C4.3 Functional Analytic Methods for PDEs
Lectures 11-12

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In previous lectures

- Sobolev spaces and their properties

This lecture

- Linear elliptic equations of second order.
- Classical and weak solutions.
- Energy estimates.
- First existence theorem: Riesz representation theorem.
- First existence theorem: Direct method of the calculus of variation.
- Second existence theorem: Fredholm alternative.

The equation of interest

- We will consider the equation

$$Lu := -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

where
  - ⋆ Ω is a domain in $\mathbb{R}^n$, which frequently has Lipschitz regularity or better,
  - ⋆ $u : \Omega \to \mathbb{R}$ is the unknown,
  - ⋆ $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are given coefficients,
  - ⋆ $f, g_i : \Omega \to \mathbb{R}$ are given sources.
- Equation (*) is said to be in divergence form. It can be written in more compact form:

$$Lu = -\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu = f + \mathrm{div} g$$

where
  - ⋆ $a = (a_{ij})$ is an $n \times n$ matrix,
  - ⋆ $b = (b_i)$ and $g = (g_i)$ are (column) vectors.

Divergence vs non-divergence form

- To dispel confusion, we note that we will not consider the equation

$$-a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega, \qquad (**)$$

which is also of importance. The equation (**) is said to be in non-divergence form.
To treat (**), we will need some preparation different from what we have had so far.

Structural assumptions

We make the following assumptions:

- The coefficients $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ belong to $L^\infty(\Omega)$.
- The coefficients a_{ij} is *symmetric*, i.e. $a_{ij} = a_{ji}$.
- The coefficients a_{ij} is *uniformly elliptic* – this will be defined on the next slide.

Ellipticity

Definition

Let $a = (a_{ij}) : \Omega \to \mathbb{R}^{n \times n}$ be symmetric and have measurable entries.

- a is *elliptic* if
$$a_{ij}(x)\xi_i \xi_j \geq 0 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$
(In other words, a is non-negative definite a.e. in Ω.)
- a is *strictly elliptic* if there exists $\lambda > 0$ such that
$$a_{ij}(x)\xi_i \xi_j \geq \lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$
- a is *uniformly elliptic* if there exist $0 < \lambda \leq \Lambda < \infty$ such that
$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i \xi_j \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

Examples

Two simplistic but important examples:

- $a_{ij} = \delta_{ij}$ in all of Ω.
- $a_{ij} = k(x)\delta_{ij}$ where $k = k_1\chi_A + k_2\chi_{\Omega\setminus A}$ for some subset A of Ω and some constants $k_1, k_2 > 0$.

The Dirichlet boundary value problem

We will write $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ to mean that

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu.$$

The Dirichlet boundary value problem for L asks to find a function u satisfying

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(BVP)}$$

where

- f and g are given sources,
- u_0 is given boundary data.

Classical solutions

$$L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c.$$

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{BVP}$$

Definition

Suppose $a \in C^1(\Omega), b, c \in C(\Omega)$. For a given $f \in C(\Omega)$, $g \in C^1(\Omega)$ and $u_0 \in C(\partial\Omega)$, a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is called a *classical solution to* the Dirichlet boundary value problem (BVP) if it satisfies (BVP) in the usual sense.

- We saw in the first lecture that the notion of classical solutions is insufficient for our need.

An observation

- Suppose $a \in C^1(\Omega)$, $b, c \in C(\Omega)$, $f \in C(\Omega)$ and $g \in C^1(\Omega)$. Suppose $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega. \qquad (*)$$

- If $\varphi \in C_c^\infty(\Omega)$ is a test function, then

$$\int_\Omega (Lu)\,\varphi\,dx = \int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx$$

and

$$\int_\Omega [f + \partial_i g_i]\,\varphi\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx.$$

- Therefore, for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx. \qquad (\Diamond)$$

An observation

- Conversely, if u is such that $(\diamondsuit)$ holds for all $\varphi \in C_c^\infty(\Omega)$, then by reversing the argument, we have

$$\int_\Omega (Lu)\,\varphi\,dx = \int_\Omega [f + \partial_i g_i]\,\varphi\,dx \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  This implies $Lu = f + \partial_i g_i$ in Ω, i.e. u satisfies (*).

- We conclude that $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  if and only if u satisfies

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i\varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx \qquad (\diamondsuit)$$

  for all $\varphi \in C_c^\infty(\Omega)$.

An observation

- We conclude that $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  if and only if u satisfies

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i \partial_i u \varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i \partial_i \varphi]\,dx \qquad (\diamondsuit)$$

  for all $\varphi \in C_c^\infty(\Omega)$.
- Key: While the formulation (*) requires u to be twice differentiable, the formulation $(\diamondsuit)$ requires u to be only once differentiable.

Weak solutions

Definition

Let $a, b, c \in L^\infty(\Omega)$ and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

- Suppose $f \in L^2(\Omega)$, $g \in L^2(\Omega)$.
  We say that $u \in H^1(\Omega)$ is a *weak solution* (or *generalized solution*) to the equation

$$Lu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  if

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i \partial_i u \varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx \qquad (\diamondsuit)$$

  holds for all $\varphi \in H_0^1(\Omega)$.
  When this holds, we also say that u satisfies (*) in the weak sense.

Weak solutions

Definition

Let $a, b, c \in L^\infty(\Omega)$ and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

- Suppose that $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$. We say that $u \in H^1(\Omega)$ is a *weak solution* (or *generalized solution*) to the Dirichlet boundary value problem

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(BVP)}$$

  if $Lu = f + \partial_i g_i$ in Ω in the weak sense and if $u - u_0 \in H_0^1(\Omega)$.

Weak solutions

- It is convenient to introduce the bilinear form $B(\cdot,\cdot)$:

$$B(u,v) = \int_\Omega [a_{ij}\partial_j u \partial_i v + b_i \partial_i u v + cuv]\,dx \qquad u,v \in H^1(\Omega).$$

  B is called the bilinear form associated with the operator L.
- Then $u \in H^1(\Omega)$ satisfies (*) in the weak sense if

$$B(u,\varphi) = \langle f,\varphi\rangle - \langle g_i, \partial_i \varphi\rangle \text{ for all } \varphi \in H_0^1(\Omega),$$

  where $\langle \cdot,\cdot\rangle$ denotes the inner product of $L^2(\Omega)$.

Energy estimate

Theorem (Energy estimates)

Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ and B is its associated bilinear form. Then there exists some large constant $C > 0$ such that

$$|B(u,v)| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$
$$\frac{\lambda}{2}\|u\|^2_{H_1(\Omega)} \leq B[u,u] + C\|u\|^2_{L^2(\Omega)}.$$

Here λ is the constant appearing in the definition of ellipticity of a.

Energy estimate

Proof

- The first estimate is clear from the definition of B and Cauchy-Schwarz's inequality:

$$\begin{aligned}|B(u,v)| &\leq \int_\Omega \Big[|a_{ij}||\partial_j u||\partial_i v| + |b_i||\partial_i u||v| + |c||u||v|\Big]\,dx\\ &\leq \|a\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2}\\ &\qquad + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}\\ &\leq C\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

Energy estimate

Proof

- For the second estimate, we start by estimating the lower order term in the same fashion while leaving the highest order term untouched:

$$B(u,u) \geq \int_\Omega \Big[a_{ij}\partial_j u \partial_i u - |b_i||\partial_i u||u| - |c||u|^2\Big]\,dx$$
$$\geq \int_\Omega a_{ij}\partial_j u \partial_i u\,dx$$
$$- \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^2} - \|c\|_{L^\infty}\|u\|_{L^2}^2.$$

- The leading term is treated using the ellipticity condition:

$$a_{ij}\partial_j u \partial_i u \geq \lambda|\nabla u|^2.$$

Energy estimate

Proof

- We thus have

$$B(u,u) \geq \lambda \|\nabla u\|_{L^2}^2 - \|b\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^2} - \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

- Using the inequality $xy \leq \frac{\lambda}{2} x^2 + \frac{1}{2\lambda} y^2$, we can absorb the quantity $\|\nabla u\|_{L^2}$ in the second term on the right hand side to the first term:

$$\begin{aligned} B(u,u) &\geq \lambda \|\nabla u\|_{L^2}^2 - \frac{\lambda}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2\lambda} \|b\|_{L^\infty}^2 \|u\|_{L^2}^2 - \|c\|_{L^\infty} \|u\|_{L^2}^2 \\ &= \frac{\lambda}{2} \|\nabla u\|_{L^2}^2 - C \|u\|_{L^2}^2. \end{aligned}$$

L as an operator on $H^1(\Omega)$

Corollary

Suppose that $a, b, c \in L^\infty(\Omega)$, *a is uniformly elliptic,*
$L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.
For every $u \in H^1(\Omega)$, *define a map* $Lu : H_0^1(\Omega) \to \mathbb{R}$ *by*

$$(Lu)(\varphi) = B(u, \varphi) \text{ for all } \varphi \in H_0^1(\Omega).$$

Then $Lu : H_0^1(\Omega) \to \mathbb{R}$ *is bounded linear, i.e.*
$Lu \in (H_0^1(\Omega))^* =: H^{-1}(\Omega)$.
Furthermore, L is a bounded linear map from $H^1(\Omega)$ *into* $H^{-1}(\Omega)$.

L as an operator on $H^1(\Omega)$

Proof

- Linearity if clear. By the energy estimate, $|(Lu)(\varphi)| \leq C\|u\|_{H^1}\|\varphi\|_{H^1}$ and so Lu belongs to $H^{-1}(\Omega)$.
- Furthermore, we have

$$\|Lu\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|_{H^1} \leq 1} |Lu(\varphi)| \leq C\|u\|_{H^1}.$$

This means $L \in \mathscr{L}(H^1(\Omega), H^{-1}(\Omega))$.

Weak sense vs H^{-1} sense

Corollary

u is a weak solution to (*) if and only if $Lu = f + \partial_i g_i$ as elements of $H^{-1}(\Omega)$.

Here $f + \partial_i g_i$ is viewed as an element of $H^{-1}(\Omega)$ by letting

$$(f + \partial_i g_i)(\varphi) = \int_\Omega [f\varphi - g_i \partial_i \varphi]\, dx.$$

$W^{1,p}$ solutions

Remark

One can similarly define a notion of $W^{1,p}$ solutions to (*) and (BVP) using $p \neq 2$. The treatment for these type of solutions is beyond the scope of this course.

An existence theorem

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, *a is uniformly elliptic,* $c \geq 0$ *a.e. in* Ω, *and* $L = -\partial_i(a_{ij}\partial_j) + c$ *(i.e.* $b \equiv 0$*). Then for every* $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ *and* $u_0 \in H^1(\Omega)$, *the Dirichlet boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{BVP}$$

has a unique weak solution $u \in H^1(\Omega)$.

An existence theorem

The above theorem is a consequence of the following statement:

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, a is uniformly elliptic, $c \geq 0$ a.e. in Ω, and $L = -\partial_i(a_{ij}\partial_j) + c$ (i.e. $b \equiv 0$). Then $L|_{H_0^1(\Omega)}$ is a bijection from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

Indeed, if we let $L^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ be the inverse of $L|_{H_0^1(\Omega)}$, then the unique solution to (BVP) is given by

$$u = u_0 + L^{-1}(-Lu_0 + f + \partial_i g_i).$$

An existence theorem

First proof: Riesz representation theorem.

- Observe that the bilinear form associated with L is positive in $H_0^1(\Omega)$:

$$B(u,u) = \int_\Omega \Big[a_{ij}\partial_j u \partial_i u + cu^2\Big]\,dx$$
$$\geq \lambda \|\nabla u\|_{L^2}^2 \geq \frac{1}{C}\|u\|_{H^1}^2 \text{ for all } u \in H_0^1(\Omega).$$

  Hence $B(\cdot,\cdot)$ defines an inner product on $H_0^1(\Omega)$, which is equivalent to the standard inner product of $H_0^1(\Omega)$.

- Thus, by the Riesz representation theorem, for every $T \in H^{-1}(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$B(u,v) = Tv \text{ for all } v \in H_0^1(\Omega).$$

  But this means precisely that $Lu = T$. We conclude that $L|_{H_0^1(\Omega)}$ is a bijection from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

An existence theorem

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, a is uniformly elliptic, $c \geq 0$ a.e. in Ω, and $L = -\partial_i(a_{ij}\partial_j) + c$ (i.e. $b \equiv 0$). Then for every $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$, the Dirichlet boundary value problem

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{BVP}$$

has a unique weak solution $u \in H^1(\Omega)$.

$$\Updownarrow$$

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, a is uniformly elliptic, $c \geq 0$ a.e. in Ω, and $L = -\partial_i(a_{ij}\partial_j) + c$ (i.e. $b \equiv 0$). Then $L|_{H_0^1(\Omega)}$ is a bijection from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

An existence theorem

First proof: Riesz representation theorem.

- The equation $Lu = T$ with $T \in H^{-1}(\Omega)$ is equivalent to

$$B(u, v) = Tv \text{ for all } v \in H_0^1(\Omega).$$

- The bilinear form $B(\cdot, \cdot)$ defines an inner product on $H_0^1(\Omega)$, which is equivalent to the standard inner product of $H_0^1(\Omega)$. The conclusion is reached using the Riesz representation theorem.

Second proof: Direct method of the calculus of variation.

We'll use the fact that $H_0^1(\Omega)$ is weakly closed in $H^1(\Omega)$. This is a consequence of the following general theorem:

Theorem (Mazur)

Let K be a closed convex subset of a normed vector space X, (x_n) be a sequence of points in K converging weakly to x. Then $x \in K$.

An existence theorem

Second proof

- Fix $T \in H^{-1}(\Omega)$ and define the 'variational energy':

$$I[v] = \frac{1}{2}B(v,v) - Tv \text{ for } v \in X := H_0^1(\Omega).$$

  The key point of the proof is the fact that: *$u \in X$ solves $Lu = T$ if u is a minimizer or I on X i.e. $I[u] \leq I[v]$ for all $v \in X$.*

- Step 1: Boundedness of minimizing sequence.
  Let $\alpha = \inf_X I \in \mathbb{R} \cup \{-\infty\}$. Note that $I[0] = 0$ and so $\alpha \leq 0$. Pick $u_m \in X$ such that $I[u_m] \to \alpha$. We show that the sequence (u_m) is bounded in $H^1(\Omega)$.
  - ⋆ By the ellipticity and the non-negativity of c, we have

$$B(u_m, u_m) = \int_\Omega [a_{ij}\partial_j u_m \partial_i u_m + c u_m^2]\,dx \geq \lambda \int_\Omega |\nabla u_m|^2\,dx.$$

An existence theorem

Second proof

- Step 1: Boundedness of minimizing sequence (u_m).
  - ⋆ Hence, by Friedrichs' inequality, $B(u_m, u_m) \geq \frac{1}{C}\|u_m\|_X^2$.
  - ⋆ It follows that

$$I[u_m] = \frac{1}{2}B(u_m, u_m) - Tu_m \geq \frac{1}{2C}\|u_m\|_X^2 - \|T\|\|u_m\|_X$$
$$\geq \frac{1}{4C}\|u_m\|_X^2 - C\|T\|^2.$$

  - ⋆ On the other hand, as $I[u_m] \to \alpha \leq 0$, we have $(I[u_m])$ is bounded from above. Therefore (u_m) is bounded in X.
- Step 2: The weak convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ Since $H^1(\Omega)$ is reflexive, the bounded sequence (u_m) has a weakly convergent subsequence.
  - ⋆ We still denote this subsequence (u_m) so that $u_m \rightharpoonup u$ in $H^1(\Omega)$.

An existence theorem

Second proof

- Step 2: The weak convergence of (u_m) along a subsequence to a minimizer of I.
  - $\star$ $u_m \rightharpoonup u$ in H^1.
  - $\star$ As X is weakly closed in H^1 and $(u_m) \in X$, we have that $u \in X$.
  - $\star$ By definition of weak convergence, we have $Tu_m \to Tu$. We claim that

$$\liminf_{m\to\infty} B(u_m, u_m) \geq B(u, u). \qquad (*)$$

Once this is shown, we have that $I[u] \leq \liminf I[u_m] = \alpha$ and so $I[u] = \alpha$.

An existence theorem

Second proof

- Step 2: The convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ We now prove (*), i.e. $\liminf_{m\to\infty} B(u_m, u_m) \geq B(u, u)$.
  - ⋆ To illustrate the idea, let us consider for now the case $c = 0$ and $a_{ij} = \delta_{ij}$. Then

$$B(u_m, u_m) - B(u, u) = \int_\Omega [|\nabla u_m|^2 - |\nabla u|^2]\, dx$$
$$= \int_\Omega |\nabla(u_m - u)|^2\, dx + 2\int_\Omega \nabla(u_m - u)\cdot\nabla u\, dx.$$

The first term is non-negative. The second term converges to 0 as $\nabla(u_m - u) \rightharpoonup 0$ in L^2. Hence

$$\liminf_{m\to\infty}[B(u_m, u_m) - B(u, u)] = \liminf_{m\to\infty} \int_\Omega |\nabla(u_m - u)|^2\, dx \geq 0.$$

An existence theorem

Second proof

- Step 2: The convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ The proof in the general case is similar. We compute

$$
\begin{aligned}
B(u_m, u_m) - B(u, u) &= \int_\Omega [a_{ij}\partial_i(u_m - u)\partial_j(u_m - u) + c(u_m - u)^2] \\
&\quad + \int_\Omega \Big[a_{ij}\partial_i(u_m - u)\partial_j u + a_{ij}\partial_i u \partial_j(u_m - u) \\
&\qquad + 2c(u_m - u)u\Big]\, dx.
\end{aligned}
$$

Again, the first integral is non-negative while the second and third terms tend to zero. The claim (*) follows, and we conclude Step 2.

An existence theorem

Second proof

- Step 3: We show that u solves $Lu = T$, i.e. $B(u,\varphi) = T\varphi$ for all $\varphi \in X$.
  - ⋆ For $t \in \mathbb{R}$, let $H(t) = I[u + t\varphi]$.
  - ⋆ As shown in Step 2, $I[u] \leq I[u + t\varphi]$ for all t. Hence H has a global minimum at $t = 0$.
  - ⋆ Now note that $H(t)$ is a quadratic polynomial in t:

$$\begin{aligned} H(t) &= \frac{1}{2}B(u + t\varphi, u + t\varphi) - T(u + t\varphi) \\ &= I[u] + \frac{1}{2}t(B(u,\varphi) + B(\varphi,u) - 2T\varphi) + \frac{1}{2}t^2 B(\varphi,\varphi). \end{aligned}$$

  - ⋆ We deduce that

$$0 = H'(0) = \frac{1}{2}(B(u,\varphi) + B(\varphi,u) - 2T\varphi).$$

  - ⋆ Since B is symmetric, we deduce that $B(u,\varphi) = T\varphi$ as wanted.

An existence theorem

Second proof

- Step 4: We prove the uniqueness: If $\bar{u}$ also solves $L\bar{u} = T$, then $\bar{u} = u$.
  - ⋆ It suffices to show that if $Lu = 0$, then $u = 0$.
  - ⋆ $Lu = 0$ means $B(u, \varphi) = 0$ for all $\varphi \in X$. In particular $B(u, u) = 0$.
  - ⋆ But we showed in Step 1 that $B(u, u) \geq \frac{1}{C}\|u\|_X^2$. Therefore $u = 0$.

An example of non-existence and non-uniqueness

We now consider a motivating example for our next discussion:

$$\begin{cases} Lu &= -u'' - u = f, \\ u(0) &= u(\pi) = 0. \end{cases} \qquad (\heartsuit)$$

- This problem has no uniqueness, as the function $v_0(x) = \sin x$ satisfies $Lv_0 = 0$ and $v_0(0) = v_0(\pi) = 0$.
- Furthermore, if $(\heartsuit)$ is solvable, then upon multiplying with v_0 and integrating we get

$$\int_0^\pi f v_0 \, dx = \int_0^\pi [-u'' v_0 - u v_0] \, dx = \int_0^\pi [u' v_0' - u v_0] \, dx$$
$$= \int_0^\pi [-u v_0'' - u v_0] \, dx = 0.$$

Hence, when $\int_0^\pi f v_0 \, dx \neq 0$, the problem $(\heartsuit)$ is not solvable.

An example of non-existence and non-uniqueness

- No uniqueness. Solvable only if $\int_0^\pi f v_0 \, dx = 0$.
- Conversely, suppose $\int_0^\pi f v_0 \, dx = 0$. If $f \in L^2(0,\pi)$, we can write $f = \sum_{n=2}^\infty f_n \sin nx$ with $(f_n) \in \ell^2$. Formally expanding $u = \sum_{n=1}^\infty u_n \sin nx$ gives

$$u_1 \text{ is arbitrary and } u_n = \frac{f_n}{n^2 - 1} \text{ for } n \geq 2.$$

An example of non-existence and non-uniqueness

- Let us check that $u_* := \sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx$ belongs to $H_0^1(0,\pi)$ and satisfies $Lu_* = f$ in the weak sense.
  - ⋆ The function $\sin nx \in H_0^1(0,\pi)$ and has norm
  $$\|\sin nx\|_{H^1}^2 = \int_0^\pi [n^2 \cos^2 nx + \sin^2 nx]\,dx = \frac{(n^2+1)\pi}{2}.$$
  - ⋆ The system $\{\sin nx\}$ is orthogonal in $H^1(0,\pi)$.
  - ⋆ It follows that
  $$\Big\|\sum_{m_1 \le n \le m_2} \frac{f_n}{n^2-1} \sin nx\Big\|_{H^1}^2 = \sum_{m_1 \le n \le m_2} \frac{f_n^2}{(n^2-1)^2} \frac{(n^2+1)\pi}{2} \le \frac{5\pi}{18} \sum_{m_1 \le n \le m_2} f_n^2 \stackrel{m_1,m_2 \to \infty}{\longrightarrow} 0.$$

An example of non-existence and non-uniqueness

- We are checking that $u_* := \sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx \in H_0^1(0,\pi)$ and $Lu_* = f$.
  - ⋆ Therefore, the series $\sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx$ converges in H^1 to $u_* \in H_0^1(0,\pi)$.
  - ⋆ To show that $Lu_* = f$, we consider the truncated series $u_{(N)} = \sum_{n=2}^{N} \frac{f_n}{n^2 - 1} \sin nx$ and $f_{(N)} = \sum_{n=2}^{N} f_n \sin nx$. These are smooth functions and satisfy $Lu_{(N)} = f_{(N)}$. The convergence of $u_{(N)}$ to u_* in H^1 and of $f_{(N)}$ to f in L^2 thus implies that $Lu_* = f$ (check this!).

An example of non-existence and non-uniqueness

$$\begin{cases} Lu &= -u'' - u = f, \\ u(0) &= u(\pi) = 0. \end{cases} \qquad (\heartsuit)$$

- We conclude that, for given $f \in L^2(0,\pi)$, $(\heartsuit)$ is solvable if and only if $\int_0^\pi f v_0\,dx = 0$. Furthermore, when that is the case, all solutions are of the form $u(x) = u_*(x) + C\sin x$ for some particular solution u_*.
- Exercise: Check that $u_* \in H^2(0,\pi)$.

An obstruction for existence and uniqueness

We now return to the general setting: $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ is a bounded linear operator from $H^1(\Omega)$ into $H^{-1}(\Omega)$.

- Uniqueness holds if and only if $L|_{H_0^1(\Omega)}$ is injective.
- Existence holds if and only if $L|_{H_0^1(\Omega)}$ is surjective.
- If $u \in H_0^1(\Omega)$ satisfies $Lu = T$, then for all $\varphi \in H_0^1(\Omega)$, we have

$$T\varphi = B(u,\varphi) = \int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i \partial_i u \varphi + cu\varphi\Big]\,dx.$$

  If we can integrate by parts once more, we then have

$$T\varphi = \int_\Omega u\Big[-\partial_j(a_{ij}\partial_i\varphi) + \partial_i(b_i\varphi) + c\varphi\Big]\,dx.$$

  Hence, if v_0 is such that $-\partial_j(a_{ij}\partial_i v_0) + \partial_i(b_i v_0) + cv_0 = 0$ in Ω, then we must necessarily have $Tv_0 = 0$.

The formal adjoint operator

Definition

Let $Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu$. The formal adjoint L^* of L is defined as the operator $L^* : H^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$L^* v = -\partial_i(a_{ij}\partial_j v) - \partial_i(b_i v) + cv,$$

$$L^* v(\psi) = \int_\Omega \Big[a_{ij}\partial_j\psi\partial_i v + b_i\partial_i\psi v + c\psi v\Big]\,dx \text{ for } \psi \in H_0^1(\Omega).$$

- The formal adjoint satisfies

$$Lu(v) = B(u,v) = L^* v(u) \text{ for all } u, v \in H_0^1(\Omega).$$

- For $v \in H^1(\Omega)$ and $T \in H^{-1}(\Omega)$, we have $L^* v = T$ if and only if

$$B(\psi, v) = T\psi \text{ for all } \psi \in H_0^1(\Omega).$$

The Fredholm alternative

Theorem (Fredholm alternative)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

(i) *The boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \qquad \text{(BVP)}$$

is uniquely solvable for each $f \in L^2(\Omega), g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$ if and only if $L|_{H_0^1(\Omega)}$ is injective.

(ii) *The kernels N of $L|_{H_0^1(\Omega)}$ and N^* of $L^*|_{H_0^1(\Omega)}$ are finite dimensional, and their dimensions are equal.*

(iii) *If N is non-trivial, (BVP) has a solution if and only if $B(u_0, v) = \langle f, v\rangle - \langle g_i, \partial_i v\rangle$ for all $v \in N^*$.*