Infinite Groups

Cornelia Druțu

University of Oxford

Part C course MT 2022

About Mathematics

Mikhail L. Gromov: "Understanding what and why did not work may be more instructive than celebrating our successes."

Carl Friedrich Gauss: "Theory attracts practice as the magnet attracts iron."

Linear groups

$\mathbb{K}$ is an algebraically closed field.

Linear group: a group G isomorphic to a subgroup of $GL(V)$.

The subalgebra of $End(V)$ generated by G is denoted by $\mathbb{K}[G]$; this is just the linear span of G over $\mathbb{K}$.

Lemma

The map $\tau : End(V) \times End(V) \to \mathbb{K}$, $\tau(A, B) = \mathrm{trace}(AB)$ is a non-degenerate bi-linear form.

Fixing a basis for V determines:

- an isomorphism of groups $GL(V) \simeq GL_n(\mathbb{K})$, where $GL_n(\mathbb{K})$ is the group of invertible $n \times n$ matrices over $\mathbb{K}$;
- an isomorphism of algebras $End(V) \simeq M_n(\mathbb{K})$, where the latter is the algebra of all $n \times n$ matrices over $\mathbb{K}$.

Irreducible, reducible and triangularizable actions

If V is a vector space and $A \leqslant End(V)$ is a subgroup, then A is said to act irreducibly on V if V contains no proper subspace $\{0\} \subsetneq V' \subsetneq V$ such that $aV' \subset V'$ for all $a \in A$.

We say that the action of A on V is completely reducible if V decomposes as a direct sum of irreducible subspaces.

A linear group $G \leq GL(V)$ is called triangularizable if there exists a basis of V with respect to which G is represented by upper-triangular matrices.

Actions of abelian groups

Lemma

If A is an abelian group acting irreducibly on V then V has dimension 1.

Proof. $\mathbb{K}$ algebraically closed $\Rightarrow$ every $a \in A$ has at least an eigenvalue.
A abelian $\Rightarrow$ the corresponding space of eigenvectors is b-invariant for every $b \in A$, hence it must coincide with V.
Thus, every $a \in A$ is a multiple of the identity map on V, hence by irreducibility V must have dimension 1. □

Proposition

If A is an abelian group acting on V then there exists a basis of V with respect to which A becomes upper triangular.

Actions of abelian groups, continued

Proof. By induction on the dimension of V. Obvious in dimension 1.
Assume true for dimension $< n$, take V of dimension n.
If A acts irreducibly apply previous Lemma.
Assume A acts reducibly, and preserves a proper subspace $V' < V$.
We obtain two induced actions of A: on V' (by restriction) and on $V'' = V/V'$ (by projection).
Both actions become actions by triangular matrices with the right choice of basis.
The combination of the two bases yields a basis in V with respect to which A becomes upper triangular. □

Our goal is to explain a generalization of this last result to solvable groups.

Burnside Theorem and applications

Theorem (Burnside's Theorem)

If $A \subset End(V)$ is a subalgebra which acts absolutely irreducibly on a finite-dimensional vector space V, then $A = End(V)$. In particular, if $G \leqslant End(V)$ is a subsemigroup acting irreducibly, then G spans $End(V)$ as a vector space, i.e. $\mathbb{K}[G] = End(V)$.

Theorem

Suppose that $G \leq \mathrm{GL}_n(\mathbb{K})$ is irreducible and that

$$|\{\mathrm{tr}(g) \mid g \in G\}| = q < \infty.$$

Then $|G| \leq q^{n^2}$.

Proof of first application to Burnside

Proof. By Burnside's Theorem, G contains $m = n^2$ linearly independent matrices $w(1), \ldots, w(m)$.
For $\underline{\mu} \in \mathbb{K}^m$ let

$$G(\underline{\mu}) = \{g \in G \mid \mathrm{tr}(w(s)g) = \mu_s \ (s = 1, \ldots, m)\}.$$

Observe that $g = (g_{ij}) \in G(\underline{\mu})$ if and only if it satisfies the equations

$$\sum_{i=1}^{n} \sum_{l=1}^{n} w(s)_{il} g_{li} = \mu_s \quad (s = 1, \ldots, m).$$

This is a system of $m = n^2$ linearly independent equations, so it has at most one solution (g_{ij}). The result follows as there are just q^{n^2} possibilites for $\underline{\mu}$. □

Second application to Burnside. Nilpotent and unipotent.

Corollary

Suppose that $G \leq \mathrm{GL}_n(\mathbb{K})$ is completely reducible and that $g^e = 1,\ \forall g \in G$. Then $|G| \leq e^{n^3}$.

Proof. See Ex. Sheet 4. □

$h \in End(V)$ is nilpotent if $h^k = 0$ for some $k > 0$
$\Leftrightarrow$ in some basis, h can be written as an upper triangular matrix with zeroes on the diagonal.
$\Leftrightarrow$ the only eigenvalue of h is 0.

$g \in End(V)$ is unipotent if $g = \mathrm{id} + h$, where h is nilpotent
$\Leftrightarrow$ the only eigenvalue of g is 1.
$g \in End(V)$ is quasi-unipotent if g^k is unipotent for some $k > 0$
$\Leftrightarrow$ all the eigenvalues of g are roots of unity.

A subgroup $G < GL(V)$ is unipotent (respectively quasi-unipotent) if every element of G is unipotent (respectively, quasi-unipotent).

Kolchin's theorem

Theorem (Kolchin's theorem)

Suppose that $\mathbb{K} = \bar{\mathbb{K}}$ and $G < GL(V)$ is a unipotent subgroup. Then, for an appropriate choice of basis, G is isomorphic to a subgroup of the group of invertible upper-triangular matrices $\mathcal{T}_n(\mathbb{K})$. In particular G is nilpotent.

Proof. The conclusion is equivalent to the statement that G preserves a full flag

$$0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V,$$

where $i = \dim(V_i)$ for each i.
The proof is by induction on the dimension n of V.
Clear for $n = 1$.
We assume that $n > 1$ and that the statement is true for dimensions $< n$.

Proof of Kolchin's theorem continued

Suppose first that the action of G on V is reducible: G preserves a proper subspace $V' \subset V$.
We obtain two induced actions of G on V' (by restriction) and on $V'' = V/V'$ (by projection).
Both actions preserve full flags in V', V'' (induction assumption), and the combination of these flags yields a full G-invariant flag in V.
Assume now that the action of G on V is irreducible.
For each $g \in G$ the endomorphism $g' = g - I$ is nilpotent, hence it has zero trace.
Therefore, for all $x \in G$, we have

$$tr(g'x) = tr(gx - x) = tr(I) - tr(I) = 0.$$

Since, by Burnside's theorem, G spans $End(V)$, we conclude that for each $x \in End(V)$ and each $g \in G$, $tr(g'x) = 0$.
$\tau : End(V) \times End(V) \to \mathbb{K}$ is nondegenerate $\Rightarrow g' = 0$ for all $g \in G$, i.e. $G = \{1\}$. □