C4.3 Functional Analytic Methods for PDEs
Lectures 13-14

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In the last lectures

- Linear elliptic equations of second order.
- Classical and weak solutions.
- Energy estimates.
- First existence theorem: Riesz representation theorem.
- First existence theorem: Direct method of the calculus of variation.
- Second existence theorem: Fredholm alternative.

This lecture

- Second existence theorem: Fredholm alternative.
- The compactness of the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.
- Third existence theorem: Spectral theory.
- H^2 regularity of weak solutions to linear elliptic equations.

The Fredholm alternative

Theorem (Fredholm alternative)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

(i) *The boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \quad \text{(BVP)}$$

is uniquely solvable for each $f \in L^2(\Omega), g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$ if and only if $L|_{H^1_0(\Omega)}$ is injective.

(ii) *The kernels N of $L|_{H^1_0(\Omega)}$ and N^* of $L^*|_{H^1_0(\Omega)}$ are finite dimensional, and their dimensions are equal.*

(iii) *If N is non-trivial, (BVP) has a solution if and only if $B(u_0, v) = \langle f, v\rangle - \langle g_i, \partial_i v\rangle$ for all $v \in N^*$.*

A consequence of the Fredholm alternative

Theorem

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. If the bilinear form B associated to L is coercive, i.e. there is a constant $C > 0$ such that

$$B(w,w) \geq C\|w\|^2_{L^2(\Omega)} \text{ for all } w \in C_c^\infty(\Omega),$$

then the boundary value problem

$$\left\{\begin{array}{rcll} Lu & = & f + \partial_i g_i & \text{in } \Omega, \\ u & = & u_0 & \text{on } \partial\Omega \end{array}\right. \tag{BVP}$$

has a unique solution for every $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$.

A consequence of the Fredholm alternative

Proof

- By density (— fill in the details —), we have

$$B(w, w) \geq C\|w\|^2_{L^2(\Omega)} \text{ for all } w \in H^1_0(\Omega).$$

- By the Fredholm alternative, it suffices to show that if $u \in H^1_0(\Omega)$ satisfies $Lu = 0$, then $u = 0$.
- By the definition of weak solution, we have $B(u, \varphi) = 0$ for all $\varphi \in H^1_0(\Omega)$. In particular $B(u, u) = 0$. By the coercivity of B, we thus have $\|u\|_{L^2} = 0$ and so $u = 0$.

A detour to FA

Definition

Let H be a Hilbert space. An bounded linear operator $K : H \to H$ is said to be *compact* if K maps bounded subset of H into pre-compact subsets of H.

Theorem (Fredholm alternative)

Let H be a Hilbert space and $K : H \to H$ be a compact bounded linear operator. Then we have the dichotomy that either $I - K$ is invertible or $\text{Ker}\,(I - K)$ is non-trivial.

Lemma

Let H be a Hilbert space and $K : H \to H$ be compact. If $Ker\,(I - K) = 0$, then $V = Im\,(I - K)$ is a closed subspace of H.

Proof

- Take $(u_m) \subset H$ such that $v_m = (I - K)(u_m) \to x$. We will show that $x \in V$ by showing that (u_m) has a convergent subsequence.
- It suffices to show that (u_m) is bounded. Indeed, once this is proved, as K is compact, there is a subsequence such that $Ku_{m_j} \to z$, and so $u_{m_j} = v_{m_j} + Ku_{m_j} \to x + z$.

A detour to FA

Proof

- Suppose by contradiction that (u_m) is not bounded, i.e. there is a subsequence (u_{m_j}) with $\|u_{m_j}\| \to \infty$.
- Let $\tilde{u}_{m_j} = \frac{u_{m_j}}{\|u_{m_j}\|}$ and $\tilde{v}_{m_j} = (I - K)\tilde{u}_{m_j} = \frac{v_{m_j}}{\|u_{m_j}\|}$.
- As (v_m) is convergent, $\tilde{v}_{m_j} \to 0$. We are thus in a similar situation as on the previous slide.
- In the same way, as $(\tilde{u}_{m_j})$ is bounded and K is compact, we can assume after passing to a subsequence if necessary that $K\tilde{u}_{m_j}$ converges to some $y \in H$.
- $\tilde{u}_{m_j} = \tilde{v}_{m_j} + K\tilde{u}_{m_j} \to y$.
- This amounts to a contradiction to the hypothesis that $Ker\,(I - K) = 0$: On one hand, as $\|\tilde{u}_{m_j}\| = 1$, we must have on $\|y\| = 1$. On the other hand, as $(I - K)\tilde{u}_{m_j} = \tilde{v}_{m_j}$, we have $(I - K)y = 0$.

A detour to FA

Theorem (Fredholm alternative)

Let H be a Hilbert space and $K : H \to H$ be a compact bounded linear operator. Then we have the dichotomy that either $I - K$ is invertible or $Ker\,(I - K)$ is non-trivial.

Proof

- Suppose by contradiction that $Ker\,(I - K) = 0$ but $Im\,(I - K)$ is a proper subspace of H.
- Let $V_0 = H$ and define inductively $V_{m+1} = (I - K)(V_m)$. We claim that V_{m+1} is a closed and proper subspace of V_m.
  - ⋆ By the lemma and the contradiction hypothesis, V_1 is a closed proper subspace of V_0.
  - ⋆ We have $(I - K)V_1 \subset (I - K)V_0 = V_1$. It follows that $KV_1 \subset V_1$. By the lemma again, $V_2 = (I - K)V_1$ is a closed subspace of V_1.

Proof

- We are proving the claim that V_{m+1} is a closed and proper subspace of V_m.
  - ⋆ V_1 is a closed proper subspace of V_0.
  - ⋆ V_2 is a closed subspace of V_1.
  - ⋆ As V_1 is a proper subspace of V_0, we can take $u \in V_0 \setminus V_1$.
  - ⋆ It is clear that $(I - K)u \in V_1$.
  - ⋆ If $(I - K)u \in V_2$, then there is some $(I - K)u = (I - K)w$ for some $w \in V_1$, contradicting the fact that $Ker\,(I - K) = 0$.
  - ⋆ We thus have $(I - K)u \in V_1 \setminus V_2$. Hence V_2 is a closed proper subspace of V_1.
  - ⋆ The claim follows by induction.

A detour to FA

Proof

- $H = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \ldots$ is a strict nested sequence of closed spaces.
- We now use the projection theorem to write $V_m = V_{m+1} \oplus W_{m+1}$ where W_{m+1} is the orthogonal complement of V_{m+1} within V_m.
- Take some $w_m \in W_{m+1} \subset V_m$ with $\|w_m\| = 1$. By the compactness of K, (Kw_m) has a convergent subsequence. To reach a contradiction, we show that $\|Kw_l - Kw_m\| \geq 1$ for $m > l$.

A detour to FA

Proof

- ... To reach a contradiction, we show that $\|Kw_l - Kw_m\| \geq 1$ for $m > l$.
  - We write
    $$Kw_l - Kw_m = \Big\{(I-K)w_m - (I-K)w_l - w_m\Big\} + w_l,$$
    and consider the terms in curly braces.
  - $w_l \in W_{l+1} \subset V_l$ and so $(I-K)w_l \subset V_{l+1}$.
  - $w_m \in W_{m+1} \subset V_m \subset V_{l+1}$.
  - $(I-K)w_m \in (I-K)(V_m) = V_{m+1} \subset V_{l+1}$.
  - So the terms in the curly braces belong to V_{l+1}.
  - As $w_l \in W_{l+1}$, we thus have by Pythagoras' theorem that $\|Kw_l - Kw_m\| \geq \|w_l\| = 1$.

  As explained earlier, this gives a contradiction to the compactness of K and thus concludes the proof.

The Fredholm alternative

Theorem (Fredholm alternative)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

1. *The boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \qquad \text{(BVP)}$$

   is uniquely solvable for each $f \in L^2(\Omega), g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$ if and only if $L|_{H_0^1(\Omega)}$ is injective.

2. *The kernels N of $L|_{H_0^1(\Omega)}$ and N^* of $L^*|_{H_0^1(\Omega)}$ are finite dimensional, and their dimensions are equal.*

3. *If N is non-trivial, (BVP) has a solution if and only if $B(u_0, v) = \langle f, v\rangle - \langle g_i, \partial_i v\rangle$ for all $v \in N^*$.*

The Fredholm alternative

Theorem (Uniqueness implies existence)

Suppose that Ω *is a bounded Lipschitz domain. Suppose that* $a, b, c \in L^\infty(\Omega)$, *a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. *Then* $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is bijective if and only if it is injective.*

Proof

- Step 1: Consideration of the top order operator L_{top} defined by $L_{top}u = -\partial_i(a_{ij}\partial_j u)$.
  - ⋆ We know from our first existence theorem that L_{top} is a bijection from $X = H_0^1(\Omega)$ in to X^*.
  - ⋆ Let $A : X^* \to X$ be the inverse of L_{top}. By the inverse mapping theorem, A is bounded linear.
  - ⋆ Let us give a direct proof for the boundedness of A. Suppose that $AT = u$, i.e. $L_{top}u = T$. Then $B_{top}(u, \varphi) = T\varphi$ where B_{top} is the bilinear form associated with L_{top}.

The Fredholm alternative

Proof

- Step 1: Consideration of the top order operator L_{top} defined by $L_{top}u = -\partial_i(a_{ij}\partial_j u)$.
  - ⋆ Using $\varphi = u$ and the ellipticity we have

$$\lambda\|\nabla u\|^2_{L^2(\Omega)} \leq \int_\Omega a_{ij}\partial_j u \partial_i u\, dx = B_{top}(u,u) = Tu \leq \|T\|\|u\|_X.$$

  - ⋆ Thus, by Friedrichs' inequality, we have

$$\|u\|^2_X \leq C\|Du\|^2_{L^2(\Omega)} \leq C\|T\|\|u\|_X,$$

and so $\|AT\|_X \leq C\|T\|$, i.e. A is bounded.

The Fredholm alternative

Proof

- Step 2: We recast the equation $Lu = T$ as an equation in the form $(I - K)u = AT$ where K is a linear operator from X into itself.
  - ⋆ We have

$$\begin{aligned} Lu = T &\Leftrightarrow L_{top}u + b_i\partial_i u + cu = T \\ &\Leftrightarrow A(L_{top}u + b_i\partial_i u + cu) = AT \\ &\Leftrightarrow u - A(-b_i\partial_i u - cu) = AT. \end{aligned}$$

  - ⋆ Hence $Lu = T$ is equivalent to $(I - K)u = AT$ with $Ku = A(-b_i\partial_i u - cu)$.
  - ⋆ We saw earlier in Lecture 11 that the map $u \mapsto -b_i\partial_i u - cu$ is a bounded linear map from X into X^*. Hence $K : X \to X$ is bounded linear.

The Fredholm alternative

Proof

- Step 3: We conclude using the Fredholm alternative for operators of the form $I - Compact$.
  - ⋆ To conclude, we need to show that $I - K$ is a bijection.
  - ⋆ Since $L : X \to X^*$ is injective, so is $I - K$. Hence, by the Fredholm alternative for operators of the form $I - Compact$, it suffices to show that K is compact, i.e. every bounded sequence $(u_m) \subset X$ has a subsequence u_{m_j} such that (Ku_{m_j}) is convergent.
  - ⋆ Suppose $(u_m) \subset X$ is bounded. As K is bounded, (Ku_m) is also bounded.
  - ⋆ As X is reflexive, we may assume after passing to a subsequence that $u_m \rightharpoonup u$ and $Ku_m \rightharpoonup w$ in $X = H_0^1(\Omega)$.
  - ⋆ In addition, by Rellich-Kondrachov's theorem, we may also assume that $u_m \to u$ and $Ku_m \to w$ in $L^2(\Omega)$.

The Fredholm alternative

Proof

- Step 3: We conclude using the Fredholm alternative...
  - ⋆ Claim: $w = Ku$.
    - ▷ We have $Ku_m = A(-b_i \partial_i u_m - cu_m)$ and so $L_{top}(Ku_m) = -b_i \partial_i u_m - cu_m$.
    - ▷ This means

$$\int_\Omega a_{ij} \partial_j (Ku_m) \partial_i \varphi \, dx = \int_\Omega (-b_i \partial_i u_m - cu_m) \varphi \, dx \text{ for all } \varphi \in H_0^1(\Omega).$$

    - ▷ Sending $m \to \infty$ using the fact that $u_m \rightharpoonup u$ and $Ku_m \rightharpoonup w$ in H^1 we get

$$\int_\Omega a_{ij} \partial_j w \partial_i \varphi \, dx = \int_\Omega (-b_i \partial_i u - cu) \varphi \, dx \text{ for all } \varphi \in H_0^1(\Omega).$$

    - ▷ This means $L_{top} w = -b_i \partial_i u - cu$, i.e. $w = L_{top}^{-1}(-b_i \partial_i u - cu) = Ku$.

The Fredholm alternative

Proof

- Step 3: We conclude using the Fredholm alternative...
  - ⋆ We thus have u_m converges weakly in H^1 and strongly in L^2 to u, and Ku_m converges weakly in H^1 and strongly in L^2 to Ku.
  - ⋆ We need to upgrade the weak convergence of Ku_m in H^1 to strong convergence. By working instead with the sequence $u_m - u$, we may assume at this point that $u = 0$.
  - ⋆ Recall that $L_{top}(Ku_m) = -b_i \partial_i u_m - cu_m$ and so

$$\int_\Omega a_{ij} \partial_j (Ku_m) \partial_i \varphi \, dx = \int_\Omega (-b_i \partial_i u_m - cu_m) \varphi \, dx \text{ for all } \varphi \in H_0^1(\Omega).$$

  - ⋆ Taking $\varphi = Ku_m$, and using ellipticity we thus find

$$\lambda \|\nabla K u_m\|^2_{L^2(\Omega)} \leq \|b_i \partial_i u_m + cu_m\|_{L^2(\Omega)} \|Ku_m\|_{L^2(\Omega)}$$

  The first factor is bounded and the second factor goes to 0.

The Fredholm alternative

Proof

- Step 3: We conclude using the Fredholm alternative...
  - ⋆ So we have proven that $\nabla K u_m \to 0$ in L^2. Together with the fact that $K u_m \to 0$ in L^2, we have that $K u_m \to 0$ in H^1.
  - ⋆ We conclude that K is compact.
  - ⋆ As $I - K$ is injective, we conclude that $I - K$ is invertible, and so is L.

Compactness of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$

Let us make a couple of remarks on the proof.

- One of the ideas in the proof is to write $Lu = T$ in the form $(I - K)u = L_{top}^{-1} \circ T$ where $K : H_0^1(\Omega) \to H_0^1(\Omega)$ is compact.
- The operator K is given by $Ku = L_{top}^{-1}(-b_i \partial_i u - cu)$. Hence $K = L_{top}^{-1} \circ B$ where $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ is given by

$$Bu = -b_i \partial_i u - cu,$$

$$\text{i.e. } Bu(\varphi) = \int_\Omega (-b_i \partial_i u - cu)\varphi \, dx \text{ for } \varphi \in H_0^1(\Omega).$$

- The operator B can be decompose further as $B = J \circ B_0$ where $B_0 : H_0^1(\Omega) \to L^2(\Omega)$ is given by $B_0 u = -b_i \partial_i u - cu$ and $J : L^2(\Omega) \to H^{-1}(\Omega)$ is the natural injection given by

$$Jv(\varphi) = \int_\Omega v\varphi \, dx \text{ for } v \in L^2(\Omega), \varphi \in H_0^1(\Omega).$$

Compactness of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$

- Altogether we have the chain $K = L_{top}^{-1} \circ J \circ B_0$:

$$K : H_0^1(\Omega) \stackrel{B_0}{\to} L^2(\Omega) \stackrel{J}{\to} H^{-1}(\Omega) \stackrel{L_{top}^{-1}}{\to} H_0^1(\Omega).$$

- We have the following compactness result for J, which also implies the compactness of K.

Theorem

Suppose that Ω is a bounded Lipschitz domain. Then the natural injection $J : L^2(\Omega) \to H^{-1}(\Omega)$ defined by

$$Jv(\varphi) = \int_\Omega v\varphi\,dx \text{ for } v \in L^2(\Omega) \text{ and } \varphi \in H_0^1(\Omega)$$

is compact, i.e. if (v_m) is bounded in $L^2(\Omega)$, then there is a subsequence (v_{m_j}) such that (Jv_{m_j}) is convergent in $H^{-1}(\Omega)$.

Compactness of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$

Proof

- Suppose (v_m) is bounded in $L^2(\Omega)$.
  Then there is a subsequence (v_{m_j}) which converges weakly in L^2 to some limit $v \in L^2(\Omega)$.
- We aim to show that (Jv_{m_j}) converges in H^{-1} to Jv.
- By working with $v_{m_j} - v$ instead of v_{m_j}, we may assume that $v = 0$.
- Suppose by contradiction that $Jv_{m_j} \not\to 0$. Passing to a subsequence, we may assume that

$$\|Jv_{m_j}\|_{H^{-1}} > \delta > 0.$$

- Let w_j be the solution to

$$\begin{cases} -\Delta w_j + w_j &= v_{m_j} \quad \text{in } \Omega, \\ w_j &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Compactness of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$

Proof

- As $Jv_{m_j} \neq 0$, we have that $w_j \neq 0$. Also, by definition of weak solution, we have

$$\int_\Omega v_{m_j}\varphi\,dx = \int_\Omega [\nabla w_j \cdot \nabla\varphi + w_j\varphi]\,dx \text{ for all } \varphi \in H_0^1(\Omega).$$

  This means

$$Jv_{m_j}(\varphi) = \langle w_j, \varphi\rangle_{H^1} \text{ for all } \varphi \in H_0^1(\Omega).$$

- Observe that if we take supremum over $\varphi \in H_0^1(\Omega)$ with $\|\varphi\|_{H_0^1(\Omega)} \leq 1$, then the supremum of the right hand side is attained exactly at $\varphi_j := \frac{w_j}{\|w_j\|_{H^1}}$.

Compactness of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$

Proof

- We thus have, for $\varphi_j = \frac{w_j}{\|w_j\|_{H^1}}$,

$$\|Jv_{m_j}\|_{H^{-1}} = Jv_{m_j}(\varphi_j) = \int_\Omega v_{m_j}\varphi_j\,dx.$$

- The sequence (φ_j) is bounded in $H^1(\Omega)$. By Rellich-Kondrachov's theorem, we may assume after passing to a subsequence, that φ_j converges strongly in L^2 to some $\varphi_* \in L^2(\Omega)$.
- Now as v_{m_j} converges weakly to $v = 0$ in $L^2(\Omega)$, we arrive at

$$\lim_{j\to\infty} \|Jv_{m_j}\|_{H^{-1}} = \lim_{j\to\infty} \int_\Omega v_{m_j}\varphi_j\,dx = \int_\Omega 0\varphi_*\,dx = 0,$$

contradicting the statement that $\|Jv_{m_j}\|_{H^{-1}} > \delta > 0$.

Spectra of elliptic operators

Theorem (Spectrum of an elliptic operator)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Then there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{EBVP}$$

has a unique solution if and only if $\lambda \notin \Sigma$. Furthermore, if Σ is infinite then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ with

$$\lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$

The set Σ is called the real spectrum of the operator L.

Spectra of elliptic operators

Proof

- Let B be the bilinear form associated with L. Recall the energy estimate: There exists $\mu > 0$ depending on the L^∞ bounds for a, b, c and the ellipticity constant λ such that

$$\frac{\lambda}{2}\|u\|_{H^1(\Omega)}^2 \leq B(u,u) + \mu\|u\|_{L^2(\Omega)}^2.$$

- If we define $L_\mu u = Lu + \mu u$ and let B_μ be the bilinear form associated with L_μ, then the right hand side above is exactly $B_\mu(u,u)$.
- So B_μ is coercive. By the Fredholm alternative, the operator $L_\mu : H_0^1(\Omega) \to H^{-1}(\Omega)$ is invertible. Denote its inverse by S_μ.

Spectra of elliptic operators

Proof

- Define an operator $K : L^2(\Omega) \to L^2(\Omega)$ by:
$$K : L^2(\Omega) \overset{J}{\hookrightarrow} H^{-1}(\Omega) \overset{S_\mu}{\to} H_0^1(\Omega) \overset{Id}{\hookrightarrow} L^2(\Omega).$$
The last leg is compact by Rellich-Kondrachov's theorem, hence K is compact.
(We also know that J is compact, but that is a harder statement.)
- Let Σ be the set of $\lambda \in \mathbb{R}$ such that (EBVP) is not always uniquely solvable. By the Fredholm alternative,
$$\begin{aligned}\lambda \in \Sigma &\Leftrightarrow (L - \lambda Id) \text{ is not injective}\\ &\Leftrightarrow (L_\mu - (\lambda + \mu)Id) \text{ is not injective}\\ &\Leftrightarrow I - (\lambda + \mu)K \text{ is not injective}\\ &\Leftrightarrow \lambda + \mu \neq 0 \text{ and } (\lambda + \mu)^{-1} \in \sigma_p(K).\end{aligned}$$

Spectra of elliptic operators

Proof

- ... $\lambda \in \Sigma$ if and only if $\lambda + \mu \neq 0$ and $(\lambda + \mu)^{-1} \in \sigma_p(K)$. The conclusion follows from a general result for spectra of compact operators, which we take for granted.

Theorem (Spectra of compact operators)

Let H be a Hilbert space of infinite dimension, $K : H \to H$ be a compact bounded linear operator and $\sigma(K)$ be its spectrum (i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda I - K$ is not invertible). Then

1. 0 *belongs to* $\sigma(K)$.
2. $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$, *i.e.* $\lambda I - K$ *has non-trivial kernel for* $\lambda \in \sigma(K) \setminus \{0\}$.
3. $\sigma(K) \setminus \{0\}$ *is either finite or an infinite sequence tending to* 0.

The question of regularity

In the rest of this course we consider regularity results for solutions to

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f \text{ in a domain } \Omega$$

with $f \in L^2(\Omega)$.

- We want to keep in mind the following two motivating examples in $1d$:

$$-u'' = f \text{ in } (-1,1) \qquad (*)$$

  and

$$-(au')' = f \text{ in } (-1,1) \text{ where } a = \chi_{(-1,0)} + 2\chi_{(0,1)}. \qquad (**)$$

- For (*), u belongs to H^2.
- For (**), au' belongs to H^1. Typically this implies u' is discontinuous and hence $u \notin H^2$. Nevertheless u is continuous.

Interior H^2 regularity

Theorem (Interior H^2 regularity)

Suppose that $a \in C^1(\Omega)$, $b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that $f \in L^2(\Omega)$.
If $u \in H^1(\Omega)$ satisfies $Lu = f$ in Ω in the weak sense then $u \in H^2_{loc}(\Omega)$, and for any open ω such that $\bar{\omega} \subset \Omega$ we have

$$\|u\|_{H^2(\omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on $n, \Omega, \omega, a, b, c$.

Global H^2 regularity

Theorem (Global H^2 regularity)

Suppose that Ω is a bounded domain and $\partial\Omega$ is C^2 regular. Suppose that $a, b, c \in C^1(\bar{\Omega})$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that $f \in L^2(\Omega)$.
If $u \in H_0^1(\Omega)$ satisfies $Lu = f$ in Ω in the weak sense then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on n, Ω, a, b, c.

Remark: If $\partial\Omega$ is C^∞, $a, b, c \in C^\infty(\bar{\Omega})$, and $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

The case of $-\Delta$

To illustrate the idea, we focus in the case a is constant, $b \equiv 0$, $c \equiv 0$. The local H^2 regularity result is equivalent to:

Theorem (Interior H^2 regularity for $-\Delta$)

Suppose $f \in L^2(B_2)$ and $u \in H^1(B_2)$. If $-\Delta u = f$ in B_2 in the weak sense, then $u \in H^2(B_1)$ and

$$\|u\|_{H^2(B_1)} \leq C(\|f\|_{L^2(B_2)} + \|u\|_{H^1(B_2)})$$

where the constant C depends only on n.

The case of $-\Delta$

The start of the proof is the following simple but important lemma:

Lemma

Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. Then

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|\Delta u\|_{L^2(\mathbb{R}^n)}.$$

The proof is a computation using integration by parts:

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \partial_i\partial_j u \partial_i\partial_j u\,dx = -\int_{\mathbb{R}^n} \partial_j u \partial_j \partial_i^2 u\,dx$$
$$= \int_{\mathbb{R}^n} \partial_j^2 u \partial_i^2 u\,dx = \|\Delta u\|_{L^2(\mathbb{R}^n)}^2.$$

The case of $-\Delta$

The following lemma is a generalisation in the weak setting:

Lemma

Suppose that $f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$ and u has compact support. Suppose that $-\Delta u = f$ in $\mathbb{R}^n$ in the weak sense. Then $u \in H^2(\mathbb{R}^n)$ and

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof of the lemma

- Take a family of mollifiers (ϱ_ε): Fix a non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and let $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.
- Set $u_\varepsilon = \varrho_\varepsilon * u$ and $f_\varepsilon = \varrho_\varepsilon * f$. Then $u_\varepsilon, f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$ and $f_\varepsilon \to f$ in $L^2(\mathbb{R}^n)$.

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ Fix $v \in C_c^\infty(\mathbb{R}^n)$ and consider $\displaystyle\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx$.
  - ⋆ Recall that, as $u \in H^1(\mathbb{R}^n)$, $\nabla u_\varepsilon = \varrho_\varepsilon * \nabla u$.
  - ⋆ Hence, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx &= \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\partial_{y_i} u(y)\,dy\Big]\partial_{x_i} v(x)\,dx\\ &= \int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\partial_{x_i} v(x)\,dx\Big]\,dy.\end{aligned}$$

  - ⋆ Integrating by parts in the inner integral we get

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \partial_{x_i}\varrho_\varepsilon(x-y) v(x)\,dx\Big]\,dy.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ $\displaystyle\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \partial_{x_i} \varrho_\varepsilon(x-y) v(x)\, dx \Big]\, dy.$
  - ⋆ Now observe that $\partial_{x_i}\varrho_\varepsilon(x-y) = -\partial_{y_i}\varrho_\varepsilon(x-y)$.
  - ⋆ We thus have, by Fubini's theorem again,

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx &= \int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \partial_{y_i} \varrho_\varepsilon(x-y) v(x)\, dx \Big]\, dy\\ &= \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \partial_{y_i} u(y) \partial_{y_i} \varrho_\varepsilon(x-y)\, dy \Big] v(x)\, dx.\end{aligned}$$

  - ⋆ As $-\Delta u = f$ in the weak sense, the inner integral is equal to $\displaystyle\int_{\mathbb{R}^n} f(y)\, \varrho_\varepsilon(x-y)\, dy$, which is $f_\varepsilon(x)$.
  - ⋆ We deduce that

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx = \int_{\mathbb{R}^n} f_\varepsilon(x) v(x)\, dx.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ As v was picked arbitrarily in $C_c^\infty(\mathbb{R}^n)$, we have that $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$ in the weak sense.
  - ⋆ As u_ε and f_ε are smooth, this equation also holds in the classical sense. (Check this!)
- Now, by the previous lemma, we have

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|\Delta u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|f_\varepsilon\|_{L^2(\mathbb{R}^n)}.$$

- Young's convolution inequality gives $\|f_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \|\varrho_\varepsilon\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$, and so

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}.$$

- Therefore, along a subsequence, $(\nabla^2 u_\varepsilon)$ converges weakly to some $A \in L^2(\mathbb{R}^n; \mathbb{R}^{n\times n})$ with $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.

The case of $-\Delta$

Proof of the lemma

- Putting things together we have $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$, $\nabla^2 u_\varepsilon \rightharpoonup A$ in $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$.
- Claim: A is the weak second derivatives of u.
  Indeed, this follows by passing $\varepsilon \to 0$ in the identity

$$\int_{\mathbb{R}^n} u_\varepsilon \partial_i \partial_j v = \int_{\mathbb{R}^n} \partial_i \partial_j u_\varepsilon v \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

- We have thus shown that $u \in H^2(\mathbb{R}^n)$ and $\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.