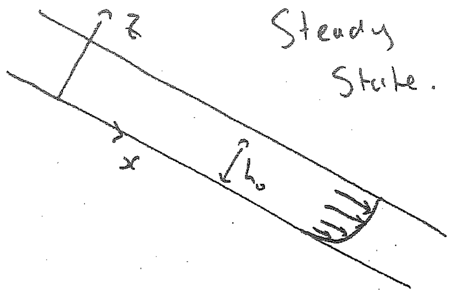
Q1)



Steady State.

$\nabla \cdot \underline{u} = 0$ N-S Eq's.

$\rho(\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}) = -\nabla p - \rho \underline{g} + \mu \nabla^2 \underline{u}$

$\underline{g} = (-\sin\alpha, \cos\alpha) g$

B.c.'s are: $u = w = 0$ @ $z = 0$

$u_z = 0, \; p = p_a$: $z = h_0$

Solution: $\bar{u} = \left(\frac{\rho g \sin\alpha}{2\mu}\right)(2h_0 z - z^2), \quad \bar{p} = p_a - \rho g \cos\alpha (z - h_0)$

Consider perturbations: $u = \bar{u} + \hat{u}, \; w = \hat{w}, \; p = \bar{p} + \hat{p}$

& $z = h(x,t)$. (Long-wave)

Non Dimensionalise: $x \sim L, \; z \sim h_0, \; u \sim U_0 = \frac{\rho g h_0^2 \sin\alpha}{2\mu}$

$w \sim \varepsilon U_0, \; p - p_a \sim \rho g h_0 \sin\alpha, \; t \sim \frac{L}{U_0}$

This results in governing Eqns with $\varepsilon = \frac{h_0}{L}, \; Re = \frac{U_0 h_0}{\nu}$

Boundary Conditions: • No Slip @ $\underline{z = 0}$: $\hat{w} = \hat{u} = 0$

<u>At $z = h$</u>: Kinematic: $\hat{w} = h_t + (\bar{u} + \hat{u}) h_x$ (1)

N.B. $\bar{u} = 2z - z^2, \; \bar{p} = -\cot\alpha (z - 1)$

(2) $\bar{p} + \hat{p} = -\frac{\gamma}{\rho g \sin\alpha L^2} h_{xx} + O(\varepsilon^2)$ or: Normal Stress. (Dynamic)

γ = Surface tension.

(3) Tangent Stress: $\bar{u}_z + \hat{u}_z = O(\varepsilon^2)$

These can also be expanded in powers of ε from σ_{nn}, σ_{nt} definition.

Now, let $\hat{u} = \hat{u}_0 + \varepsilon \hat{u}_1 + \dots$ etc..

At leading order: $\hat{u}_{0zz} = 0$, $\hat{p}_{0z} = 0$

$$\Rightarrow \hat{u}_0 = 2z(h-1), \ \hat{w}_0 = -z^2 h_x$$

$$\hat{p}_0 = (h-1)\cot\alpha - S h_{xx}$$

$(h_t + 2h^2 h_x = 0)$

where $S = \gamma / \rho g L^2 \sin\alpha = O(1)$

At 1^{st} order: $\hat{u}_{1zz} = 2\hat{p}_{0x}$

$$+ Re\left[\hat{u}_{0t} + (\bar{u} + \hat{u}_0)\hat{u}_{0x} + \hat{w}_0(\bar{u}_z + \hat{u}_{0z})\right]$$

$(\hat{u}_{0t} = 2z h_t)$

$$\Rightarrow \hat{u}_{1zz} = 2(h_x \cot\alpha - S h_{xxx}) - 4Re\, z h^2 h_x + 2 Re\, z^2 h h_x$$

B.C. : $\hat{u}_{1z} = 0$ @ $z = h$

$$\Rightarrow \hat{u}_1 = (h_x \cot\alpha - S h_{xxx})(z^2 - 2zh) + \tfrac{1}{6} Re\, h h_x (z^4 - 4z^3 h + 8 z h^3)$$

Cons. of Mass: $h_t + \frac{\partial}{\partial x}\int_0^h \bar{u} + \underbrace{\hat{u}}_{= \hat{u}_0 + \varepsilon \hat{u}_1 + \dots} \, dz = 0$

$\Rightarrow$ This gives the Benney Eqn after simplification.

Let $\varepsilon \to 0 \Rightarrow h_t + 2h^2 h_x = 0$

$h = 1 + \eta \rightarrow \eta_t + 2\eta_x = 0$ STABLE

$\eta \ll 1$

Advection @ speed = 2.