Infinite Groups

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Variation of Kolchin's theorem

Theorem (Variation of Kolchin's theorem)

Suppose $\mathbb{K} = \bar{\mathbb{K}}$, $G < GL(V)$ quasiunipotent and, moreover, there exists an upper bound α on the orders of all eigenvalues of elements $g \in G$. Then G contains a finite index subgroup isomorphic to a subgroup of the group $\mathcal{U}_n(\mathbb{K})$ of upper triangular matrices with 1 on the diagonal. The index depends only on V and on α. In particular G is virtually nilpotent.

Proof. By induction on the dimension of V.

As before, it suffices to consider the case when G acts irreducibly on V. The orders of the eigenvalues of elements of G are uniformly bounded $\Rightarrow$ the set of traces of elements of G is a certain finite set $C \subset \mathbb{K}$ of cardinality $q = q(\alpha)$.

Using a previous theorem, we conclude that the group G is finite, of cardinality at most q^{n^2}, where $n = \dim V$. □

Solvable linear groups

Theorem (Lie-Kolchin-Mal'cev Theorem)

Let $G \leqslant GL(V)$ be solvable linear, with $n = \dim V$ (G is solvable linear of degree n). Then G has a triangularizable normal subgroup K of finite index at most $\mu(n)$, a number that depends only on n.

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of groups.
A group G is $\mathcal{X}$-by-$\mathcal{Y}$ if there exists a short exact sequence

$$\{1\} \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow \{1\},$$

such that $N \in \mathcal{X}$ and $Q \in \mathcal{Y}$.

Solvable linear groups

Corollary

Let G be a solvable linear group of degree n.

(i) *G is virtually unipotent-by-abelian;*

(ii) *(the Zassenhaus Theorem) the derived length of G is at most $\beta(n) := n + \log_2 \mu(n)$.*

This can be combined with the following general result.

Theorem

Every nilpotent subgroup of $GL(n, \mathbb{Z})$ is finitely generated.

The two previous results imply the following.

Corollary

Every finitely generated solvable group linear over $\mathbb{Z}$ is polycyclic.

Solvable groups linear over $\mathbb{Z}$

Proof. Such a group G has a finite index subgroup that is unipotent-by-abelian, hence a finite index normal subgroup N that is polycyclic.

The quotient G/N is solvable and finite, hence noetherian, hence polycyclic. Hence G is polycyclic. □

The converse is also true.

Theorem (Auslander's Theorem)

Every polycyclic group is linear over $\mathbb{Z}$.

Corollary

Every polycyclic group is virtually (finitely generated nilpotent)-by-(f.g. abelian).

Another comparison between solvable and polycyclic

We have thus obtained the following way of distinguishing polycyclic groups among solvable groups.

Theorem

Given a f.g. solvable group G, the following are equivalent:

- *G is polycyclic;*
- *G is linear over $\mathbb{Z}$.*

Comparison between solvable and nilpotent: growth

We now arrive at the final topic of this course: the way to distinguish nilpotent groups in the larger class of solvable groups *via* their growth.

This is the celebrated Milnor-Wolf Theorem.

Along the way we will discover new features that allow to distinguish between solvable and polycyclic, polycyclic and nilpotent.

Let $G = \langle S \rangle$, where S finite, $S^{-1} = S$, $1 \notin S$.
Let dist_S be the word metric associated to S.
The growth function of G with respect to S is

$$\mathfrak{G}_{G,S}(R) := \mathrm{card}\, \bar{B}(1, R).$$

Sometimes, when there is no risk of confusion, we write simply $\mathfrak{G}_S(R)$.
In order to understand how much does $\mathfrak{G}_{G,S}$ depend on S, and how much on G we introduce the following relations.

Growth functions

Definition

Given $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$, we define an asymptotic inequality $f \preceq g$ if there exist $a, b > 0$, $c \geqslant 0$ and $x_0 \in \mathbb{R}$ such that for all $x \in X$, $x \geqslant x_0$, we have $bx + c \in X$ and $f(x) \leqslant ag(bx + c)$. If $f \preceq g$ and $g \preceq f$ then we write $f \asymp g$ and we say that f and g are asymptotically equal.

Lemma

Assume that (G, dist_S) and (H, dist_X) are bi-Lipschitz equivalent, i.e. $\exists L > 0$ and a bijection $f : G \to H$ such that

$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_X(f(g), f(g')) \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \tag{1}$$

Then $\mathfrak{G}_{G,S} \asymp \mathfrak{G}_{H,X}$.
This is in particular true when $(H, \mathrm{dist}_X) = (G, \mathrm{dist}_{S'})$, for another finite set S' generating G.

Growth functions

Corollary

If S, S' are two finite generating sets of G then $\mathfrak{G}_S \asymp \mathfrak{G}_{S'}$. Thus one can speak about the growth function $\mathfrak{G}_G$ of a group G, well defined up to the equivalence relation $\asymp$.

Examples

1. If $G = \mathbb{Z}^k$ then $\mathfrak{G}_S \asymp x^k$ for every finite generating set $S = S^{-1}$.
2. If $G = F_k$ is the free group of finite rank $k \geqslant 2$ and S is the set of k generators then

$$\mathfrak{G}_{S \sqcup S^{-1}}(n) = 1 + (q^n - 1)\frac{q+1}{q-1}, \quad q = 2k - 1.$$

Growth functions: properties

Proposition

1. *If G is infinite, $\mathfrak{G}_G|_{\mathbb{N}}$ is strictly increasing.*
2. *If $H \leqslant G$ then $\mathfrak{G}_H \preceq \mathfrak{G}_G$.*
3. *If $H \leqslant G$ finite index then $\mathfrak{G}_H \asymp \mathfrak{G}_G$.*
4. *If $N \lhd G$ then $\mathfrak{G}_{G/N} \preceq \mathfrak{G}_G$.*
5. *If $N \lhd G$, N finite, then $\mathfrak{G}_{G/N} \asymp \mathfrak{G}_G$.*
6. *For each finitely generated group G, $\mathfrak{G}_G(r) \preceq 2^r$.*
7. *The growth function is sub-multiplicative:*

$$\mathfrak{G}_{G,S}(r+t) \leqslant \mathfrak{G}_{G,S}(r)\mathfrak{G}_{G,S}(t)\,.$$

$\mathfrak{G}_{G,S}$ sub-multiplicative $\Rightarrow \ln \mathfrak{G}_{G,S}(n)$ sub-additive.

By Fekete's Lemma, there exists a (finite) limit

$$\lim_{n\to\infty} \frac{\ln \mathfrak{G}_{G,S}(n)}{n}.$$

Hence, we also have a finite limit

$$\gamma_{G,S} = \lim_{n\to\infty} \mathfrak{G}_{G,S}(n)^{\frac{1}{n}},$$

called growth constant. The property (1) implies that $\mathfrak{G}_{G,S}(n) \geqslant n$; whence, $\gamma_{G,S} \geqslant 1$.

Definition

If $\gamma_{G,S} > 1$ then G is said to be of exponential growth. If $\gamma_{G,S} = 1$ then G is said to be of sub-exponential growth.

Note that if there exists a finite generating set S such that $\gamma_{G,S} > 1$ then $\gamma_{G,S'} > 1$ for every other finite generating set S'. Likewise for the equality to 1.