mathcal{P}(\mathbb{C})$ is a multi-valued function defined on an open subset U of $\mathbb{C}$. We say that $z_0 \in U$ is not a branch point of f if there is an open disk $D \subseteq U$ containing z_0 such that there is a holomorphic branch of f defined on $D \backslash \{z_0\}$. We say z_0 is a branch point otherwise.



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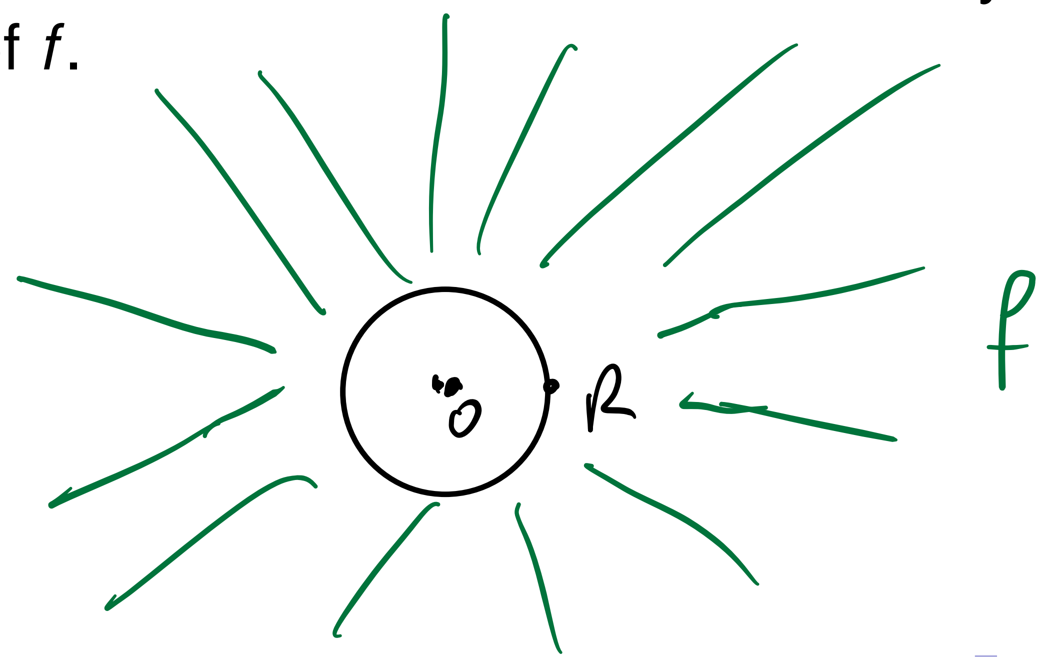
When $\mathbb{C} \setminus U$ is bounded, we say that f does not have a branch point at ∞ if there is a holomorphic branch of f defined on $\mathbb{C} \setminus B(0, R) \subseteq U$ for some $R > 0$. Otherwise we say that ∞ is a branch point of f.



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For example $0, \infty$ are the branch points of $[z^{1/2}]$.

The Logarithm

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$[\text{Log}(z)] = \{\log(|z|) + i(\theta + 2n\pi) : n \in \mathbb{Z}\}$ where $z = |z|e^{i\theta}$.

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$$L(z) = \log(|z|) + i\arg(z), \quad \text{where } \arg(z) \in (-\pi, \pi)$$

this is called the principal branch of Log.

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The branch points of $[\mathrm{Log}(z)]$ are 0 and ∞, as it is not possible to make a continuous choice of logarithm on any circle $S(0, r)$.

We note that $L(z)$ is also holomorphic. Indeed for small $h \neq 0$, $L(a+h) \neq L(a)$ and

$$\frac{L(a+h)-L(a)}{h} = \frac{L(a+h)-L(a)}{\underbrace{exp(L(a+h))}_{=\,a+h} - \underbrace{exp(L(a))}_{=\,a}},$$

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We have

$$\lim_{h \to 0} \frac{exp(L(a+h)) - exp(L(a))}{L(a+h) - L(a)} = exp'(L(a)) = a$$

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We note that the same argument applies to any continuous branch of the logarithm.

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any holomorphic branch of $[\mathrm{Log}(z)]$ gives a holomorphic branch of $[z^\alpha]$.
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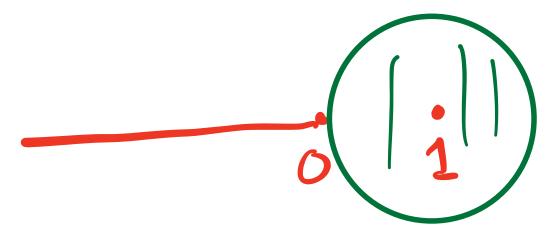
Note $(z_1 z_2)^\alpha \neq z_1^\alpha z_2^\alpha$ in general!

Binomial theorem for complex powers

$$[(1+z)^{\alpha}] = \{\exp(\alpha \cdot w) : w \in \mathbb{C}, \exp(w) = 1 + z\}.$$

Using $L(z)$ we obtain a branch

$$f(z) = \exp(\alpha \cdot L(1+z))$$



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By the ratio test, $s(z)$ has radius of convergence equal to 1, so that $s(z)$ defines a holomorphic function in $B(0,1)$.

$$\binom{\alpha}{k} \Big/ \binom{\alpha}{k+1} = \left| \frac{k+1}{\alpha - k} \right| \xrightarrow[k \to \infty]{} 1$$

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Differentiating term by term: $(1+z)s'(z) = \alpha \cdot s(z)$.

$s'(z) = \sum k \binom{\alpha}{k} z^{k-1} = \sum (\alpha - k + 1)\binom{\alpha}{k-1} z^{k-1}$, $\quad zs'(z) = \sum (k-1)\binom{\alpha}{k-1} z^{k-1}$

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Differentiating term by term: $(1+z)s'(z) = \alpha \cdot s(z)$.
Now $f(z)$ is defined on all of $B(0,1)$. We claim that $f(z) = s(z)$ on $B(0,1)$.

$$\text{Let } g(z) = \frac{s(z)}{f(z)} = s(z)\exp(-\alpha \cdot L(1+z))$$

then $g(z)$ is holomorphic for every $z \in B(0,1)$ and by the chain rule

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since $s'(z) = \frac{\alpha \cdot s(z)}{1+z}$.
Also $g(0) = 1$ so, since $B(0,1)$ is connected g is constant and $s(z) = f(z)$.

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Let's say $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is a continuous branch of the argument.

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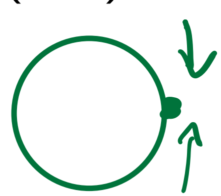
$\lim_{\theta \to 0^+} f(e^{i\theta}) = 2n\pi$, $\lim_{\theta \to 0^-} f(e^{i\theta}) = (2n+2)\pi$, so f is not continuous.



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Riemann surfaces

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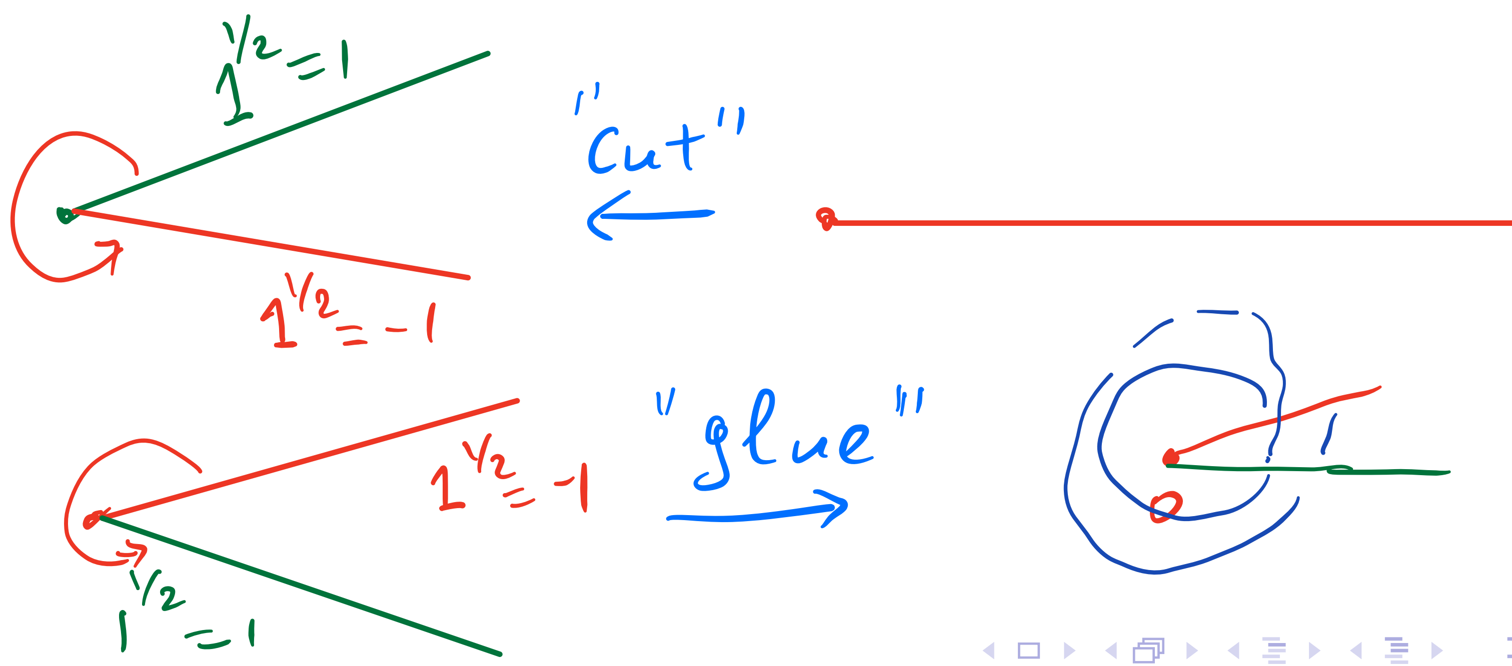
Consider $[z^{1/2}]$. We can ‘join’ the two branches of $[z^{1/2}]$ to obtain a function from a Riemann surface to $\mathbb{C}$.

2 copies of the sliced open plane



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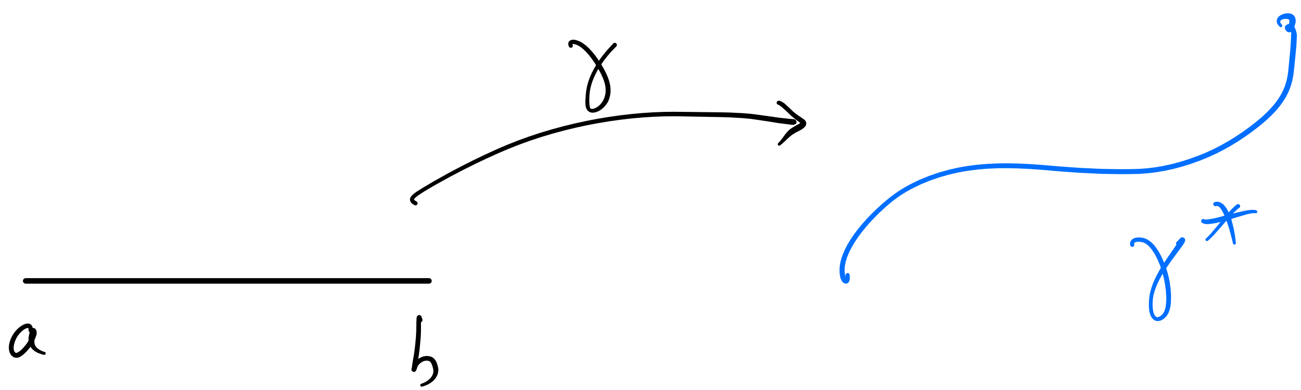

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Definition

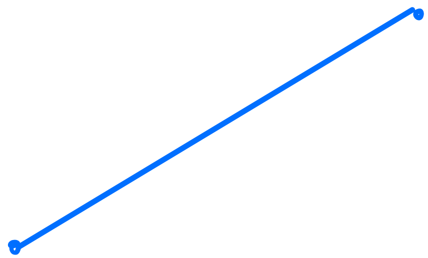
A path $\gamma\colon [a, b] \to \mathbb{C}$ is differentiable if its real and imaginary parts are differentiable. Equivalently, γ is differentiable at $t_0 \in [a, b]$ if

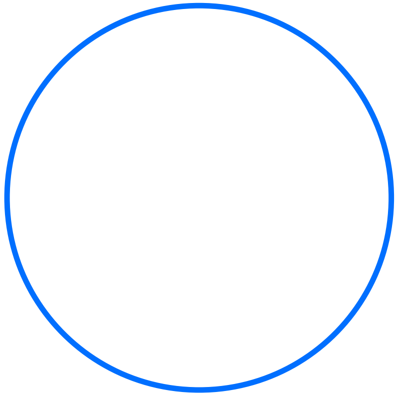
$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. Notation: $\gamma'(t_0)$. (If $t = a$ or b then we take the one-sided limit.) A path is C^1 if it is differentiable and its derivative $\gamma'(t)$ is continuous.

EXAMPLES:

1. Line segment: $t \mapsto a + t(b - a) = (1 - t)a + tb, \;\; t \in [0, 1]$,
2. circle: $z(t) = z_0 + re^{2\pi it}$, $t \in [0, 1]$ a closed path.





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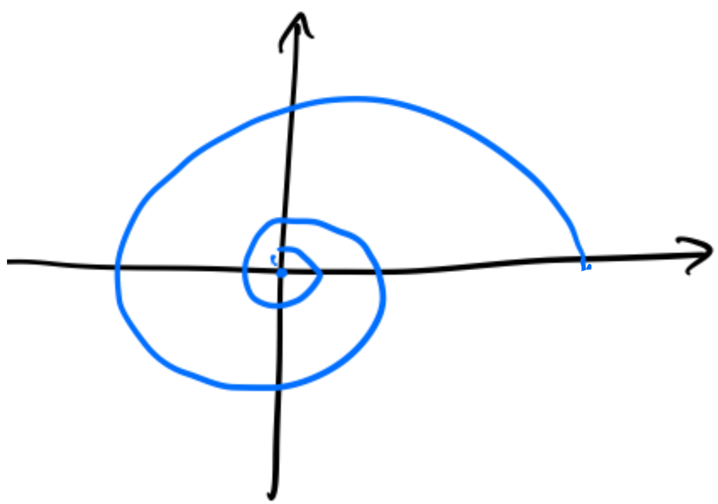
3. spiral

$$\gamma(t) = t^3 e^{2\pi i/t}, \quad t \in [0,1]$$



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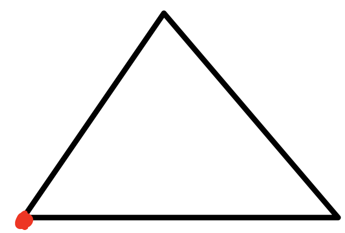
NON EXAMPLES:

Peano curves, triangles, $\gamma(t) = t e^{2\pi i/t}$, $t \in [0,1]$.

$\alpha : I \longrightarrow$ 



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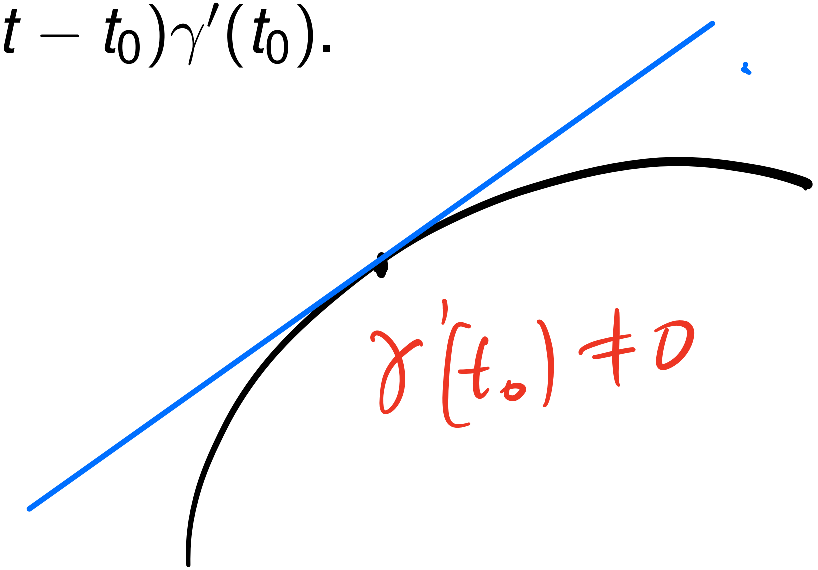
Remarks: If γ is a C^1 path and $\gamma'(t_0) \neq 0$ then γ has a tangent at t_0: $L(t) = \gamma(t_0) + (t - t_0)\gamma'(t_0)$.



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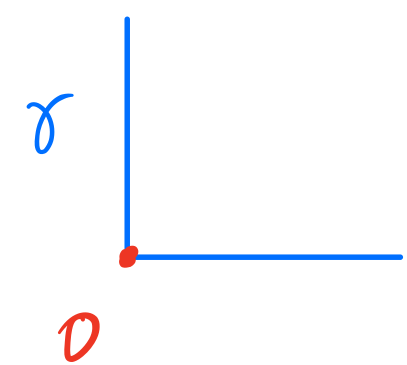
However a C^1 path might not have a tangent at every point, eg $\gamma : [-1,1] \to \mathbb{C}$

$$\gamma(t) = \begin{cases} t^2 & -1 \leq t \leq 0 \\ it^2 & 0 \leq t \leq 1. \end{cases}$$



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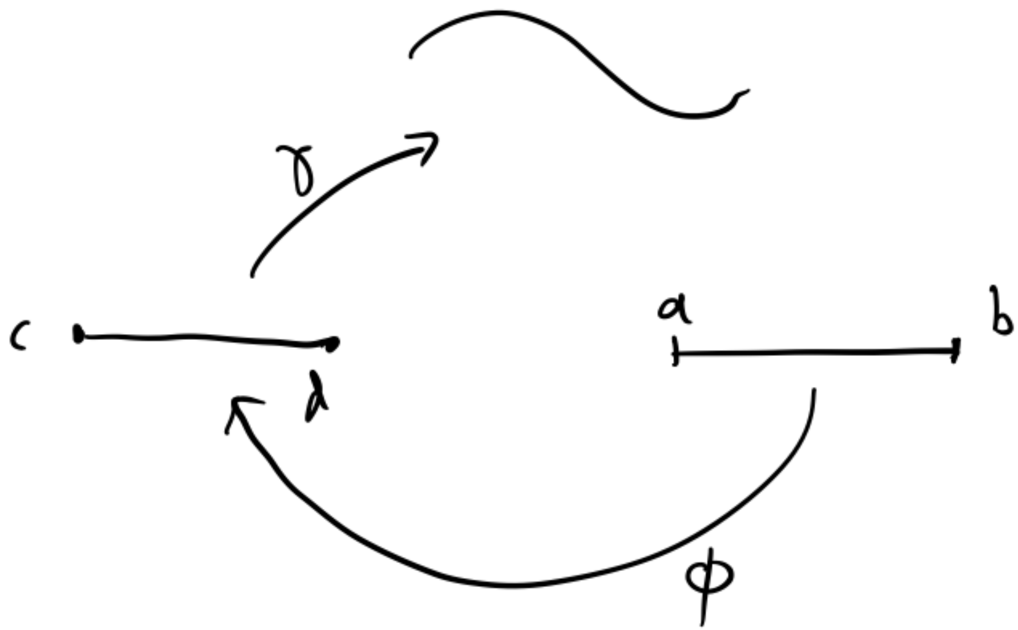
Let $\gamma\colon [c, d] \to \mathbb{C}$ be a C^1-path. If $\phi\colon [a, b] \to [c, d]$ is continuously differentiable with $\phi(a) = c$ and $\phi(b) = d$, then we say that $\tilde{\gamma} = \gamma \circ \phi$, is a reparametrization of γ.



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$\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are equivalent if there is a continuously differentiable bijective function $s\colon [a,b] \to [c,d]$ such that $s'(t) > 0$ for all $t \in [a,b]$ and $\gamma_1 = \gamma_2 \circ s$.

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Equivalence classes: oriented curves in the complex plane.
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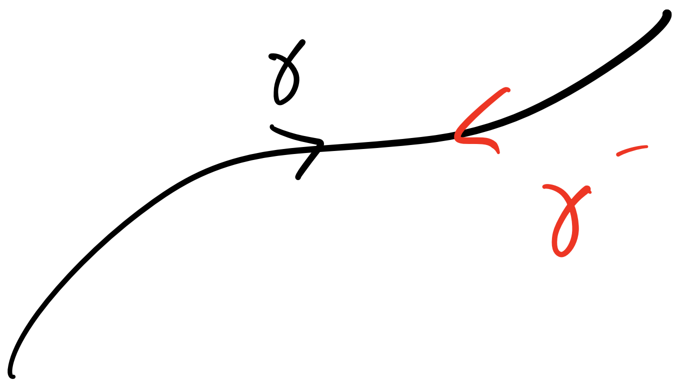
$s'(t) > 0$: the path is traversed in the same direction for each of the parametrizations γ_1 and γ_2. If $\gamma : [a, b] \to \mathbb{C}$ then the opposite path is $\gamma^-(t) = \gamma(a + b - t)$.



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Definition

If $\gamma: [a,b] \to \mathbb{C}$ is a C^1 path then we define the length of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Using the chain rule one sees that the length of a parametrized path is also constant on equivalence classes of paths.

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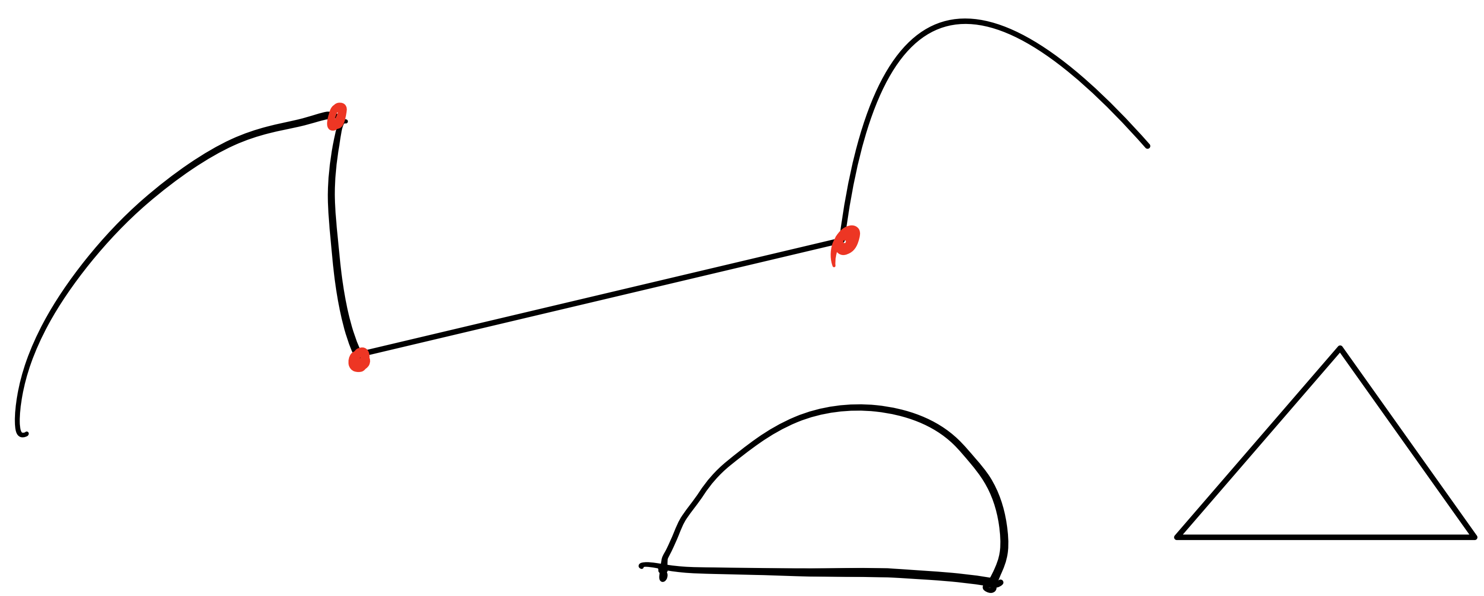
We will say a path $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise C^1 if it is continuous on $[a,b]$ and the interval $[a,b]$ can be divided into subintervals on each of which γ is C^1.
So there are $a = a_0 < a_1 < \ldots < a_m = b$ such that $\gamma_{|[a_i,a_{i+1}]}$ is C^1.



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Two paths $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ can be *concatenated* to give a path $\gamma_1 \star \gamma_2$, defined by

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If $\gamma, \gamma_1, \gamma_2$ are piecewise C^1 then so are γ^- and $\gamma_1 \star \gamma_2$.

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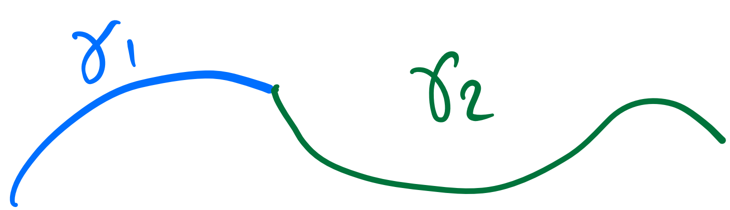

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A piecewise C^1 path is precisely a finite concatenation of C^1 paths.

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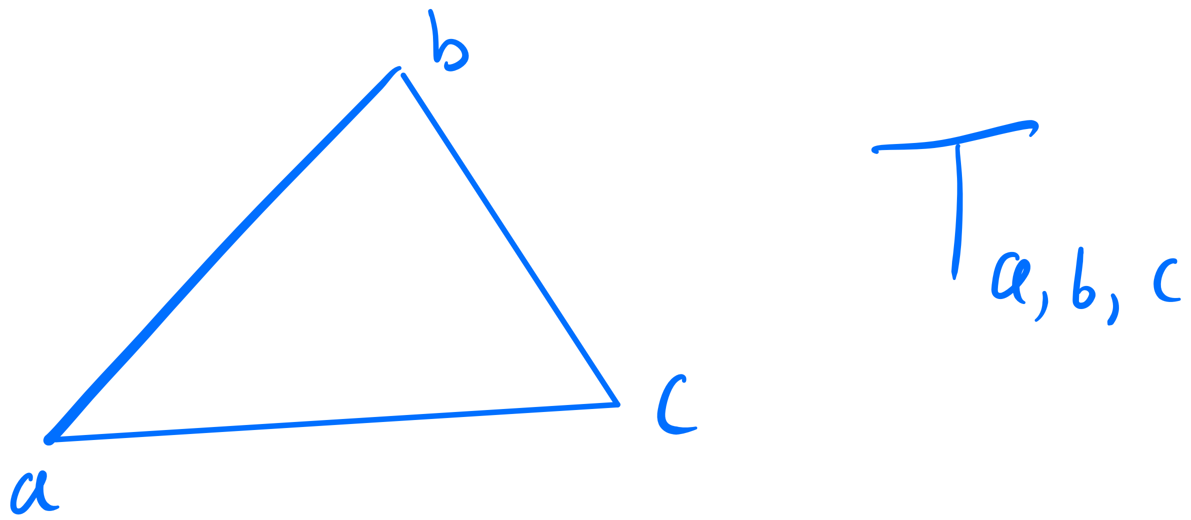
Example: If $a, b, c \in \mathbb{C}$, we define the triangle:
$T_{a,b,c} = \gamma_{a,b} \star \gamma_{b,c} \star \gamma_{c,a}$ where $\gamma_{x,y}$ is the line segment joining x, y.



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Recall the definition of Riemann integrable functions. We have the following:

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Let $[a, b]$ be a closed interval and $S \subset [a, b]$ a finite set. If f is a bounded continuous function (taking real or complex values) on $[a, b] \backslash S$ then it is Riemann integrable on $[a, b]$.

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Let $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ be any partition of $[a, b]$ which includes the elements of S.
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Integral along a path

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Definition

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-C^1 path and $f : \mathbb{C} \to \mathbb{C}$, then we define the integral of f along γ to be

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We note that if γ is a concatenation of the C^1 paths $\gamma_1, ..., \gamma_n$ then $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + ... + \int_{\gamma_n} f(z)dz$.

Example

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the path $\gamma(t) = \exp(2\pi it)$ (a circle). Then

$$\int_\gamma z^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{Z})$$

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If $n = -1$ we get $2\pi i \int_0^1 1 \, dt = 2\pi i$.

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So the integral only depends on the oriented curve $[\gamma]$.

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substitution $s = s(t)$

$ds = s'(t)\,dt$

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If $a = x_0 < x_1 < \ldots < x_n = b$ such that γ is C^1 on $[x_i, x_{i+1}]$ we have a corresponding decomposition of $[c, d]$ given by the points $s^{-1}(x_0) < \ldots < s^{-1}(x_n)$, and $\int_{\tilde{\gamma}} f(z)dz = \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt$

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We define also the integral *with respect to arc-length* of a function $f\colon U \to \mathbb{C}$ such that $\gamma^* \subseteq U$ to be

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This integral is invariant with respect to C^1 reparametrizations $s\colon [c,d] \to [a,b]$ if we require $s'(t) \neq 0$ for all $t \in [c,d]$. Note that in this case

$$\int_\gamma f(z)|dz| = \int_{\gamma^-} f(z)|dz|.$$

Properties of the integral

Properties of the integral

Let $f, g \colon U \to \mathbb{C}$ be continuous functions on an open subset $U \subseteq \mathbb{C}$ and $\gamma, \eta \colon [a, b] \to \mathbb{C}$ be piecewise-C^1 paths whose images lie in U. Then we have the following:

1. (*Linearity*): For $\alpha, \beta \in \mathbb{C}$,
$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$
2. If γ^- denotes the opposite path to γ then
$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$
3. (*Additivity*): If $\gamma \star \eta$ is the concatenation of the paths γ, η in U, we have
$$\int_{\gamma\star\eta} f(z)dz = \int_\gamma f(z)dz + \int_\eta f(z)dz.$$
4. (Estimation Lemma.) We have

length of γ
$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).$$

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Proposition

Let $f_n \colon U \to \mathbb{C}$ be a sequence of continuous functions. Suppose that $\gamma \colon [a, b] \to U$ is a piecewise C^1 path. If (f_n) converges uniformly to a function f on the image of γ then

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$\sup\{|f(z) - f_n(z)| : z \in \gamma^*\} \to 0$ as $n \to \infty$ which implies the result.

Example. Let's say

$$\sum_{n=1}^{\infty} a_n z^n$$

converges on $B(0, R)$. Then convergence is uniform on $B(0, r)$ for $r < R$. So if γ is a piecewise C^1 path in $B(0, r)$ we have

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Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a continuous function. If there exists a differentiable function $F\colon U \to \mathbb{C}$ with $F'(z) = f(z)$ then we say F is a primitive for f on U.

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Theorem

(Fundamental theorem of Calculus): Let $U \subseteq \mathbb{C}$ be a open and let $f \colon U \to \mathbb{C}$ be a continuous function. If $F \colon U \to \mathbb{C}$ is a primitive for f and $\gamma \colon [a, b] \to U$ is a piecewise C^1 path in U then we have

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular the integral of such a function f around any closed path is zero.

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If γ is only piecewise C^1, then take a partition $a = a_0 < a_1 < \ldots < a_k = b$ such that γ is C^1 on $[a_i, a_{i+1}]$ for each $i \in \{0, 1, \ldots, k-1\}$. Then we obtain a telescoping sum:

$$\int_\gamma f(z) = \int_a^b f(\gamma(t))\gamma'(t)dt = \sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$$

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$$\int_\gamma f(z) = \int_a^b f(\gamma(t))\gamma'(t)dt = \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$$

$$= \sum_{i=0}^{k-1}(F(\gamma(a_{i+1})) - F(\gamma(a_i))) = F(\gamma(b)) - F(\gamma(a))$$

Finally, γ is closed iff $\gamma(a) = \gamma(b)$ so the integral of f along a closed path is zero.

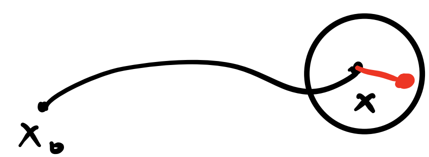
Corollary

Let U be a domain and let $f: U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.

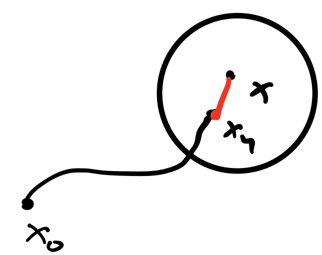
Recall: If $U \subseteq \mathbb{C}$ is open, connected (a domain) then for any $x, y \in U$ $\exists$ piecewise C^1-path from x to y.

Sketch of proof: Fix $x_0 \in U$. Let $S = \{x : \exists$ piecewise C^1-path from x_0 to $x\}$

Then 1) S is open:



2) S is closed, since if $x_n \to x$ then $x \in S$
$\uparrow$
S



So $S = U$.

Corollary

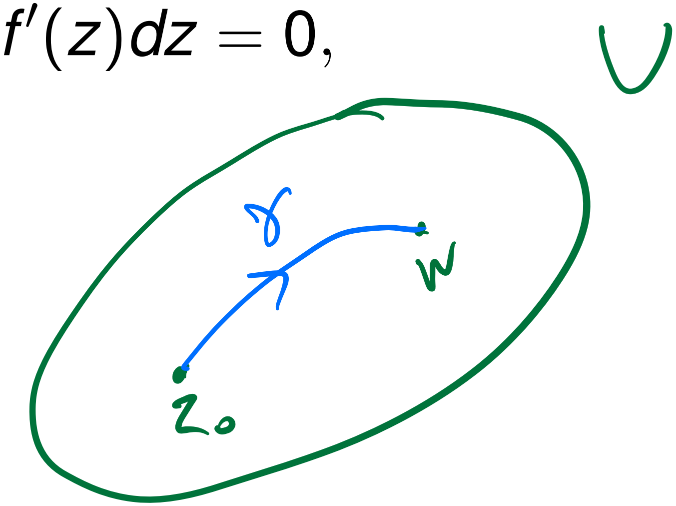
Let U be a domain and let $f\colon U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.

Proof.

Pick $z_0 \in U$. Since U is path-connected, if $w \in U$, we may find a piecewise C^1-path $\gamma\colon [0,1] \to U$ such that $\gamma(0) = z_0$ and $\gamma(1) = w$. Then by the previous Theorem

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = 0,$$

so that f is constant. □



Example

piecewise C^1

Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed curve such that $a \notin \gamma^*$. Show that

$$\int_\gamma (z-a)^n \, dz = 0 \text{ for } n \neq -1.$$

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Theorem

If U is a domain and $f\colon U \to \mathbb{C}$ is a continuous function such that for any closed path in U we have $\int_\gamma f(z)dz = 0$, then f has a primitive.

piecewise C^1

Proof.

Fix z_0 in U, and for any $z \in U$ set $F(z) = \int_\gamma f(z)dz$.
where $\gamma\colon [a, b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.

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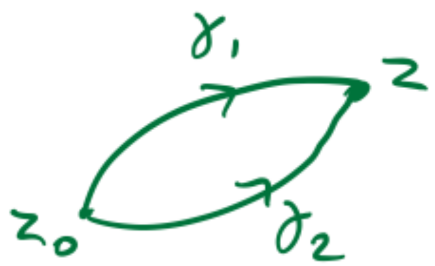
$F(z)$ is independent of the choice of γ:

Suppose γ_1, γ_2 are two paths joining z_0, z

The path $\gamma = \gamma_1 \star \gamma_2^-$ is closed so



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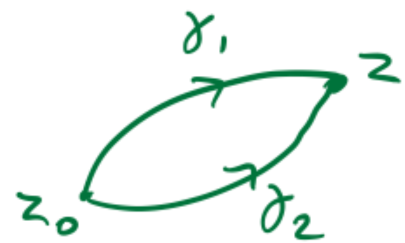

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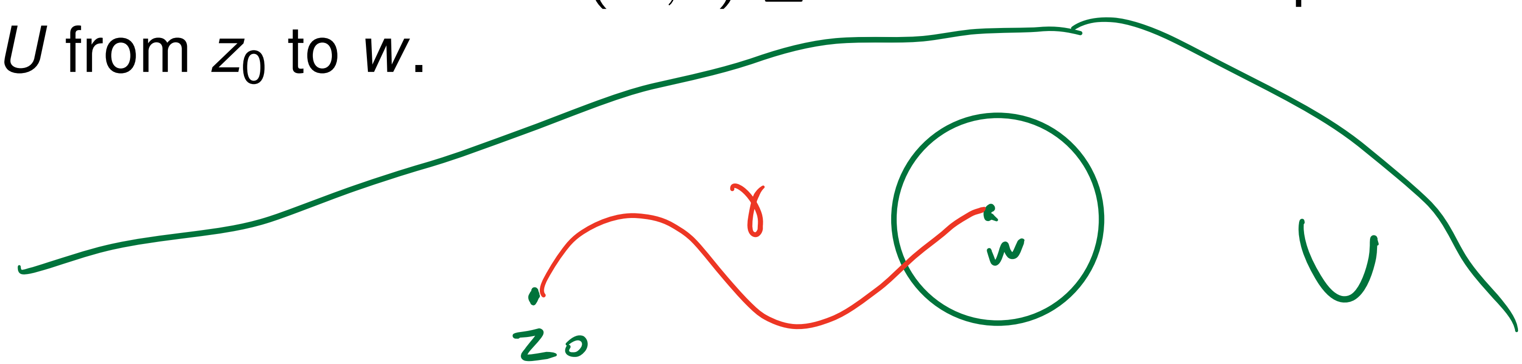
Claim: F is differentiable and $F'(z) = f(z)$.
Fix $w \in U$ and $\epsilon > 0$ such that $B(w,\epsilon) \subseteq U$ and choose a path $\gamma\colon [a,b] \to U$ from z_0 to w.



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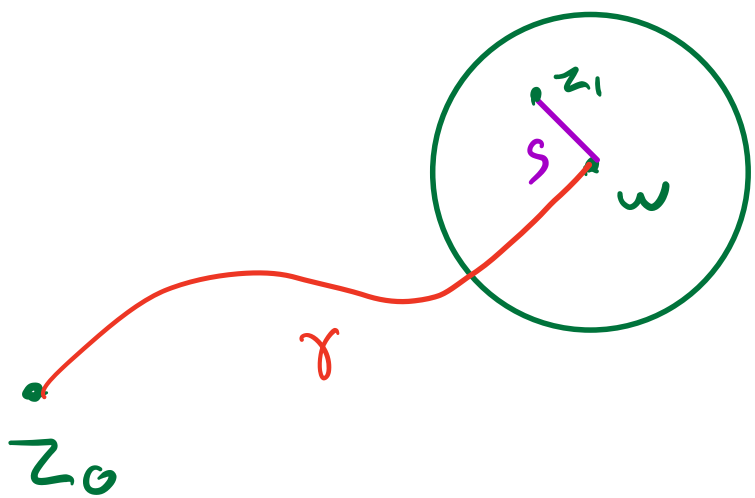
If $z_1 \in B(w,\epsilon) \subseteq U$, then the concatenation of γ with the straight-line path $s\colon [0,1] \to U$ given by $s(t) = w + t(z_1 - w)$ from w to z_1 is a path γ_1 from z_0 to z_1. It follows that

$$F(z_1) - F(w) = \int_{\gamma_1} f(z)dz - \int_{\gamma} f(z)dz$$



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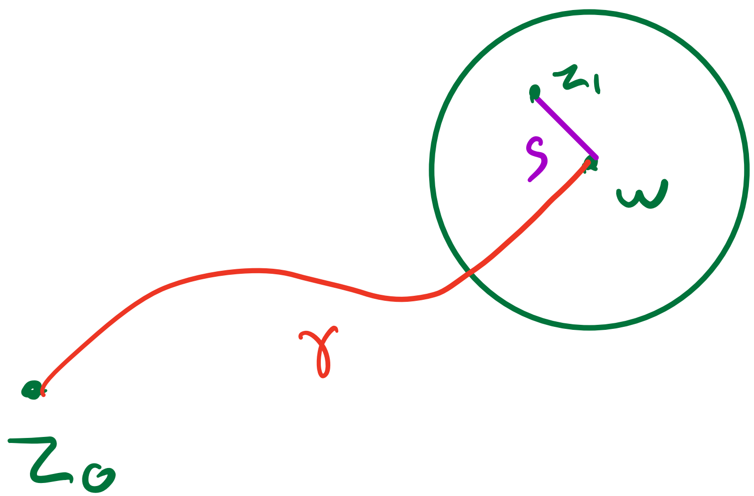

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Remark: $1/z$ does have a primitive on any domain D where we can chose a branch of [Log(z)]:

If we have $e^{L(z)} = z$ on D by the chain rule

$$\exp(L(z)) \cdot L'(z) = 1 \Rightarrow L'(z) = 1/z.$$

Cauchy's theorem

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This is the single most important theorem of the course. Almost all important facts about holomorphic functions follow from it. Sample applications:

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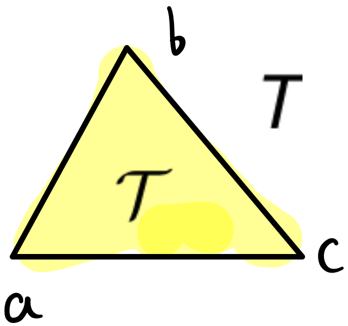
For most of our applications we will need a simpler case of the theorem for starlike domains. We defer the discussion of the general case to later lectures.

Definition

A triangle or triangular path T is a path of the form $\gamma_1 \star \gamma_2 \star \gamma_3$ where $\gamma_1(t) = a + t(b-a)$, $\gamma_2(t) = b + t(c-b)$ and $\gamma_3(t) = c + t(a-c)$ where $t \in [0,1]$ and $a, b, c \in \mathbb{C}$. (Note that if $\{a, b, c\}$ are collinear, then T is a degenerate triangle.) That is, T traverses the boundary of the triangle with vertices $a, b, c \in \mathbb{C}$. The solid triangle $\mathcal{T}$ bounded by T is the region



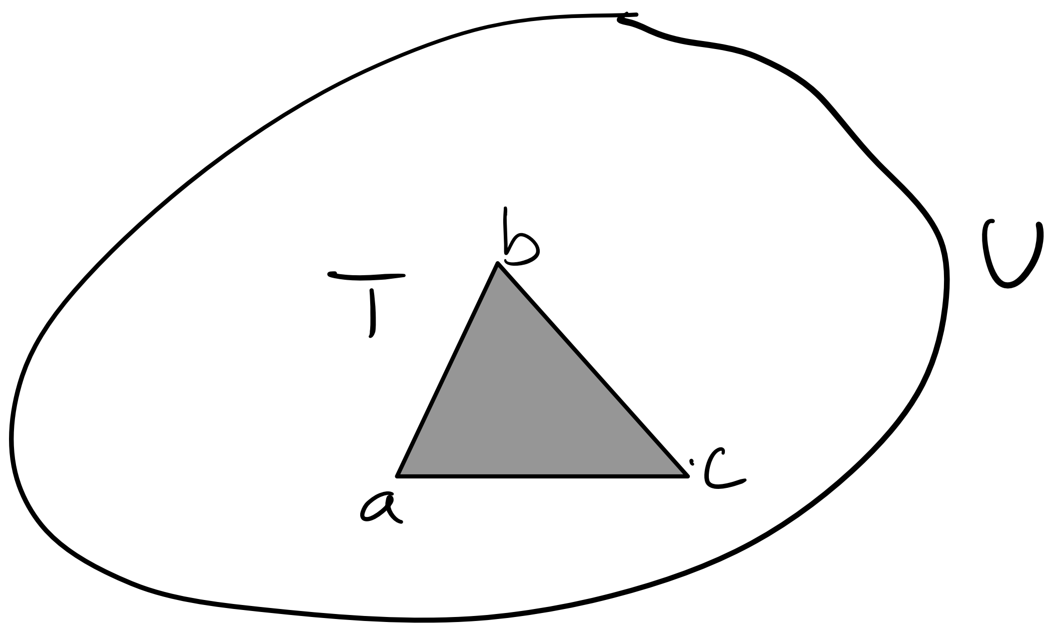
$$\mathcal{T} = \{t_1 a + t_2 b + t_3 c : t_i \in [0,1], \sum_{i=1}^{3} t_i = 1\},$$

with the points in the interior of $\mathcal{T}$ corresponding to the points with $t_i > 0$ for each $i \in \{1, 2, 3\}$. We will denote by $[a, b]$ the line segment $\{a + t(b-a) : t \in [0,1]\}$, the side of T joining vertex a to vertex b. When we need to specify the vertices a, b, c of a triangle T, we will write $T_{a,b,c}$.

Theorem

(Cauchy's theorem for a triangle): Suppose that $U \subseteq \mathbb{C}$ is an open subset and let $T \subseteq U$ be a triangle whose interior is entirely contained in U. Then if $f : U \to \mathbb{C}$ is holomorphic we have

$$\int_T f(z)dz = 0$$



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2. Assuming that $I = |\int_T f(z)dz| \neq 0$ we will subdivide T into 4 smaller triangles and represent the integral as sum of the integrals on the smaller triangles. One of the integrals of the smaller triangles will be at least $I/4$. We will keep subdividing till we get a very small triangle where by part 1 the integral will be smaller than expected, contradiction.

Suppose $I = |\int_T f(z)dz| > 0$. We build a sequence of smaller and smaller triangles T^n, as follows: Let $T^0 = T$, and suppose that we have constructed T^i for $0 \leq i < k$. Then take the triangle T^{k-1} and join the midpoints of the edges to form four smaller triangles, which we will denote S_i $(1 \leq i \leq 4)$.
Then $I_k = \int_{T^{k-1}} f(z)dz = \sum_{i=1}^{4} \int_{S_i} f(z)dz$, since the integrals around the interior edges cancel.

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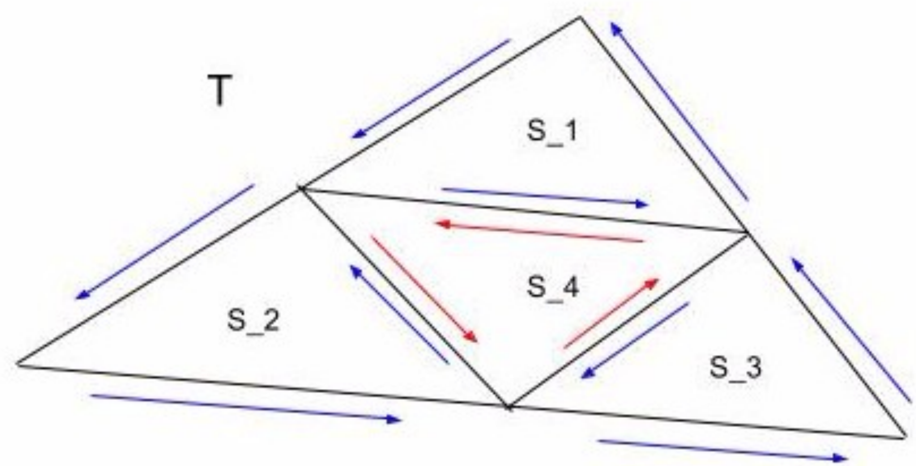

Figure: Subdivision of a triangle

$I_{k-1} = |\int_{T^{k-1}} f(z)dz| \leq \sum_{i=1}^{4} |\int_{S_i} f(z)dz|$, so that for some i we must have $|\int_{S_i} f(z)dz| \geq I_{k-1}/4$. Set T^k to be this triangle S_i. Then by induction we see that $\ell(T^k) = 2^{-k}\ell(T)$ while $I_k \geq 4^{-k}I$.

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Let $\mathcal{T}^k$ be the solid triangle with boundary T^k. The sets $\mathcal{T}^k$ are nested and their diameter tends to 0, so there is a unique point z_0, lying in all of them.

<u>Recall</u> If K_i are closed, $K_{i+1} \subseteq K_i$ and $\operatorname{diam}(K_i) \to 0$ then $\bigcap K_i = \{p\}$

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Let $\mathcal{T}^k$ be the solid triangle with boundary T^k. The sets $\mathcal{T}^k$ are nested and their diameter tends to 0, so there is a unique point z_0, lying in all of them.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z),$$

where $\psi(z) \to 0 = \psi(z_0)$ as $z \to z_0$.

$$\int_{T^k} f(z)dz = \int_{T^k} (z - z_0)\psi(z)dz$$

and if z is on T^k, we have $|z - z_0| \leq \text{diam}(\mathcal{T}^k) = 2^{-k}\text{diam}(T)$.

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$$I_k = \left| \int_{T^k} (z - z_0)\psi(z)dz \right| \leq \eta_k \cdot \text{diam}(T^k)\ell(T^k)$$

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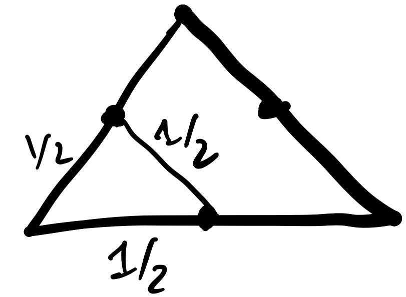

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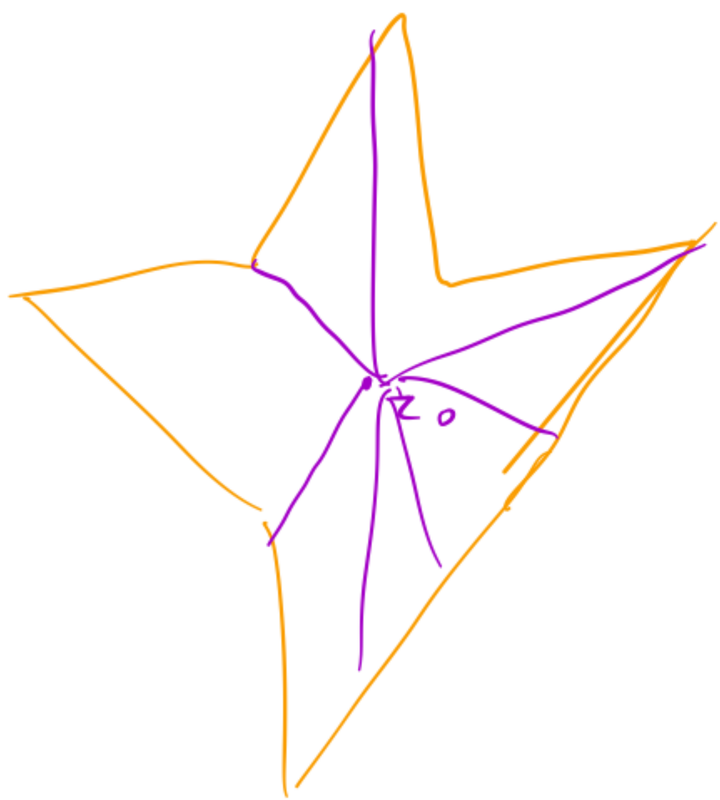
So $4^k I_k \to 0$ as $k \to \infty$. On the other hand, by construction $I_k \geq I/4^k \Rightarrow 4^k I_k \geq I > 0$, contradiction. $\square$

Definition

Let X be a subset in $\mathbb{C}$. We say that X is *convex* if for each $z, w \in U$ the line segment between z and w is contained in X. We say that X is star-like if there is a point $z_0 \in X$ such that for every $w \in X$ the line segment $[z_0, w]$ joining z_0 and w lies in X. We will say that X is star-like with respect to z_0 in this case. Thus a convex subset is thus starlike with respect to every point it contains.



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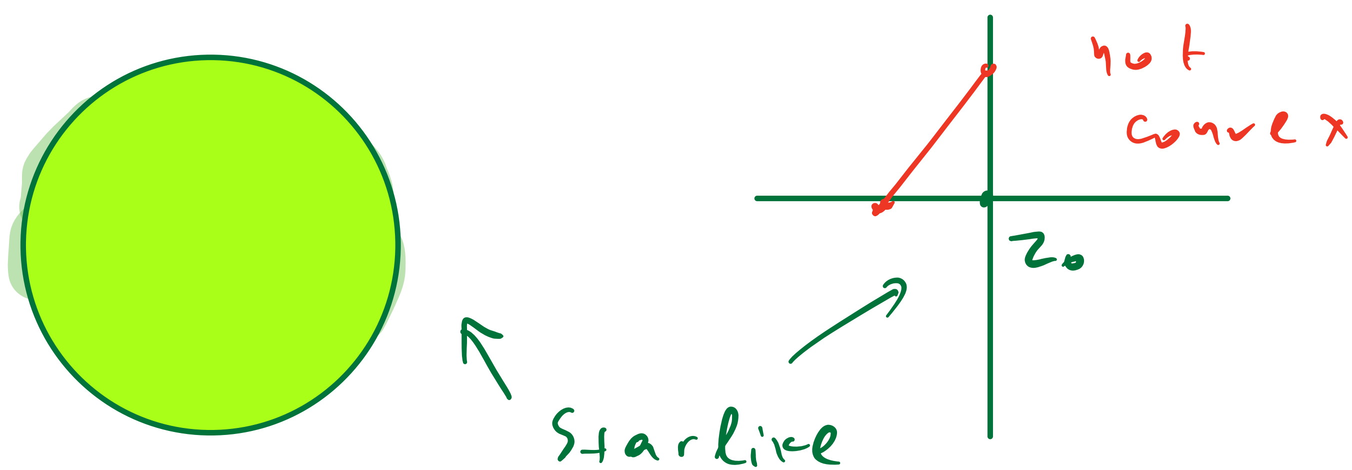
Example. A disk (open or closed) is convex, as is a solid triangle or rectangle. On the other hand the union of the xy-axes is starlike with respect to 0 but not convex.



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Theorem

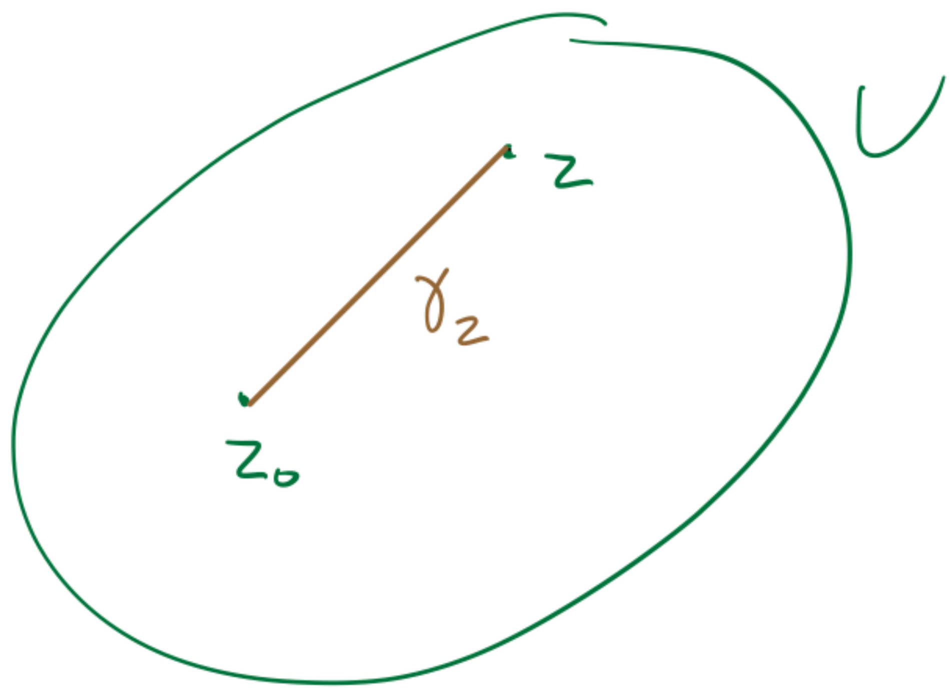
(Cauchy's theorem for a star-like domain): Let U be a star-like domain. Then if $f: U \to \mathbb{C}$ is holomorphic and $\gamma: [a, b] \to U$ is a closed path in U we have

piecewise-C^1

$$\int_\gamma f(z)dz = 0.$$

Proof. It suffices to show that f has a primitive in U.
Let $z_0 \in U$ such that for every $z \in U$, $\gamma_z = z_0 + t(z - z_0)$, $t \in [0, 1]$ is contained in U. We claim that

$$F(z) = \int_{\gamma_z} f(\zeta) d\zeta$$



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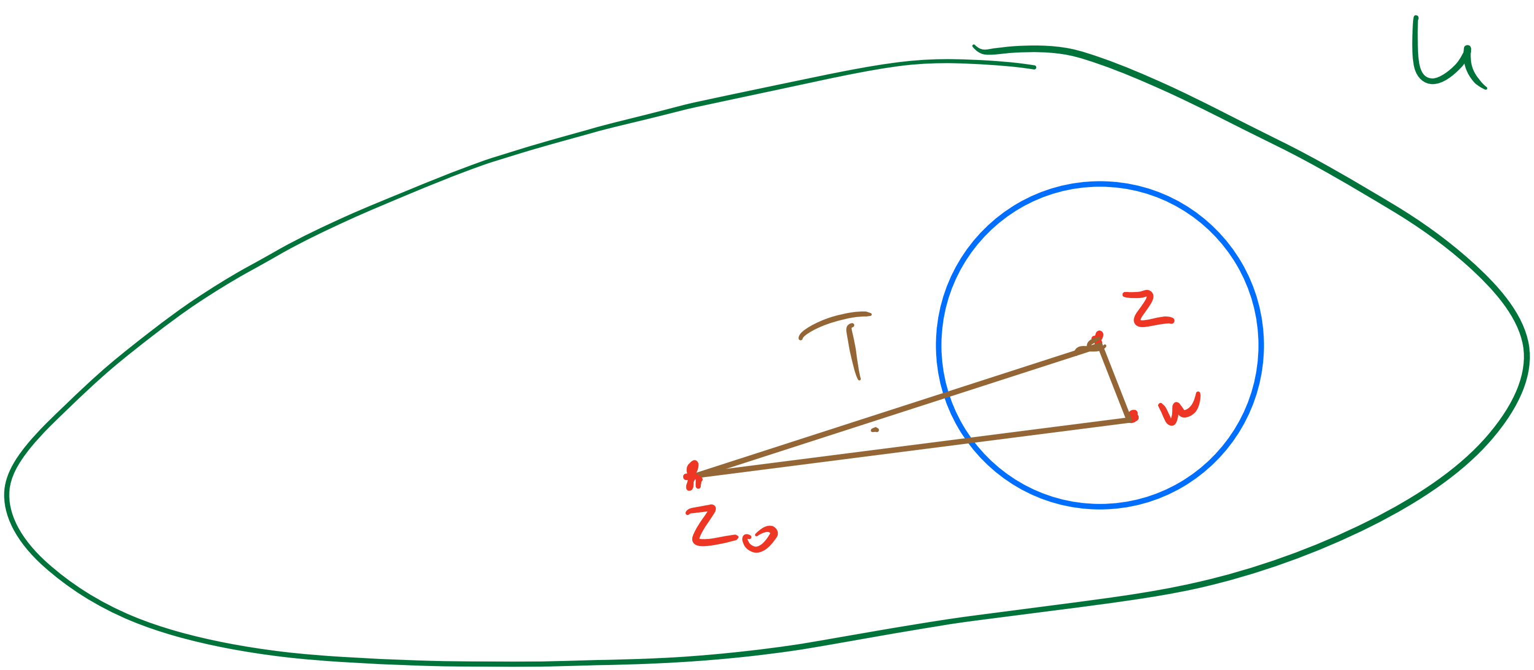

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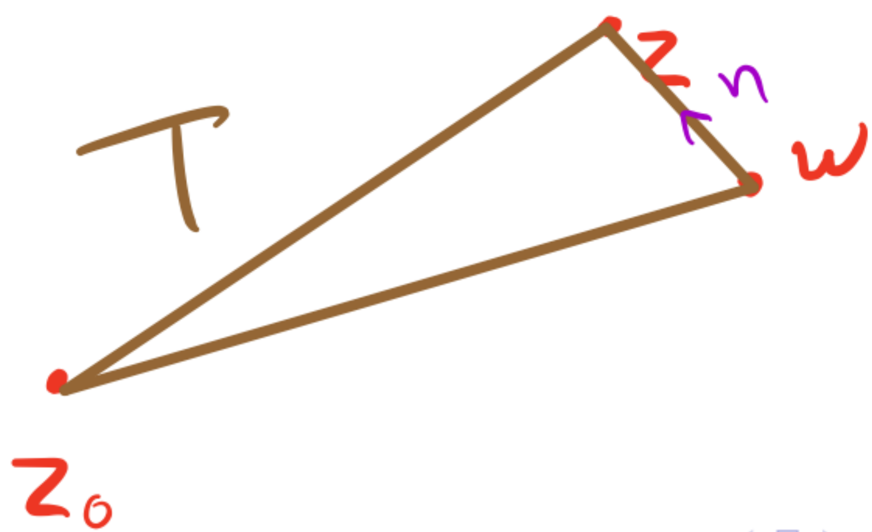

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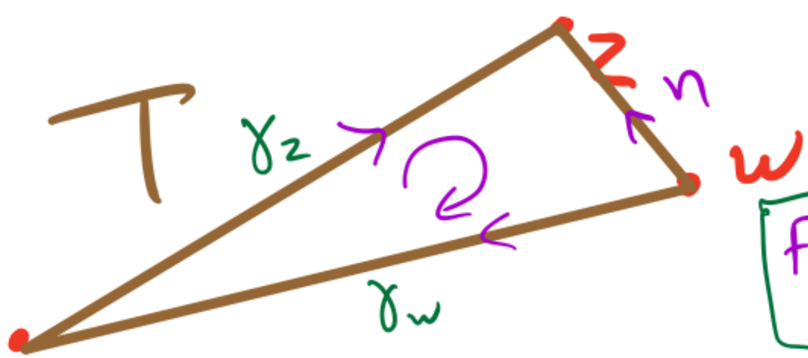

$$\boxed{F(z) - F(w) + \int_{-\eta} f = 0}$$

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$$\leq \sup_{t \in [0,1]} |f(w + t(z - w)) - f(z)| \to 0 \text{ as } w \to z.$$

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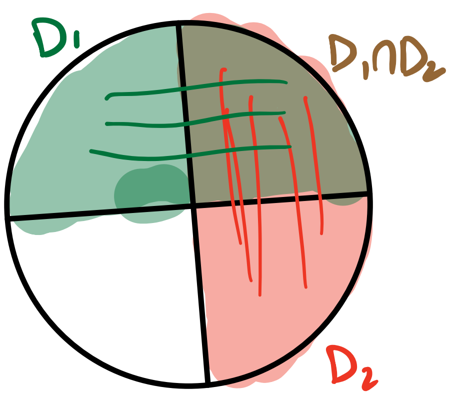
Suppose that D_1 and D_2 are primitive domains and $D_1 \cap D_2$ is connected. Then $D_1 \cup D_2$ is primitive.



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The union of two open intersecting half-discs D_1, D_2 of a disc $B(0, r)$ is primitive.

Indeed each D_1, D_2 are convex, so they are primitive. $D_1 \cap D_2$ is connected so by the lemma $D_1 \cup D_2$ is primitive.

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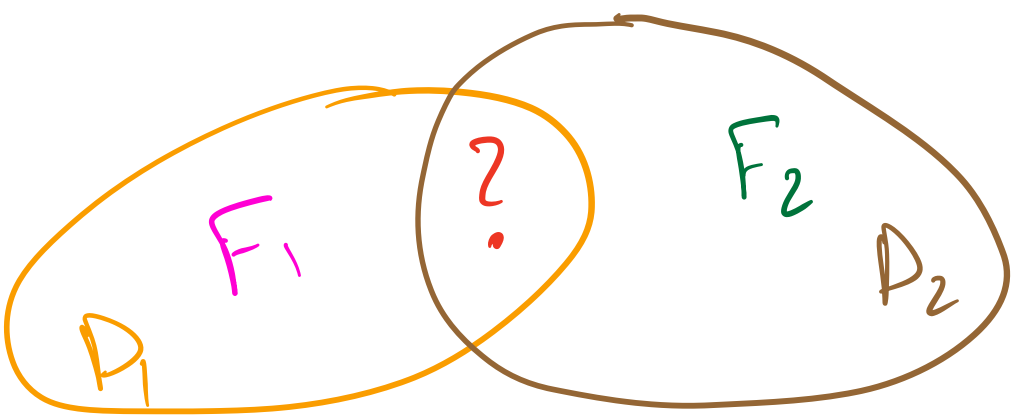

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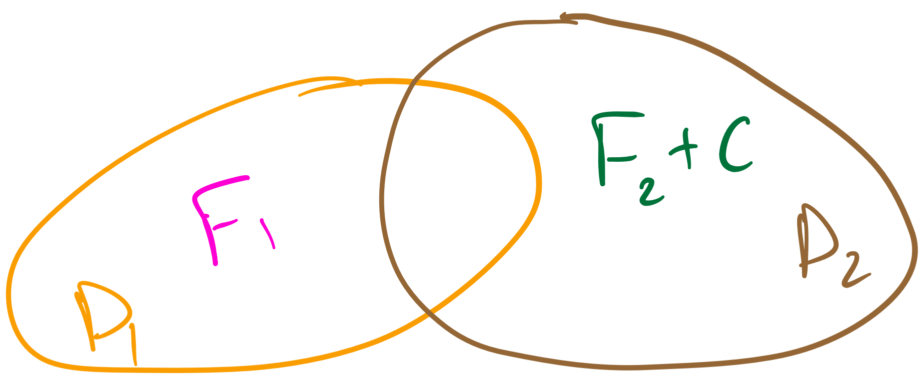

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We will need the following simple calculation: Let $\gamma = \gamma(a, r)$ be the path $t \mapsto a + re^{2\pi i t}$. We have then

$$\int_\gamma \frac{1}{z-a} dz = \int_0^1 \frac{1}{\exp(2\pi i t)} \cdot (2\pi i \exp(2\pi i t)) dt = 2\pi i.$$

Theorem

(Cauchy's Integral Formula.) Suppose that $f\colon U \to \mathbb{C}$ *is a holomorphic function on an open set* U *which contains the disc* $\bar{B}(a,r)$*. Then for all* $w \in B(a,r)$ *we have*

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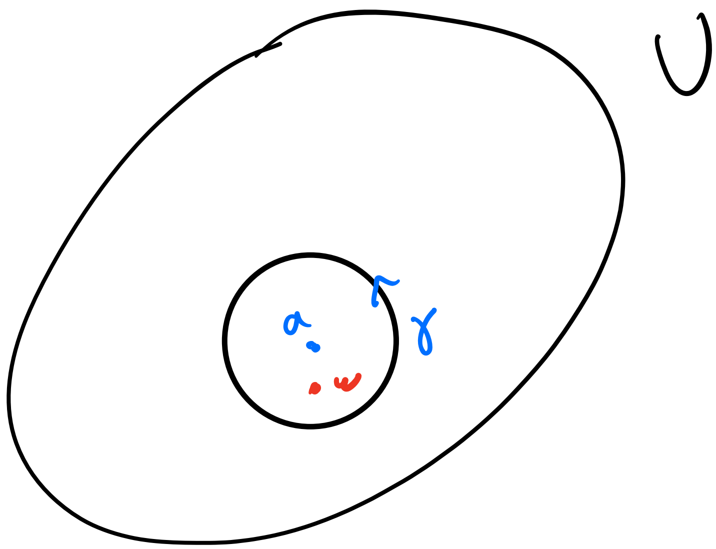

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Consider a circle $\gamma(w,\epsilon)$ centered at w and contained in $B(a,r)$. Pick two anti-diametric points on $\gamma(w,\epsilon)$ and join them by straight segments to points on γ.

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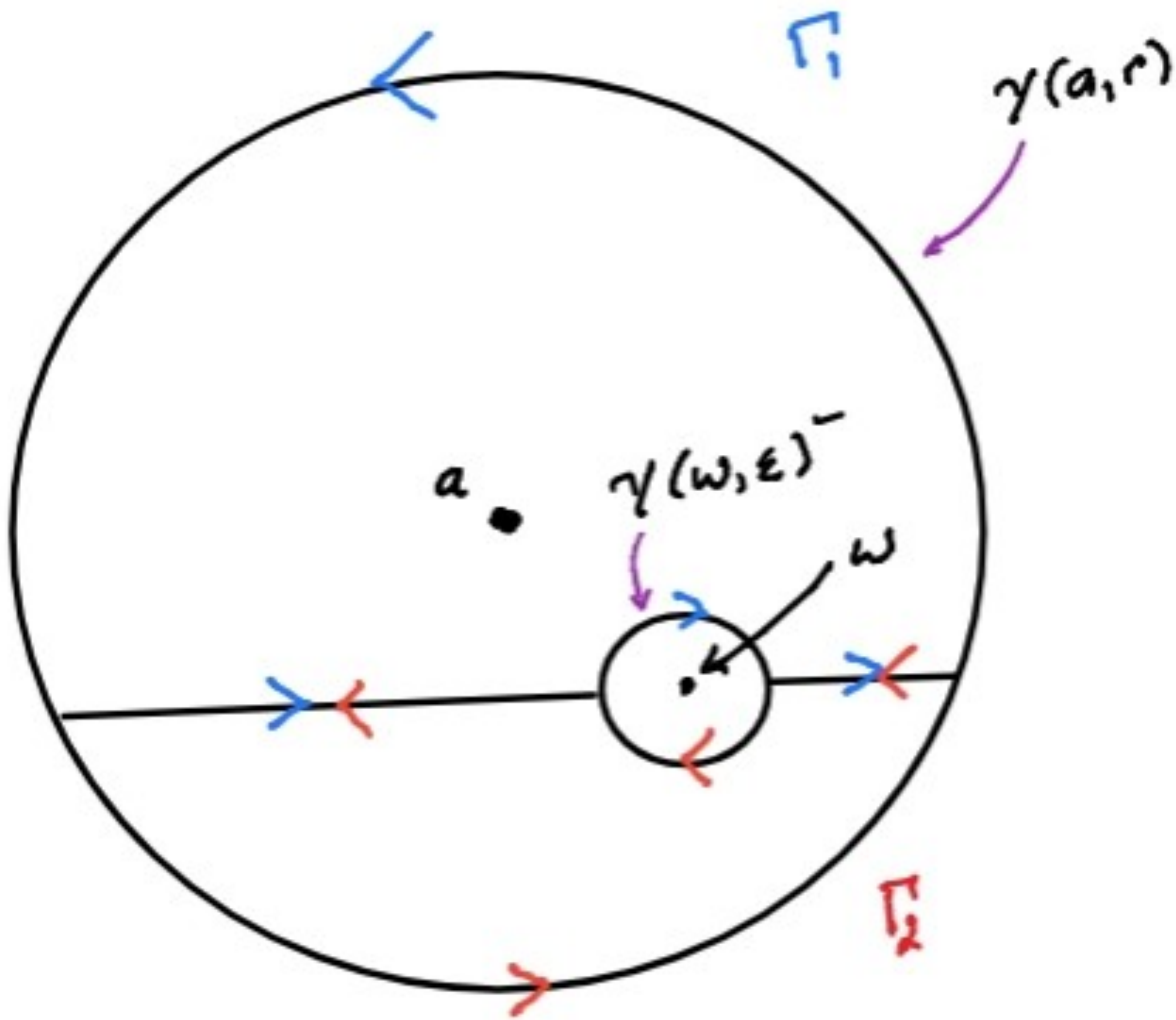

$$\int_{\Gamma_1} \frac{f(z)}{z-w} dz + \int_{\Gamma_2} \frac{f(z)}{z-w} dz = \int_{\gamma(a,r)} \frac{f(z)}{z-w} dz - \int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w} dz.$$

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$\Downarrow$

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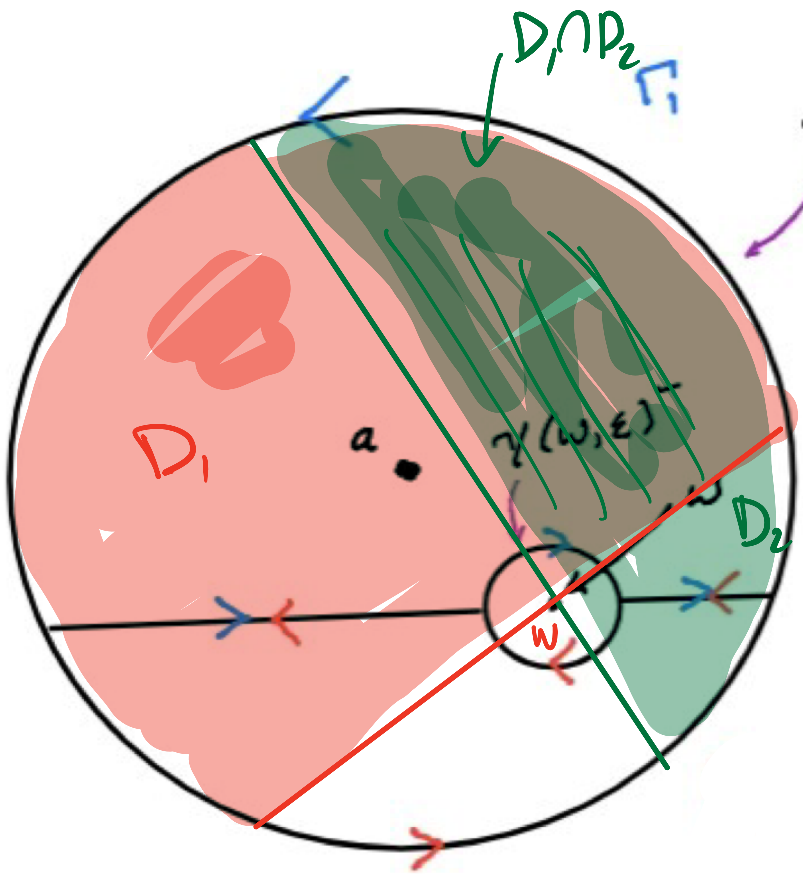

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$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w}dz &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)-f(w)}{z-w}dz + \frac{f(w)}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{dz}{z-w} \\ &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)-f(w)}{z-w}dz + f(w)\end{aligned}$$

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It follows that

$$\frac{1}{2\pi i} \int_{\gamma(w,\epsilon)} \frac{f(z) - f(w)}{z - w} dz = 0$$

and

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□

Definition

If $f\colon U \to \mathbb{C}$ is a function on an open subset U of $\mathbb{C}$, then we say that f is analytic on U if for every $z_0 \in \mathbb{C}$ there is an $r > 0$ with $B(z_0, r) \subseteq U$ such that there is a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ with radius of convergence at least r and $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$. An analytic function is holomorphic, as any power series is (infinitely) complex differentiable.

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We will use the integral expression $f(w) = \frac{1}{2\pi i} \int_{\gamma(z_0, r)} \frac{f(z)}{z - w} dz$.

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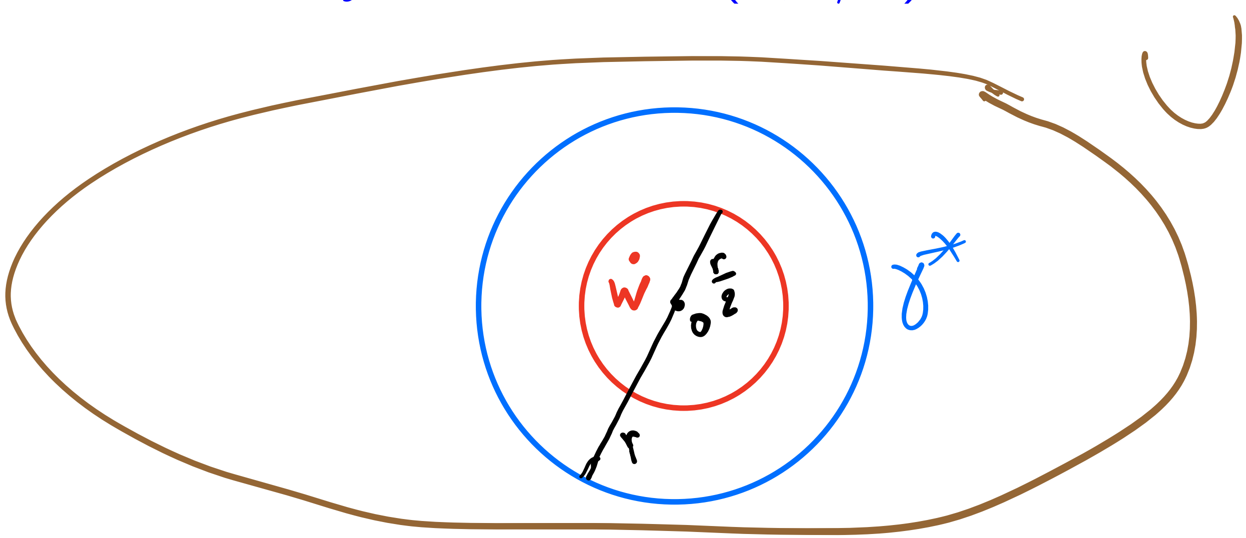

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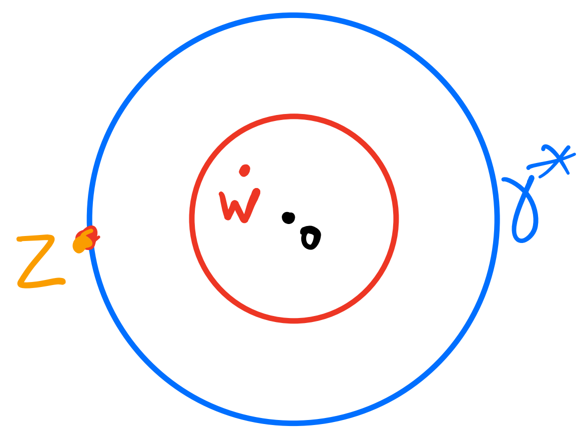

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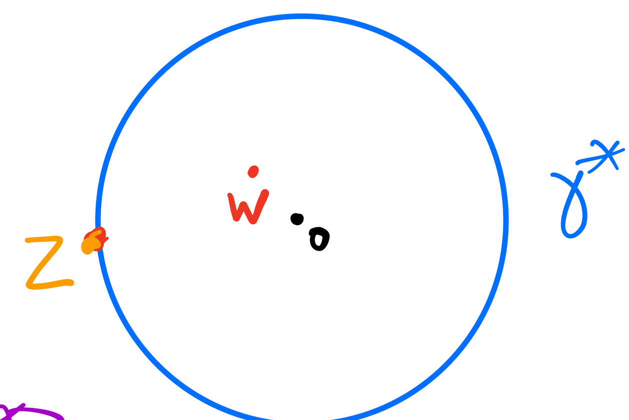
Recall

Weierstrass M-test:

$\sum f_n(z)$ conv. uniformly if $|f_n(z)| \leq M_n$ $(\forall z)$ and $\sum M_n < \infty$



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Since γ^* is compact, $M = \sup\{|f(z)| : z \in \gamma^*\}$ is finite. We apply Weierstrass M-test:
$|f(z) \cdot w^n/z^{n+1}| = |f(z)||z|^{-1}|w/z|^n < \frac{M}{2r}(1/2)^n, \quad \forall z \in \gamma^*.$

Uniform convergence implies that for all $w \in B(0, r)$ we have

$$\sum_{n=0}^{\infty} \overbrace{\left(\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz \right)}^{= a_n} w^n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-w} = f(w)$$

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$$\int_\gamma \sum_{n=0}^{\infty} \frac{f(z) \cdot w^n}{z^{n+1}}$$

Recall

$$\sum_{n=0}^{N} f_n(z) \xrightarrow{\text{unif}} f$$

$$\Downarrow$$

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Remark. If $z_0 \neq 0$ then the formula above applies to $g(w) = f(w + z_0)$ and we obtain:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma(0,r)} \frac{f(z + z_0)}{z^{n+1}} dz \right) w^n = f(w + z_0)$$

and setting $u = w + z_0$ and substituting $v = z + z_0$ in the integral we get

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma(z_0,r)} \frac{f(v)}{(v - z_0)^{n+1}} dv \right) (u - z_0)^n = f(u)$$

Corollary

(Taylor Series Expansion) If $f\colon U \to \mathbb{C}$ is holomorphic on an open set U, then for any $z_0 \in U$, and for any open disc $B(z_0, r)$ centred at z_0 and lying in U we have the Taylor series expansion

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Moreover

$$a_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for any $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ with $z_0 \in B(a, r)$, and we obtain the Cauchy Integral Formulas for the derivatives of f at z_0:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Proof.

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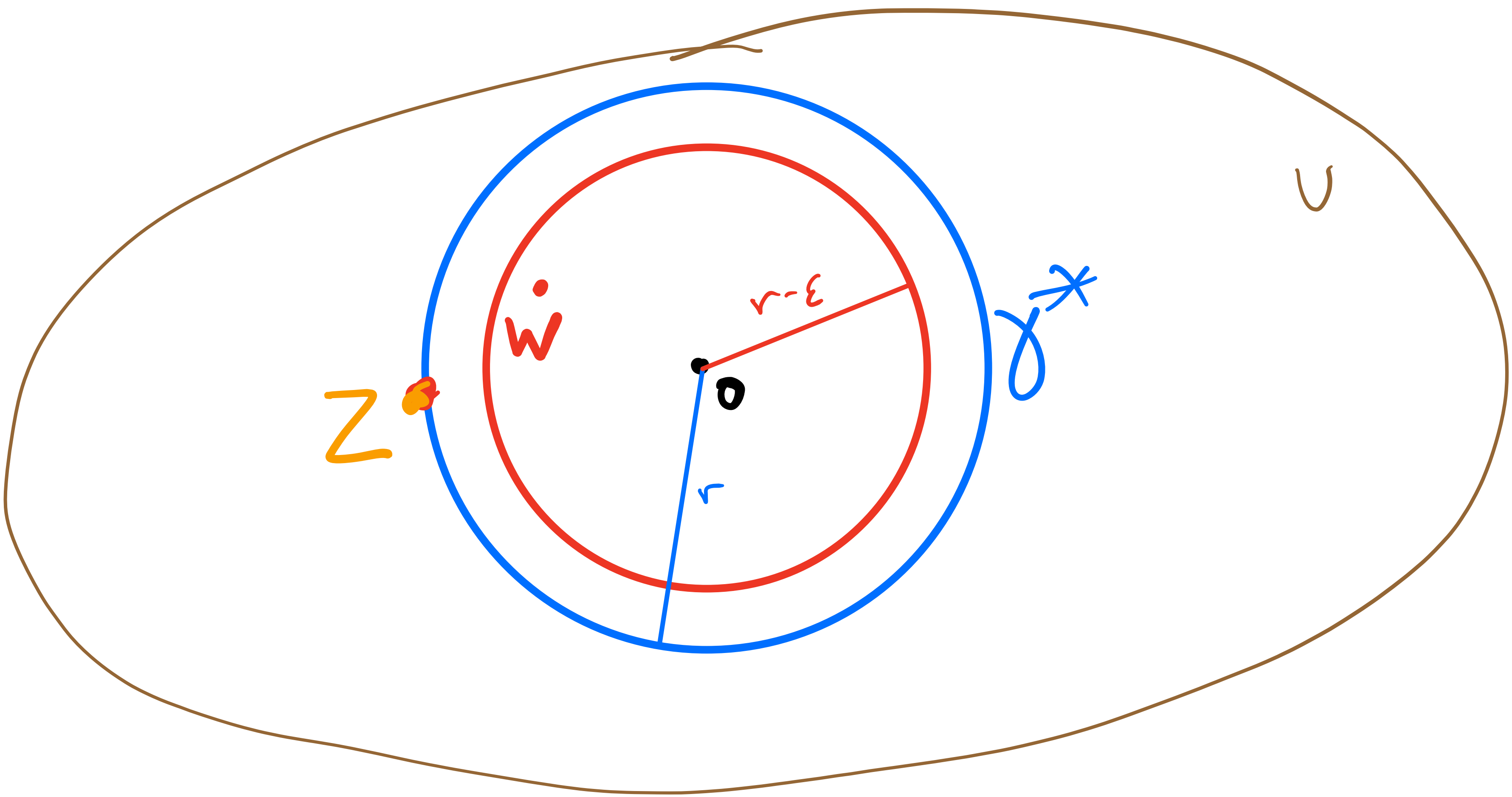

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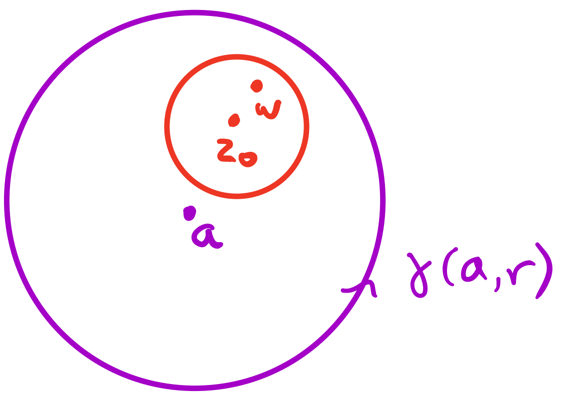

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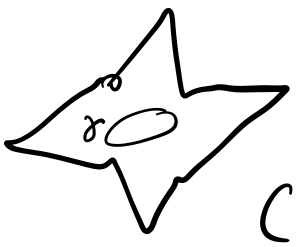
Recap

1) $\oint f = 0 \xLeftarrow{\text{FTC}} \exists F$ primitive

2) $\int_\gamma f = 0 \ \forall \gamma \Longrightarrow \exists F$ primitive

3) $\triangle$ $\quad \int_\Delta f = 0$

$\Downarrow$

4) 

f has primitive $\Longrightarrow \int_\gamma f = 0$

Cauchy for starlike

$\Downarrow$

5) Cauchy's integral formula

$\Downarrow$

6) f analytic

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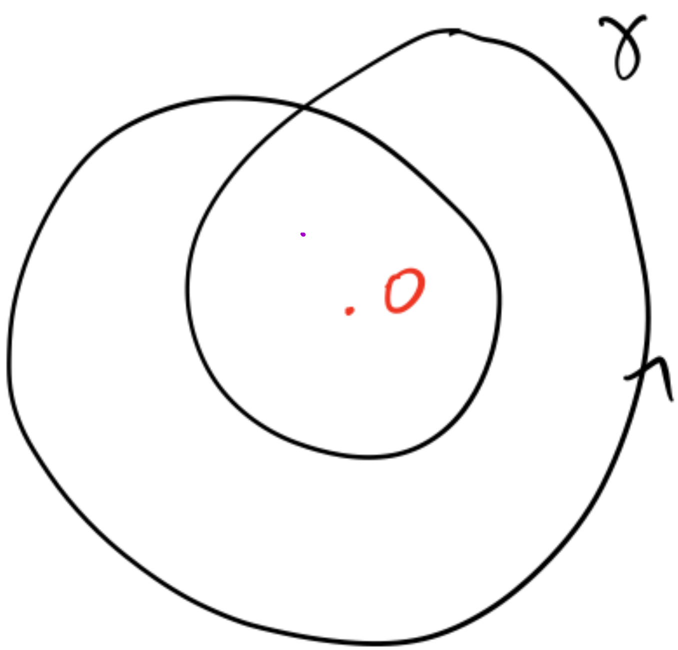
Let $\gamma\colon [0,1] \to \mathbb{C}$ closed path which does not pass through 0. We will give a rigorous definition of the number of times γ "goes around the origin".



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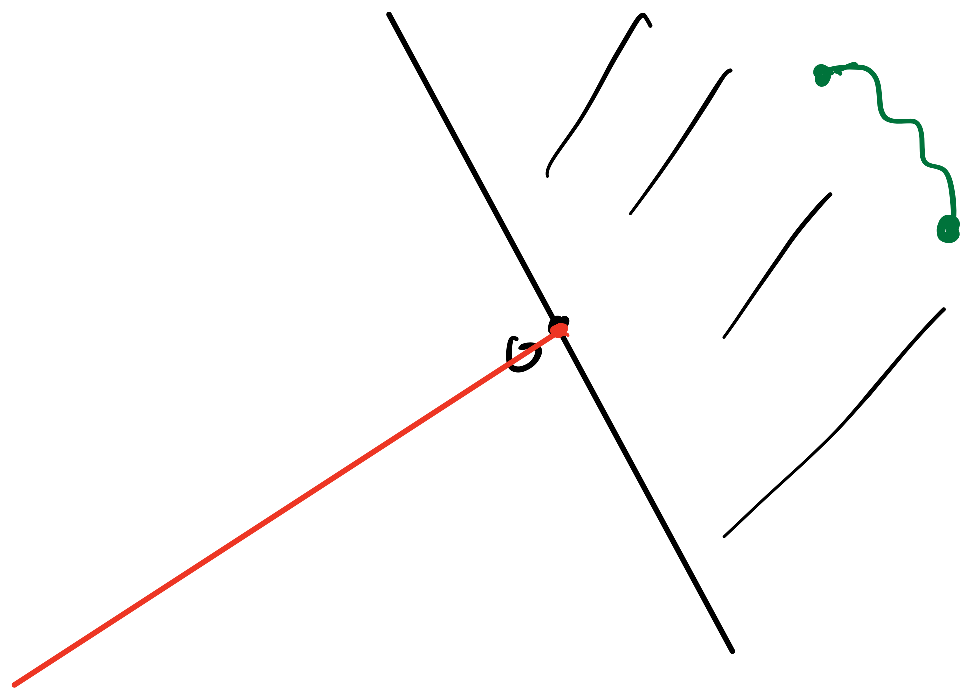

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Proposition

Let $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ be a path. Then there is continuous function $a\colon [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}.$$

Moreover, if a and b are two such functions, then there exists $n \in \mathbb{Z}$ such that $a(t) = b(t) + n$ for all $t \in [0,1]$.

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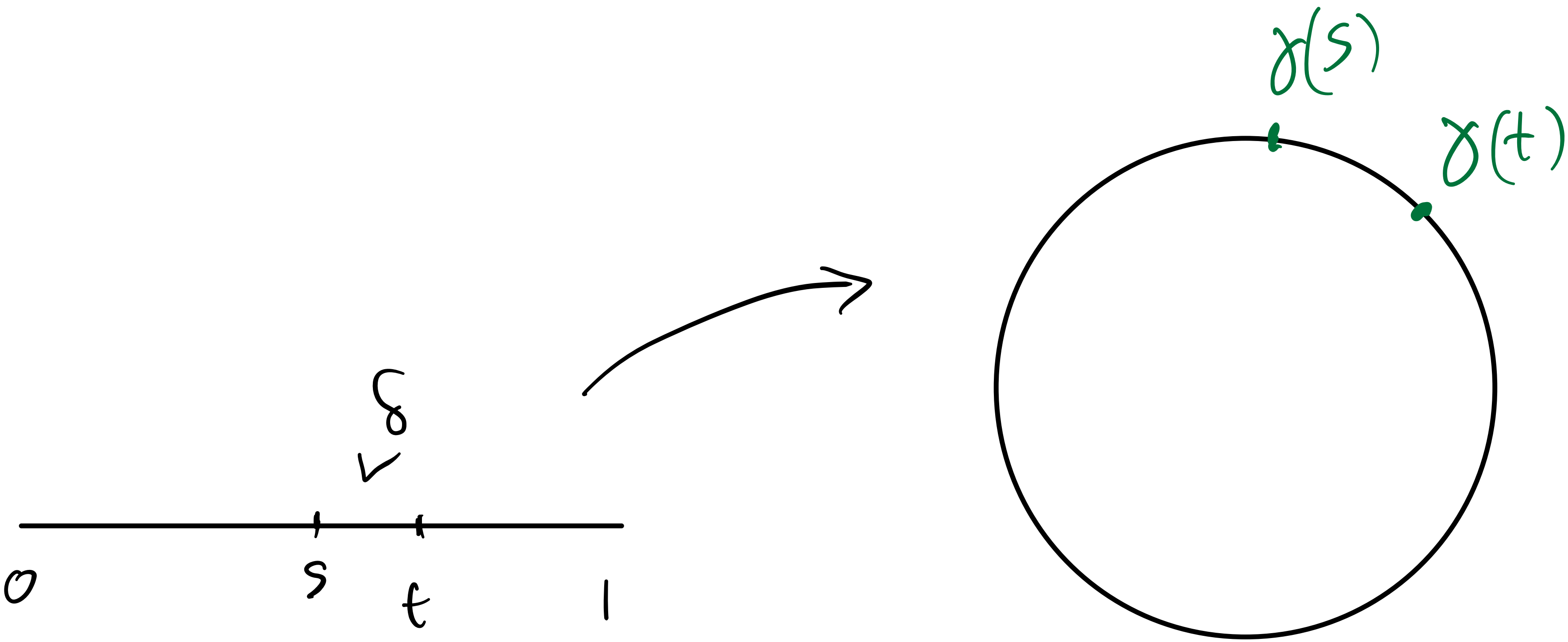

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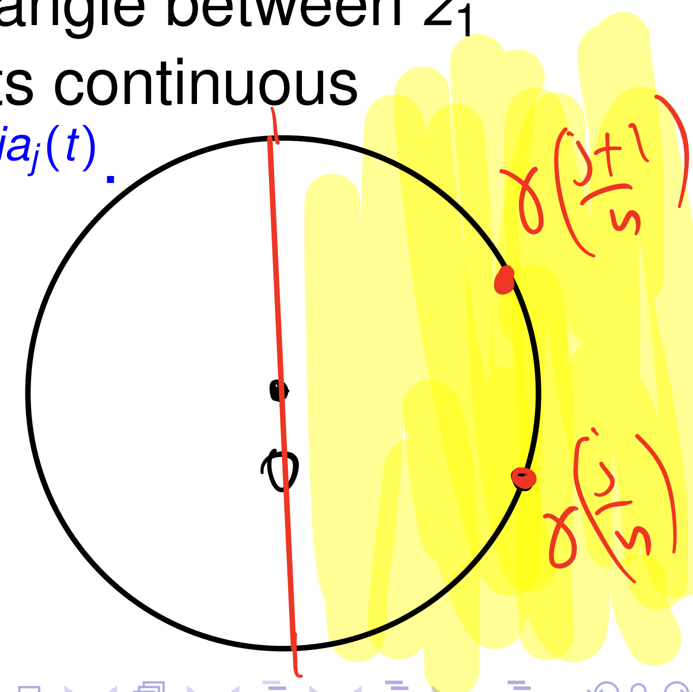

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Uniqueness: $e^{2\pi i(a(t)-b(t))} = 1$, hence $a(t) - b(t) \in \mathbb{Z}$, but $[0,1]$ is connected so $a(t) - b(t)$ is constant. $\square$

Definition

If $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ is a closed path and $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}$ as in the previous lemma, then $a(1) - a(0) \in \mathbb{Z}$. This integer is called the winding number $I(\gamma, 0)$ of γ around 0.

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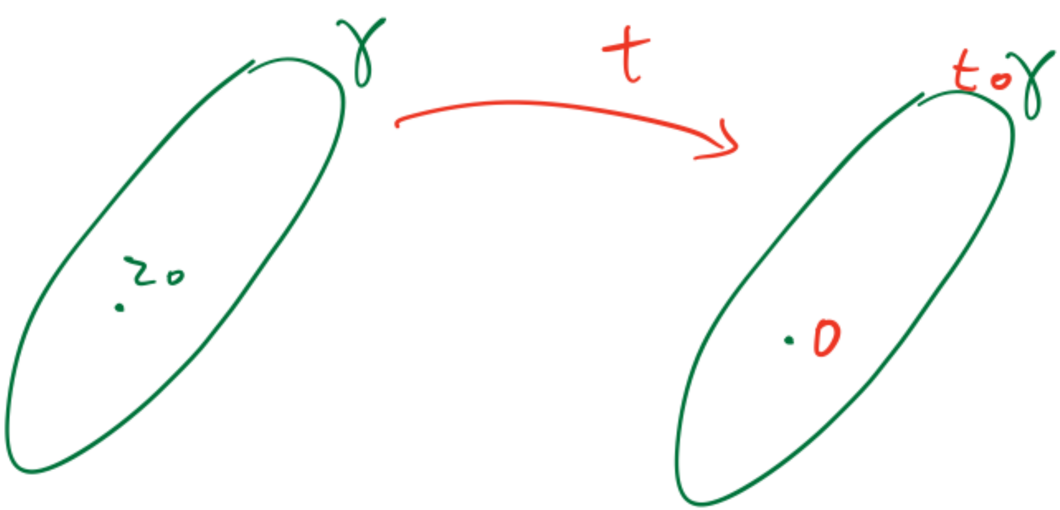

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Remarks: 1. The definition of the winding number only requires the closed path γ to be continuous, not piecewise C^1.

Definition

If $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ is a closed path and $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}$ as in the previous lemma, then $a(1) - a(0) \in \mathbb{Z}$. This integer is called the winding number $I(\gamma, 0)$ of γ around 0.
It is uniquely determined by the path γ because the function a is unique up to an integer.
If z_0 is not in the image of γ, we may define the winding number $I(\gamma, z_0)$ of γ about z_0 similarly:
Let $t\colon \mathbb{C} \to \mathbb{C}$ be given by $t(z) = z - z_0$, we define $I(\gamma, z_0) = I(t \circ \gamma, 0)$.

Remarks: 1. The definition of the winding number only requires the closed path γ to be continuous, not piecewise C^1.

2. if $\gamma\colon [0,1] \to U$ where $0 \notin U$ and there exists a holomorphic branch $L\colon U \to \mathbb{C}$ of [Log(z)] on U, then $I(\gamma, 0) = 0$. Indeed in this case we may define $a(t) = \Im(L(\gamma(t)))$, and since $\gamma(0) = \gamma(1)$ it follows $a(1) - a(0) = 0$.

The winding number for C^1 paths can be expressed using integrals:

Lemma

Let γ be a piecewise C^1 closed path and $z_0 \in \mathbb{C}$ a point not in the image of γ. Then the winding number $I(\gamma, z_0)$ of γ around z_0 is given by

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

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Proof.

If $\gamma\colon [0,1] \to \mathbb{C}$ we may write $\gamma(t) = z_0 + r(t)e^{2\pi i a(t)}$. Then

$$\int_\gamma \frac{dz}{z - z_0} = \int_0^1 \frac{1}{r(t)e^{2\pi i a(t)}} \cdot \left(r'(t) + 2\pi i r(t) a'(t)\right) e^{2\pi i a(t)} dt$$

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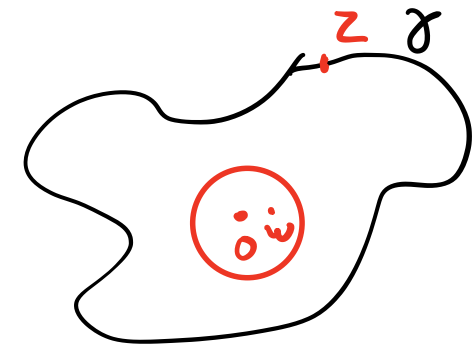
$= 2\pi i(a(1) - a(0))$, since $r(1) = r(0) = |\gamma(0) - z_0|$. $\square$

Corollary (of the proof: holomorphic $\Rightarrow$ analytic)

Let U be an open set in $\mathbb{C}$ and let $\gamma : [0,1] \to U$ be a closed piecewise C^1-path *path. If $f(z)$ is a continuous function on γ^* then the function*

$$I_f(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz,$$



is analytic in w.

Proof. This follows by the same argument that we used to show that holomorphic functions are analytic.

In the proof we only used that f is continuous on γ^*.

If w_0 is not on γ^* then for some $\epsilon > 0$ we have that $|\frac{w}{z}| < \frac{1}{2}$ for all $w \in B(w_0, \epsilon)$ and this suffices to show that $I_f(\gamma, w)$ is analytic.

Proposition

Let U be an open set in $\mathbb{C}$ and let $\gamma\colon [0,1] \to U$ be a closed piecewise C^1 path. Then the function $w \mapsto I(\gamma, w)$ is a continuous function on $\mathbb{C}\backslash\gamma^*$, hence constant on the connected components of $\mathbb{C}\backslash\gamma^*$.

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Proof.

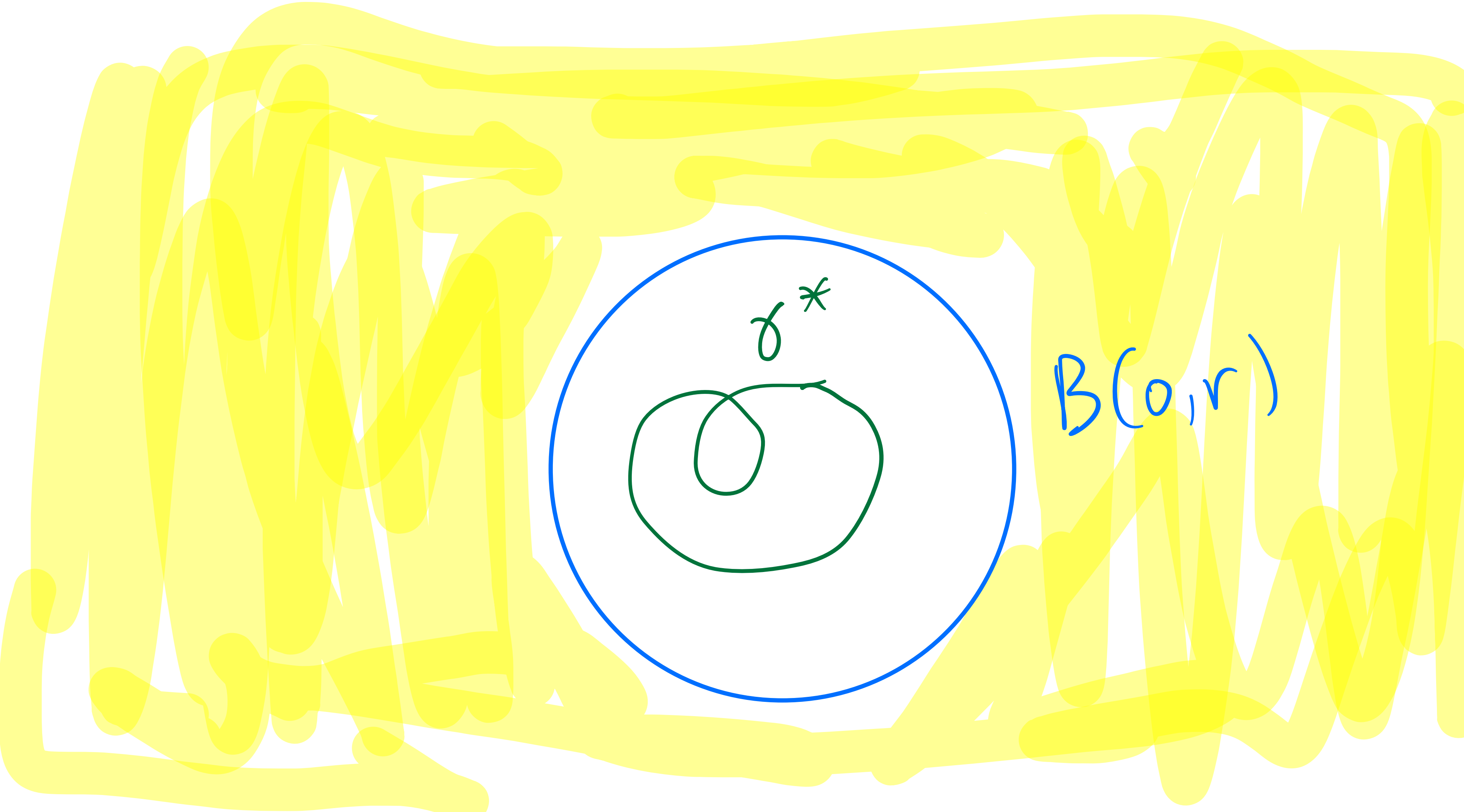
We showed earlier that the function

$$I(\gamma, w) = \int_\gamma \frac{1}{z-w}\,dz$$

is analytic, so it is continuous.

$\square$

If γ is a closed path then γ^* is compact and hence bounded. Thus there is an $R > 0$ such that the connected set $(\mathbb{C}\backslash B(0, R)) \cap \gamma^* = \emptyset$. It follows that $\mathbb{C}\backslash\gamma^*$ has exactly one unbounded connected component.



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Since

$$| \int_\gamma \frac{d\zeta}{\zeta - z} | \leq \ell(\gamma). \sup_{\zeta \in \gamma^*} |1/(\zeta - z)| \to 0$$

as $z \to \infty$ it follows that $I(\gamma, z) = 0$ on the unbounded component of $\mathbb{C} \backslash \gamma^*$.

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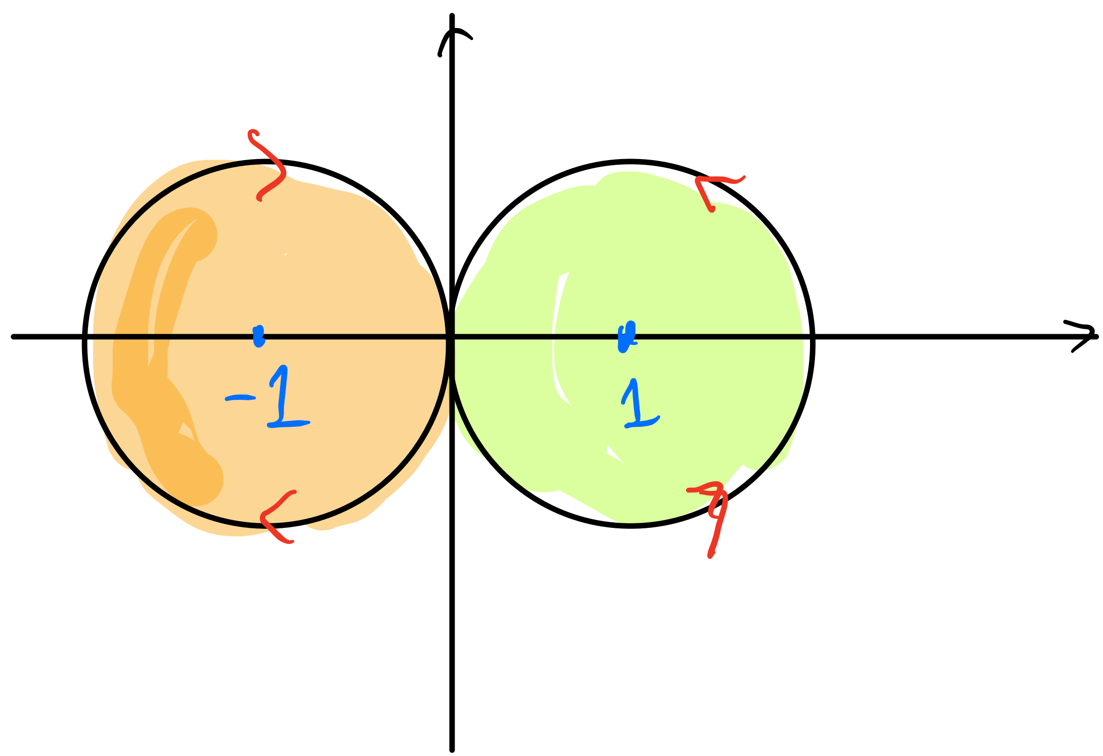
Definition

Let $\gamma \colon [0, 1] \to \mathbb{C}$ be a closed path. We say that a point z is in the inside of γ if $z \notin \gamma^*$ and $I(\gamma, z) \neq 0$. The previous remark shows that the inside of γ is a union of bounded connected components of $\mathbb{C} \backslash \gamma^*$. (We don't, however, know that the inside of γ is necessarily non-empty.)

Example

Suppose that $\gamma_1 : [-\pi, \pi] \to \mathbb{C}$ is given by $\gamma_1 = 1 + e^{it}$ and $\gamma_2 : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma_2(t) = -1 + e^{-it}$. Then if $\gamma = \gamma_1 \star \gamma_2$, γ traverses a figure-of-eight and it is easy to check that the inside of γ is $B(1,1) \cup B(-1,1)$ where $I(\gamma, z) = 1$ for $z \in B(1,1)$ while $I(\gamma, z) = -1$ for $z \in B(-1,1)$.



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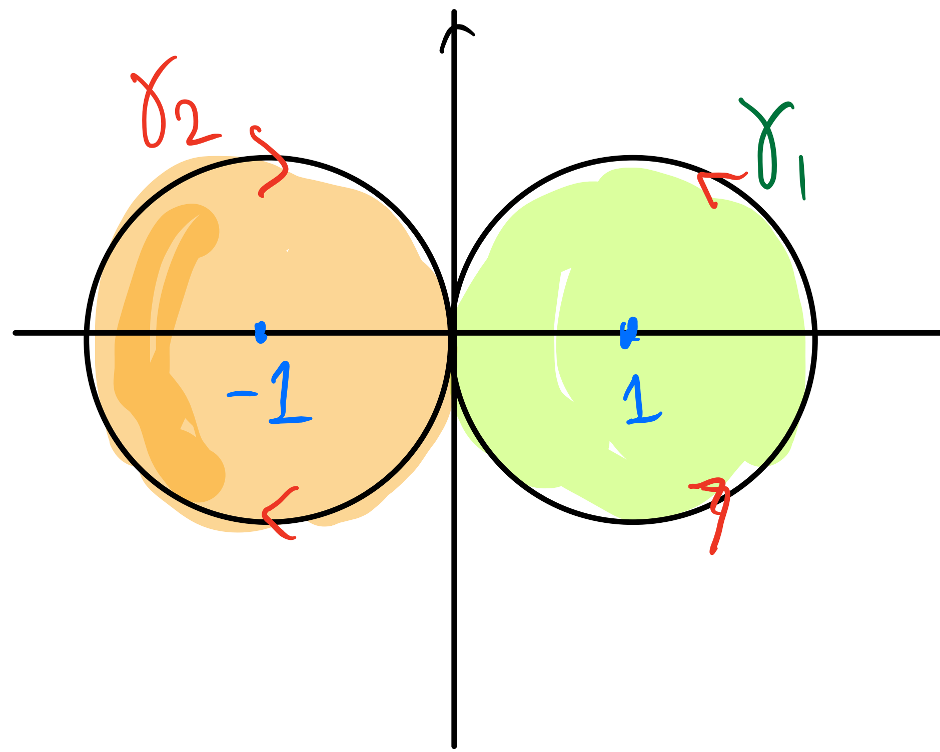
$$I(\gamma, 1) = \frac{1}{2\pi i}\left(\underbrace{\int_{\gamma_1} \frac{1}{z-1}\,dz}_{= 1} + \underbrace{\int_{\gamma_2} \frac{1}{z-1}\,dz}_{= 0} \right)$$



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Remark.

It is a theorem, known as the Jordan Curve Theorem, that if $\gamma \colon [0,1] \to \mathbb{C}$ is a simple closed curve, so that $\gamma(t) = \gamma(s)$ if and only if $s = t$ or $s, t \in \{0, 1\}$, then $\mathbb{C} \backslash \gamma^*$ is the union of precisely one bounded and one unbounded component, and on the bounded component $I(\gamma, z)$ is either 1 or -1. If $I(\gamma, z) = 1$ for z on the inside of γ we say γ is positively oriented and we say it is negatively oriented if $I(\gamma, z) = -1$ for z on the inside.