Infinite Groups

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Comparison between solvable and polycyclic

We now proceed to compare the class of solvable groups with the smaller class of polycyclic groups. In order to do this, we need the following notion.

Definition

A group is said to be noetherian, or to satisfy the maximal condition if for every increasing sequence of subgroups

$$H_1 \leqslant H_2 \leqslant \cdots \leqslant H_n \leqslant \cdots \qquad (1)$$

there exists N such that $H_n = H_N$ for every $n \geqslant N$.

Proposition

A group G is noetherian if and only if every subgroup of G is finitely generated.

Proof of characterization of noetherian

Proof $\Rightarrow$ Assume there exists $H \leqslant G$ which is not finitely generated. Pick $h_1 = H \setminus \{1\}$ and let $H_1 = \langle h_1 \rangle$. Inductively, assume that

$$H_1 < H_2 < \dots < H_n$$

is a strictly increasing sequence of finitely generated subgroups of H, pick $h_{n+1} \in H \setminus H_n$, and set $H_{n+1} = \langle H_n, h_{n+1} \rangle$.

We thus have a strictly increasing infinite sequence of subgroups of G, contradicting the assumption that G is noetherian.

$\Leftarrow$ Assume that all subgroups of G are finitely generated.
Consider an increasing sequence of subgroups as in (1). Then $H = \bigcup_{n \geqslant 1} H_n$ is a subgroup, hence generated by a finite set S. There exists N such that $S \subseteq H_N$, hence $H_N = H = H_n$ for every $n \geqslant N$. $\square$

Back to the comparison between solvable and polycyclic

Proposition

A solvable group is polycyclic if and only if it is noetherian.

Proof The 'only if' part follows immediately from the fact that every polycyclic group is solvable, and its subgroups are polycyclic hence finitely generated.

To prove the 'if' part, let G be a noetherian solvable group.

We prove by induction on the derived length k that G is polycyclic.

For $k = 1$ the group is abelian, and since, being noetherian, G is finitely generated, it is polycyclic.

Comparison between solvable and polycyclic, continued

Assume the statement is true for k, consider a solvable group G of derived length $k+1$.

The commutator subgroup $G' \leqslant G$ is also noetherian and solvable of derived length k.

By the induction hypothesis, G' is polycyclic.

The abelianization $G_{ab} = G/G'$ is finitely generated (because G is), hence it is polycyclic.

It follows that G is polycyclic. □

Remarks

1. *There are noetherian groups that are not virtually polycyclic, e.g. Tarski monsters: finitely generated groups such that every proper subgroup is cyclic, constructed by Al. Olshanskii.*
2. *Polycyclic groups are noetherian $\Rightarrow$ given any property $(*)$ satisfied by the trivial group $\{1\}$, a polycyclic group contains a maximal subgroup with property $(*)$.*

Noetherian induction for polycyclic groups

We introduce a third type of inductive argument for polycyclic groups: the noetherian induction.

Assume that we have to prove that every polycyclic group has a certain property P. It suffices to check that:

- the trivial group $\{1\}$ has property P (initial case);
- a group G such that all its proper quotients G/N have P must have property P (inductive step).

Indeed, assume that, once all the above was checked, one finds a group G that does not have property P.

Let $(*)$ be the property " K is a normal subgroup K such that G/K does not have property P", and let N be a maximal subgroup satisfying $(*)$.

Then G/N is polycyclic, without property P, such that all its proper quotients have property P, contradicting the inductive step.

Example of f.g. solvable non-polycyclic group

Example

Recall that the lamplighter group is the wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$, and that it is finitely generated (Ex. Sheet 1).

- *The commutator subgroup G' coincides with the following subgroup of $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$:*

$$C = \{f : \mathbb{Z} \to \mathbb{Z}_2 \mid \mathrm{Supp}(f) \text{ has even cardinality}\}\,, \tag{2}$$

  where $\mathrm{Supp}(f) = \{n \in \mathbb{Z} \mid f(n) = 1\}$.
  [NB. The notation here is additive, the identity element is 0.]
  In particular, G' is not finitely generated. The group G is metabelian (since G' abelian).
- *Not all the subgroups in the lamplighter group G are finitely generated: G' is not, $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ is not.*

The lamplighter group continued

- G is not virtually torsion-free: For any finite-index subgroup $H \leqslant G$, $H \cap \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ has finite index in $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$; in particular this intersection is infinite and contains elements of order 2.
- G is not finitely presented.

  The last three statements imply that the lamplighter group is not polycyclic.

The Baumslag-Solitar group

An example of solvable (even metabelian) finitely presented group that is not polycyclic is the Baumslag–Solitar group.

$$G = BS(1,p) = \langle a, b | aba^{-1} = b^p \rangle \text{ for } |p| \geqslant 2.$$

The matrices

$$a = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate a subgroup of $SL(2, \mathbb{R})$ isomorphic to $BS(1, p)$.

Baumslag-Solitar group continued

- The derived subgroup G' of G is (isomorphic to)

$$G' = \left\{ \begin{pmatrix} 1 & mp^k \\ 0 & 1 \end{pmatrix} \; ; \; m, k \in \mathbb{Z} \right\}.$$

- Therefore $G = BS(1, p)$ is metabelian.
- The derived subgroup G' is not finitely generated. Hence G is not polycyclic.

Nilpotency class and derived length

Every nilpotent group is solvable. A natural question is to find a relationship between nilpotency class and derived length.

Proposition

1. *For every group G and every $i \geqslant 0$,*

$$G^{(i)} \leqslant C^{2^i} G.$$

2. *If G is a k-step nilpotent group then its derived length is at most*

$$[\log_2 k] + 1 .$$

Proof (1) by induction on $i \geqslant 0$.

The statement is obviously true for $i = 0$. Assume that it is true for i.

Then

$$G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right] \leqslant \left[C^{2^i}G, C^{2^i}G\right] \leqslant C^{2^{i+1}}G.$$

(2) follows immediately from (1). □

Remark

The derived length can be much smaller than the nilpotency class: the dihedral group D_{2n} with $n = 2^k$ is k-step nilpotent and metabelian.

Linear groups

In what follows $\mathbb{K}$ is an algebraically closed field (e.g. $\mathbb{C}$), V is a finite-dimensional vector space over $\mathbb{K}$.

$End(V)$ is the algebra of (linear) endomorphisms of V.

$GL(V)$ is the group of invertible endomorphisms of V.

Linear actions of groups G on V are called representations of G on V.

A group G that is isomorphic to a subgroup of $GL(V)$ for some V, is called a matrix group or a linear group.

The subalgebra of $End(V)$ generated by a linear group G will be denoted by $\mathbb{K}[G]$; this is just the linear span of G over $\mathbb{K}$.

Trace, $GL(V)$ and $End(V)$

Lemma

The map $\tau : End(V) \times End(V) \to \mathbb{K}$, $\tau(A, B) = \mathrm{trace}(AB)$ is a non-degenerate bi-linear form.

Fixing a basis for V determines:

- an isomorphism of groups $GL(V) \simeq GL_n(\mathbb{K})$, where $GL_n(\mathbb{K})$ is the group of invertible $n \times n$ matrices over $\mathbb{K}$;
- an isomorphism of algebras $End(V) \simeq M_n(\mathbb{K})$, where the latter is the algebra of all $n \times n$ matrices over $\mathbb{K}$.