SUPPLEMENTARY APPLIED MATHS

Sturm-Liouville form: $Ly = -(p(x)y')' + q(x)y = \lambda r(x)y$, $r(x) \geq 0$ weight function. For evals and evecs use homogeneous BCs, then orthogonality relation is $\int y_m(x)y_n(x)r(x)dx = 0$ for $m \neq n$.

General L with BC and adjoint $L^\star$ with BC*: $Ly_k = \lambda_k y_k$ with BC=0 and adjoint eigenfunctions $L^\star w_k = \lambda_k w_k$ with BC*=0. Orthogonality: $\int y_m(x)w_n(x)dx = 0$ for $m \neq n$.

Solve $Ly = f$ with inhomogeneous BCs: $\langle f, w_k\rangle = \langle Ly, w_k\rangle = \langle y, L^\star w_k\rangle + BT = \langle y, \lambda_k w_k\rangle + BT$. Subst $y = \sum c_n y_n$: $\langle f, w_k\rangle = \sum c_n \lambda_k \langle y_n, w_k\rangle + BT = c_k \lambda_k \langle y_k, w_k\rangle + BT$ (by orthogonality). Gives $c_k = (\langle f, w_k\rangle + BT)/(\lambda_k\langle y_k, w_k\rangle)$ if $\lambda_k \neq 0$.

Green's function: $Lg = \delta(x-\xi)$ with homogeneous BCs. Then $y(x) = \int g(x;\xi)f(\xi)d\xi + y_h(x)$ where $Ly_h = 0$ and y_h satisfies inhomogeneous BCs. If $Ly = p(x)y'' + \ldots$, then $[g']_{\xi-}^{\xi+} = 1/p(\xi)$.

Distributions: T is a distribution if it is linear and continuous, i.e.

(i) $\langle T, \alpha\phi_1 + \beta\phi_2\rangle = \alpha\langle T, \phi_1\rangle + \beta\langle T, \phi_2\rangle$, $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$.

(ii) If $\phi_n(x)$ is a sequence of test functions s.t. $\phi_n(x) \to 0$ as $n \to \infty$ then $\langle T, \phi_n\rangle \to 0$ as a sequence of real numbers. Then $\lim_{n\to\infty}\langle T, \phi_n\rangle = \langle T, \lim_{n\to\infty}\phi_n\rangle$.

Equivalent continuity condition (for checking): $\forall L > 0$, $\exists C > 0$ and $N \geq 0$ s.t. $|\langle T, \phi\rangle| \leq C\sum_{m\leq N}\max_{x\in\mathbb{R}}|\phi^{(m)}(x)|$, $\forall\phi$ s.t. supp $\phi \subset [-L, L]$.

Translation property: $\langle T(x+\alpha), \phi(x)\rangle = \langle T(x), \phi(x-\alpha)\rangle$, $\forall\phi \in C_0^\infty(\mathbb{R})$.

Distributional derivative: $\langle T', \phi\rangle = -\langle T, \phi'\rangle$, $\forall\phi \in C_0^\infty(\mathbb{R})$.

Convergence of T_j to T as $j \to \infty$ means $\lim_{j\to\infty}\langle T_j, \phi\rangle = \langle T, \phi\rangle$, $\forall\phi \in C_0^\infty(\mathbb{R})$.

If $T(\alpha)$ is a family of distributions with continuous parameter α then $T(\alpha) \to T(\alpha_0)$ for $\alpha \to \alpha_0$ means $\lim_{\alpha\to\alpha_0}\langle T(\alpha), \phi\rangle = \langle T(\alpha_0), \phi\rangle$, $\forall\phi \in C_0^\infty(\mathbb{R})$.

APPLIED PDEs